

# Solutions of the system of d'Alembert and eikonal equations, and classification of reductions of PDEs

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**Abstract.** We present an approach to systematic description and classification of solutions of partial differential equations that are obtained by means of reduction of these equations to other equations with smaller number of independent variables. We propose to classify such reductions by means of classification of reduction conditions. The approach is illustrated by an example of the system of d'Alembert and eikonal equations. Solutions of this system were used to outline classification of reductions for the general nonlinear d'Alembert equation, with generalisation to arbitrary Poincaré invariant equations.

## 1. Introduction

In this paper we propose a new view to a problem that seemed to be solved long ago – how to classify solutions of partial differential equations (PDEs), obtained by means of reduction to new equations with smaller number of independent variables. We will discuss from this point some well-known methods, such as Lie algorithm of symmetry reduction and various non-classical approaches.

This paper continues development of ideas presented in the talks at the previous GADEIS conferences [1–3]. In the papers [1,2] we found and studied reduction conditions of the nonlinear d'Alembert equations and possible reduced equations. In [3] we considered classification of equations with respect to their possible reductions.

Here we consider a new idea – classification of reductions of PDEs to new PDEs with smaller number of independent variables or to ODEs. This idea stems from observation that many methods for searching of such reductions produce redundant results. A generally accepted method for classification of symmetry-induced reductions provides for using of subalgebras inequivalent up to conjugacy (see [4,5]). First, it is necessary to find the invariance algebra of the equation under consideration. Subalgebras of Lie algebras can be classified with respect to conjugacy (see, e.g., [6] for classification of subalgebras of the Poincaré algebra). Then the original equation can be reduced with respect to each of the obtained inequivalent subalgebras, and solutions of the reduced equations are constructed. It is usually assumed that this method gives reductions that are inequivalent.

However, this method may produce the same reduction conditions and whence the same reductions for different non-conjugate subalgebras, and it does not work well for the general reductions of PDEs (then we have an arbitrary number of independent variables).

We propose to classify reductions of PDEs by means of symmetry classification of the reduction conditions. Thus, inequivalent solutions of the reduction conditions will provide



inequivalent reductions. This way we may obtain fewer reductions than by means of the standard Lie procedure, cutting some reductions obtainable from subalgebras of the invariance algebra, and more reductions due to nonclassical reductions if they exist.

The only problem with such equivalence and further classification of reductions – it is very difficult to solve equations of the reduction conditions. However, in many interesting cases, e.g., for the general nonlinear d'Alembert equation, it appears possible.

It is worth noting that the direct method presented by Clarkson and Kruskal [7] actually gives classification of reductions even if it gives no new solutions, though this particular advantage of their method was not pointed out by its authors.

Previously [8] we presented application of such direct reduction method by means of the ansatz

$$u = \exp\{if(t, \vec{x})\} \varphi(\omega)$$

to the nonlinear Schrödinger equation with an arbitrary function on  $|u|$  in the nonlinear part

$$2iu_t + \Delta u - uF(|u|) = 0,$$

where  $u$  is a complex valued function,  $u = u(t, \vec{x})$ , is an  $n$ -dimensional vector of space variables  $\vec{x} = (x_1, x_2, x_3)$ ,  $|u| = \sqrt{uu^*}$ , an asterisk designates complex conjugation,

$$\Delta u = \sum_{a=1}^3 \frac{\partial^2 u}{\partial x_a^2}, \quad a = 1, 2, 3.$$

Though no new reductions were obtained as compared to the Lie algorithm, we may see that the listed reductions are inequivalent with respect to equivalence transformations of the reduction conditions.

In this paper we present an approach to classification of reductions of the nonlinear d'Alembert equation with arbitrary number of independent variables

$$u_{\mu\mu} = F(u). \quad (1)$$

Here and below the indices mean differentiation with respect to the relevant independent variables,  $u_\mu = \frac{\partial u}{\partial x_\mu}$ ,  $u_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}$ , Greek indices run from 0 to  $n$ , Latin indices run from 1 to  $n$  and summation over repeated indices is assumed if nothing is indicated otherwise. Here we have  $n$  space variables and  $n + 1$  independent variables.

We consider classification of reductions of the equation (1) to ODEs by means of the ansatz with one new independent variable

$$u = \varphi(v), \quad (2)$$

where  $v$  is a new variable.

Further we give some basic definitions.

## 2. What are reduction and classification of reductions

The equation

$$\Phi(x, u, u_1, \dots, u_l) = 0, \quad (3)$$

where  $u_k$  is the set of all  $k$ th-order partial derivatives of the function  $u = (u^1, u^2, \dots, u^m)$ , is called reducible by means of an ansatz  $u = \varphi(\omega)$ , if substitution of this ansatz into the equation gives an equation on  $\omega$ . Here  $x = (x_1, x_2, \dots, x_n)$ , and in this section we present definitions for arbitrary equations with  $n$  independent and  $m$  dependent variables.

There could be various formats and generalisations of the above ansatz, but for presentation of the general idea this simplest form is sufficient. The resulting equations are called reduced equations and are used to find exact solutions. Ansatzes that reduce equations are connected with the sets of operators of conditional invariance, as explained in [9].

We have the equation (3) and  $k$  first-order operators in involution  $\langle Q_1, Q_2, \dots, Q_k \rangle$  (in the general case these operators do not form an algebra). These operators reduce the equation  $\Phi = 0$ , if the system

$$\Phi = 0, \quad Q_i u = 0 \quad (4)$$

is compatible. Technically, we take the general solution of the system  $Q_i u = 0$  – the ansatz, and substitute it into the equation  $\Phi = 0$ , obtaining an equation with smaller number of independent variables.

The concept of “reduction” includes both the reduced equation and the set of new variables, as well as the ansatz. We can consider as classification of reductions either symmetry classification of the system (4) or classification of solutions of its compatibility conditions (reduction conditions).

### 3. Review of some methods for PDEs reduction

Let us list some symmetry-related methods used to find exact solutions of PDEs by means of their reduction to equations with smaller number of independent variables or to ordinary differential equations:

1. Symmetry reduction or Lie algorithm (for the detailed description of this method see, e.g., the books by Ovsyannikov [4] or Olver [5]). This approach is quite algorithmic and thus the most popular for the systematic search for families of exact solutions of PDEs.

2. The direct method (often giving wider classes of solutions than the symmetry reduction) was proposed by P. Clarkson and M. Kruskal [7]. This method for majority of equations results in considerable difficulties as it requires investigation of compatibility and solution of cumbersome reduction conditions of the initial equation. These reduction conditions are much more difficult for investigation and solution in the case of equations containing second and/or higher derivatives for all independent variables, and for multidimensional equations – e.g., in the situation of the nonlinear wave equations.

This approach is equivalent to finding of  $Q$ -conditional symmetries of PDEs (for the proof of this statement see [9]) and methods for finding of conditional and nonclassical symmetries (the same reductions may be found by both methods).

3. The direct method for finding of classical symmetry solutions. It is possible to find reductions using not arbitrary operators of  $Q$ -conditional symmetry, but only operators from the invariance algebra. This way we also will obtain inequivalent reductions.

This method (without identification as a distinct method and without pointing out classification of reductions) was actually used in the paper by Fushchych and Serov [10].

4. Other methods (including ad-hoc methods and guessing) that produce specific reductions or exact solutions, e.g., methods named after specific ansatzes being used, or methods similar to the method of differential constraints. Many famous exact solutions of various famous PDEs were found by just these methods. Ad-hoc methods include many methods that work only for some equations such as substitution of some prescribed ansatzes and looking for equations that they reduce. More solutions and reductions are suggested every day in multiple papers.

However, we need to determine whether these reductions (and thus solutions) are not equivalent to known ones; to find interesting solutions and reductions within equivalence classes; to systematise known solutions.

#### 4. Example of reductions of the nonlinear d'Alembert equation

Here we consider conditions of reduction of the multidimensional d'Alembert equation (1) by means of the ansatz (2) with one new independent variable,  $u_{\mu\mu} = u_{00} - u_{11} - \dots - u_{nn}$ .

The reduction conditions are a system of the d'Alembert and eikonal equations (sometimes they are also called Hamilton equations) [11,12] and have the following form:

$$v_{\mu\mu} = R(v), \quad v_\mu v_\mu = r(v). \quad (5)$$

Sufficient conditions of reduction of the wave equation to an ODE and the general solution of the system (5) in the case of three spatial dimensions were found by Fushchych, Zhdanov, Revenko in [13].

It is evident that the d'Alembert–eikonal system (5) may be reduced by local transformations to the form

$$v_{\mu\mu} = f(v), \quad v_\mu v_\mu = \lambda, \quad \lambda = 0, 1. \quad (6)$$

Let us note that the case of  $\lambda = 0$  leads only to a degenerated reduction equation  $F(v) = 0$  that may give solutions of the original equation  $u = \varphi(v)$ ,  $\varphi(v)$  being an arbitrary function, that we will not consider here. Only the case of  $\lambda = 1$  in the system (6) provides reductions of equation (1) to ODEs.

It is possible to generalise statements on compatibility of the d'Alembert–eikonal system (6) formulated in [13] for three space variables to an arbitrary number of space variables.

**Statement.** *For the system (6) ( $v = v(x_0, x_1, \dots, x_n)$ ) to be compatible it is necessary and sufficient that the function  $f$  have the following form:*

$$f = \frac{\lambda N}{v + C}, \quad N = 0, 1, \dots, n.$$

Proof of this statement can be obtained as a generalisation of the proof adduced in [13].

#### 5. General solution of the d'Alembert–eikonal system

We will consider in more detail solutions of the system

$$v_{\mu\mu} = \frac{N}{v + C}, \quad N = 0, 1, \dots, n, \quad v_\mu v_\mu = 1. \quad (7)$$

These systems for different  $N$  are inequivalent with respect to local transformations.

Our results on compatibility of the reduction conditions actually mean that any multidimensional (in  $n$  space variables) nonlinear wave equation of the form (1) can be reduced by ansatzes (2) only to ODEs equivalent up to local transformations of dependent and independent variables to equations of the following form:

$$\varphi'' + \varphi' \frac{N}{v} = F(\varphi),$$

where  $N$  is an integer,  $N = 0, 1, \dots, n$ .

We presented classification of the reduction conditions (5) up to the local transformations. For the case of  $n$  independent variables we receive  $n + 1$  inequivalent systems of reduction conditions of the form (7). Thus, we can receive only  $n + 1$  inequivalent reduced equations by means of the ansatz (2). However, classification of reductions requires also to classify possible new variables in the ansatz (2).

The solutions of the system (7) can be classified by their ranks. The rank of the solution is the rank of the  $(n + 1) \times (n + 1)$  matrix  $(v_{\mu\nu})$  of the second derivatives of the function  $v$ . It

follows from the condition  $v_\mu v_\mu = 1$  that  $\det(v_{\mu\nu}) = 0$ , so we cannot have solutions of the rank  $n + 1$ . It is obvious that solutions of different ranks cannot be equivalent. Examples of solutions of the rank  $N$  are

$$v = (x_0^2 - (x_1^2 + x_2^2 + \dots + x_N^2))^{1/2}.$$

Examples of the zero-rank solutions are linear functions.

**Statement.** Rank of solutions of the system (7) is equal to  $N$ . These solutions can be written in parametric form similarly to [13] as follows:

$$\begin{aligned} v &= x_0(1 + \tau_k^2 + B_l^2(\tau_k))^{1/2} + x_k \tau_k + x_l B_l + \Phi(\tau_k), \\ l &= N + 1, \dots, n, \quad \tau_k = \tau_k(x), \quad k = 1, \dots, N, \\ x_k + x_0 \left( \tau_k + B_l \frac{\partial B_l}{\partial \tau_k} \right) (1 + \tau_k^2 + B_l(\tau_k))^{-1/2} + x_l \frac{\partial B_l}{\partial \tau_k} + \frac{\partial \Phi}{\partial \tau_k} &= 0, \end{aligned}$$

with functions  $\Phi$  having certain special forms (examples are adduced below).  $B_l(\tau_k)$  are arbitrary properly differentiable functions of the parameter functions  $\tau_k$ .

Further classification of solutions of the reduction conditions will involve the substitution

$$z_k = v_a(1 + \tau_m^2)^{-1/2}, \quad p(z_k) = (1 + \tau_m^2)^{-1/2} \Phi(\tau_k).$$

Conditions for the functions  $\Phi$  are transformed into conditions for  $p$ , and further classification of solutions of the d'Alembert–eikonal system (7) is done in accordance to the rank of the matrix  $(p_{z_k z_m})$ .

Examples of the form of the function  $\Phi(\tau)$ :

$$\begin{aligned} \Phi(\tau) &= C_k \tau_k + C_0(1 + \tau_k^2)^{1/2}, \\ \Phi(\tau) &= B_m(\rho) \tau_m + \rho \tau_N - (1 + \tau_k^2)^{1/2} Q(\rho), \quad m = 1, \dots, N - 1, \\ \dot{B}_m(\rho) \tau_m + \tau_N - (1 + \tau_k^2)^{1/2} \dot{Q}(\rho) &= 0, \end{aligned}$$

where  $B_m(\rho)$ ,  $Q(\rho)$  are arbitrary properly differentiable functions of the parameter function  $\rho$  for the ranks of the matrix  $(p_{z_k z_m})$  greater than 1.

Thus, we showed classification of solutions of the system (7) by their ranks and by the ranks of parameter functions  $p(z_k)$  in the presentation of the general solution.

## 6. Conclusions and further work

The idea of classification of reductions was presented with the objective of giving hints and directions for further research – classification of reductions for equations of mathematical physics, and studying of different reduction methods with consideration of equivalence analysis of the resulting reductions.

Another direction of further work may be based upon the fact that ansatzes and methods used for reduction of the d'Alembert ( $n$ -dimensional wave) equation can be also used for arbitrary Poincaré-invariant equations. Arbitrary ansatzes of the form (2) that reduce the equation (1) will also reduce all Poincaré-invariant scalar equations. It can be easily proved based upon the fact that all Poincaré-invariant scalar equations can be represented as functions of differential invariants of the Poincaré algebra. If we substitute the ansatz (2) into any invariant from the basis of differential invariants of the form  $u_\mu u_\nu u_{\mu\nu}$ ,  $u_{\mu\nu} u_{\mu\nu}$  etc. we will receive the product of derivatives of the function  $\varphi(v)$  and the same forms of differential invariant of the function  $v$ .

From the reduction conditions (7) on the function  $v$  we get

$$v_\mu v_\nu v_{\mu\nu} = 0, \quad v_{\mu\nu} v_{\mu\nu} = -\frac{N}{v^2}, \quad v_\mu v_\nu v_{\mu\alpha} v_{\nu\alpha} = 0, \quad v_{\mu\nu} v_{\mu\alpha} v_{\nu\alpha} = \frac{2N}{v^3}$$

and so on, thus, the original equation will be reduced to an ODE including only new variable  $v$  and derivatives of the new function  $\varphi(v)$ .

Another idea for further research – it may be interesting to look at connections between “old” view to equivalence of symmetry-induced solutions and classification of solutions of the reduction conditions. One such link is obvious – the reduced equations obtained from subalgebras that include the operator  $P_0 + P_3$  would correspond to solutions of the reduction conditions

$$v_{\mu\mu} = 0, \quad v_\mu v_\mu = 1.$$

It will be also extremely interesting to study equivalence of reductions of PDEs that were obtained by various ad-hoc and other methods involving reduction of PDEs.

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