Laplace type invariants for variable coefficient mKdV equations

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Abstract. We consider a class of variable-coefficient mKdV equations. We derive the equivalence transformations in the infinitesimal form and we employ them to construct differential invariants of the respective equivalence algebra. Operators of invariant differentiation are also constructed. Applications, similar to Laplace invariants, are presented.

1. Introduction

Laplace [1] in his general theory of integration of linear hyperbolic partial differential equations

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0 \tag{1}$$

derived the quantities

$$h = a_x + ab - c, \quad k = b_y + ab - c$$

known as *Laplace invariants*. The expressions h and k do not change under the linear transformation of the dependent variable,

$$u' = \phi(x,y)u. \tag{2}$$

These invariants are useful in various problems, for example in the group classification of differential equations [2] and the solution of initial value problems for hyperbolic equations by Riemann's method [3].

We recall the following simple but fundamental applications of the Laplace invariants:

1. A hyperbolic equation of the form (1) can be transformed into $u_{xy} = 0$ by means of (2) iff $h = k = 0$.
2. A hyperbolic equation of the form (1) can be transformed into $u_{xy} + c(x,y)u = 0$ by means of (2) iff $h = k$.
3. A hyperbolic equation of the form (1) can be transformed into $u_{xy} + cu = 0$, $c =$const, by means of (2) iff $h = k = f(x)g(y)$.
4. A hyperbolic equation of the form (1) can be factorized iff $h = 0$ or $k = 0$. That is, if $L = \partial_x\partial_y + a(x,y)\partial_x + b(x,y)\partial_y + c(x,y)$ then

$$L = [\partial_x + \alpha(x,y)][\partial_y + \beta(x,y)] \quad \text{iff } h = 0$$

and

$$L = [\partial_y + \beta(x, y)][\partial_x + \alpha(x, y)] \quad \text{iff } k = 0.$$

The proofs of the above statements can be found in [4, 5].

The differential invariants of the Lie groups of continuous transformations play important role in mathematical modelling, non-linear science and differential geometry. First it was noted by S. Lie [6], who showed that every invariant system of differential equations [7], and every variational problem [8], could be directly expressed in terms of differential invariants. Lie also demonstrated [7], how differential invariants can be used to integrate ordinary differential equations, and succeeded in completely classifying all the differential invariants for all possible finite-dimensional Lie groups of point transformations in the case of one independent and one dependent variable. Lie's preliminary results on invariant differentiations and existence of finite bases of differential invariants were generalized by Tresse [9] and Ovsiannikov [11]. The general theory of differential invariants of Lie groups including algorithms of construction of differential invariants can be found in [10, 11].

A simple method for constructing differential invariants of families of linear and nonlinear differential equations admitting infinite equivalence transformation groups was developed by Ibragimov [14] (see also [15]). This method was adopted by various scientists and it was then applied to several linear and nonlinear equations with interesting results. [4,16–32]. For example, Ibragimov [16] gave a solution to the Laplace problem which consists of finding all invariants of the hyperbolic equations (1). Namely, in addition to Ovsiannikov's invariants [2], he found three new invariants together with invariant differentiations and he constructed a basis of all invariants. The Laplace problem was also proved by Mahomed and coauthors [33].

We point out that other approaches also exist. See for example, ref [34, 35]. For instance, Yehorchenko [34] introduced a method where the initial basis operators contains no arbitrary functions. In this method we search for differential operators of any specific finite order and hence, we deal with finite dimensional algebra. The arbitrary functions are expanded into Taylor series,

$$A(t) = \sum_{m=0}^{\infty} a_m t^m, \quad A_t(t) = \sum_{m=1}^{\infty} m a_m t^{m-1}, \quad \text{etc.}$$

More detail of this approach can be found in [34].

Here we apply Ibragimov's method for the general class of variable-coefficient mKdV equations

$$u_t + f(t)u^2 u_x + g(t)u_{xxx} + h(t)u + (p(t) + q(t)x)u_x + k(t)uu_x + l(t) = 0, \tag{3}$$

where all the parameters are smooth functions of t and $f(t)g(t) \neq 0$. We derive the equivalence transformations which are employed to construct differential invariants. Certain applications, similar to Laplace invariants, are presented. Finally, we construct an operator of invariant differentiation.

2. Equivalence transformations

Equivalence transformations play an important part in the theory of invariants. Derivation of equivalence transformations for the class of equations under consideration is the first step towards to the target which is the determination of differential invariants. The set of all equivalence transformations of a given family of differential equations forms a group which is called the equivalence group. There exist two methods for calculation of equivalence transformations, the direct which was used first by Lie [7] and the Lie infinitesimal method which was introduced by Ovsyannikov [11]. Although, the direct method involves considerable computational difficulties, it has the benefit of finding the most general equivalence group and also unfolds all form-preserving [12] (also known as admissible [13]) transformations admitted by this class of equations. For recent applications of the direct method one can refer, for example, to references [36–39]. More detailed description and examples of both methods can be found in [40]. The method that we employ here to determine differential invariants requires the equivalence transformations to be in the infinitesimal form. Hence, we use the infinitesimal method to derive the desired equivalence transformations. We search for the equivalence operator X in the following form:

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \sum_{i=1}^{7} \mu_i \frac{\partial}{\partial \zeta_i},$$

where ζ_i correspond to the functions $f, g, \ldots, l$. The functions ξ^1, ξ^2 and η depend on x, t and u, while μ_i depend on x, t, u, f, g, h, p, q, k and l. Without presenting any detailed analysis, we find that

$$\begin{gathered}
\xi^1 = A(t), \quad \xi^2 = B(t)x + \Gamma(t), \quad \eta = \Theta(t)u + \Psi(t), \\
\mu_1 = (B - 2\Theta - A_t)f, \quad \mu_2 = (3B - A_t)g, \quad \mu_3 = -A_t h - \Theta_t, \\
\mu_4 = (B - A_t)p - \Gamma q - \Psi k + \Gamma_t, \quad \mu_5 = -A_t q + B_t, \\
\mu_6 = -2\Psi f + (B - \Theta - A_t)k, \quad \mu_7 = -\Psi h + (\Theta - A_t)l - \Psi_t.
\end{gathered}$$

That is, the class (3) admits an infinite-dimensional continuous group $\mathcal{E}$ of equivalence transformations generated by Lie algebra $L_{\mathcal{E}}$ spanned by the operators:

$$\begin{aligned}
X_A &= A\frac{\partial}{\partial t} - A_t f\frac{\partial}{\partial f} - A_t g\frac{\partial}{\partial g} - A_t h\frac{\partial}{\partial h} - A_t p\frac{\partial}{\partial p} - A_t q\frac{\partial}{\partial q} - A_t k\frac{\partial}{\partial k} - A_t l\frac{\partial}{\partial l}, \\
X_B &= xB\frac{\partial}{\partial x} + Bf\frac{\partial}{\partial f} + 3Bg\frac{\partial}{\partial g} + Bp\frac{\partial}{\partial p} + B_t\frac{\partial}{\partial q} + Bk\frac{\partial}{\partial k}, \\
X_\Gamma &= \Gamma\frac{\partial}{\partial x} + (\Gamma_t - \Gamma q)\frac{\partial}{\partial p}, \\
X_\Theta &= u\Theta\frac{\partial}{\partial u} - 2\Theta f\frac{\partial}{\partial f} - \Theta_t\frac{\partial}{\partial h} - \Theta k\frac{\partial}{\partial k} + \Theta l\frac{\partial}{\partial l}, \\
X_\Psi &= \Psi\frac{\partial}{\partial u} - \Psi k\frac{\partial}{\partial p} - 2\Psi f\frac{\partial}{\partial k} - (\Psi h + \Psi_t)\frac{\partial}{\partial l}.
\end{aligned}$$

The direct method leads to the following equivalence transformations for the class (3) [41]:

$$\tilde{t} = \alpha(t), \quad \tilde{x} = \beta(t)x + \gamma(t), \quad \tilde{u} = \theta(t)u + \psi(t), \tag{4}$$

where α, β, γ, θ and ψ run through the set of smooth functions of t, $\alpha_t\beta\theta \neq 0$. The arbitrary elements of (3) are transformed by the formulas

$$\begin{gathered}
\tilde{f} = \frac{\beta}{\alpha_t\theta^2}f, \quad \tilde{g} = \frac{\beta^3}{\alpha_t}g, \quad \tilde{h} = \frac{1}{\alpha_t}\left(h - \frac{\theta_t}{\theta}\right), \\
\tilde{p} = \frac{1}{\alpha_t}\left(\beta p - \gamma q + \beta\frac{\psi^2}{\theta^2}f - \beta\frac{\psi}{\theta}k + \gamma_t - \gamma\frac{\beta_t}{\beta}\right), \quad \tilde{q} = \frac{1}{\alpha_t}\left(q + \frac{\beta_t}{\beta}\right), \\
\tilde{k} = \frac{\beta}{\alpha_t\theta}\left(k - 2\frac{\psi}{\theta}f\right), \quad \tilde{l} = \frac{1}{\alpha_t}\left(\theta l - \psi h - \psi_t + \psi\frac{\theta_t}{\theta}\right).
\end{gathered}$$

3. Differential invariants

A function of the form

$$I(x, t, u, \zeta_i(t), \zeta_{it}(t), \zeta_{itt}(t), \dots)$$

which remains invariant under the equivalence group $\mathcal{E}$ is called differential invariant of order s of equation (3), where s denotes the maximal order derivative of $\zeta_i(t)$. If no derivatives appear, then it is called differential invariant of order zero. An equation

$$E(x, t, u, \zeta_i(t), \zeta_{it}(t), \zeta_{itt}(t), \dots) = 0$$

that satisfies the conditions

$$X_k^{(s)}(E)\Big|_{E=0} = 0, \quad k = A,\ B,\ \Gamma,\ \Theta,\ \Psi,$$

is called an invariant equation of order s.

In order to determine the differential invariants of order s, we need to calculate the prolongations of the operator X. The procedure for determining the prolongations can be found in [15]. We do not find invariant of zero order. However we find the following invariant equations of zero order:

$$f = 0, \quad g = 0.$$

Although we have taken the functions $f(t)$ and $g(t)$ to be nonzero, the above equations state that there do not exist point transformations which map an equation of the class (3) into an equation of the same class with either $f(t) = 0$ or $g(t) = 0$.

Next step is to derive differential invariants of first order. We introduce the quantities

$$A_1 = fk_t - kf_t + fhk - 2lf^2, \quad A_2 = 2ghf - 2fqg - gf_t + fg_t.$$

We find that the class of equations (3) admits one differential invariant of first order

$$I^{(1)} = \frac{A_1 g^{7/6}}{{A_2}^{4/3} f^{1/6}}$$

and two invariant equations

$$A_1 = 0, \quad A_2 = 0.$$

For the differential invariants of second order we introduce the quantities

$$\begin{aligned} A_3 &= 8h^2gf^2 - 8hfgf_t - 10hf^2qg + 6hf^2g_t + 2gq^2f^2 + 5gqff_t + 3gf_t^2 \\ &\quad - 3qg_tf^2 - ff_{tt}g + 2f^2gh_t - 2f^2gq_t + f^2g_{tt} - 3f_tg_tf, \\ A_4 &= -3kf_t^2 + 3f_tfk_t + 5f_tfhk - 2f_tf^2l + f_tfkq - qf^2hk + 2qf^3l - qf^2k_t \\ &\quad - 3h^2kf^2 + 6lhf^3 - 4hk_tf^2 + ff_{tt}k - f^2kh_t + 2f^3l_t - f^2k_{tt}. \end{aligned}$$

We find two differential invariants of second order

$$I_1^{(2)} = \frac{gA_3}{{A_2}^2}, \quad I_2^{(2)} = \frac{g^{13/6}A_4}{f^{1/6}{A_2}^{7/3}}$$

and two invariant equations

$$A_3 = 0, \quad A_4 = 0.$$

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4. Applications

Similar to the applications of Laplace invariants that stated in the Introduction, we have the following results.

Theorem 1. *Equation* (3) *can be transformed into*

$$u_t + u_{xxx} + c_1 u^2 u_x + c_2 u u_x = 0$$

if and only if $A_1 = A_2 = 0$. *That is, if and only if invariant equations are satisfied.*

Equation (3) *can be transformed into*

$$u_t + u_{xxx} + c_1 u^2 u_x + \phi(t) u u_x = 0$$

if and only if $A_2 = 0$ *and* $A_1 \neq 0$.

Equation (3) *can be transformed into*

$$u_t + \phi(t) u_{xxx} + u^2 u_x + c_1 u u_x = 0$$

if and only if $A_1 = 0$ *and* $A_2 \neq 0$.

Equation (3) *can be transformed into*

$$u_t + u_{xxx} + \phi(t) u^2 u_x + c_1 u u_x = 0$$

if and only if $A_1 - c_1 A_2 = 0$ *and* $h = 2q$.

The first part of the Theorem 1 is also presented in the work [41]. Below we present the mappings that connect equation (3) with each one of the four equations that appear in the Theorem 1.

The equation

$$\tilde{u}_{\tilde{t}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + c_1 \tilde{u}^2 \tilde{u}_{\tilde{x}} + c_2 \tilde{u}\tilde{u}_{\tilde{x}} = 0$$

is connected with equation (3) under the mapping

$$\begin{aligned}
\tilde{t} &= \int \sqrt{\frac{f^3}{c_1^3 g}} \mathrm{e}^{-3\int h dt} dt, \\
\tilde{x} &= \sqrt{\frac{f}{c_1 g}} \mathrm{e}^{-\int h dt} x + \int \left\{ \frac{1}{4\sqrt{c_1^5 f g}} \mathrm{e}^{-3\int h dt} \left[c_1^2 \left(k^2 - 4fp\right) \mathrm{e}^{2\int h dt} - c_2^2 f^2 \right] \right\} dt, \\
\tilde{u} &= \mathrm{e}^{\int h dt} u + \frac{k}{2f} \mathrm{e}^{\int h dt} - \frac{c_2}{2c_1},
\end{aligned}$$

where the invariant equations $A_1 = 0$, $A_2 = 0$ must hold. We point out that c_2 can be taken equal to zero.

The equation

$$\tilde{u}_{\tilde{t}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + c_1 \tilde{u}^2 \tilde{u}_{\tilde{x}} + \phi(\tilde{t}) \tilde{u}\tilde{u}_{\tilde{x}} = 0$$

is connected with equation (3) under the mapping

$$\begin{aligned}
\tilde{t} &= \int \frac{g}{c_1^{3/2}} \mathrm{e}^{-3\int q dt} dt, \\
\tilde{x} &= \frac{1}{\sqrt{c_1}} \mathrm{e}^{-\int q dt} x + \int \frac{g}{\sqrt{c_1} f} \mathrm{e}^{-3\int q dt} \left[k\sqrt{\frac{f}{g}} \mathrm{e}^{\int q dt} \int l \sqrt{\frac{f}{g}} \mathrm{e}^{\int q dt} dt \right. \\
&- \left. \frac{pf}{g} \mathrm{e}^{2\int q dt} - f \left(\int l \sqrt{\frac{f}{g}} \mathrm{e}^{\int q dt} dt \right)^2 \right] dt, \\
\tilde{u} &= \sqrt{\frac{f}{g}} \mathrm{e}^{\int q dt} u + \int l \sqrt{\frac{f}{g}} \mathrm{e}^{\int q dt} dt,
\end{aligned}$$

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where the invariant equation $A_2 = 0$ must hold and

$$\phi = \frac{c_1}{f}\left(k\sqrt{\frac{f}{g}}\mathrm{e}^{\int q dt} - 2f\int l\sqrt{\frac{f}{g}}\mathrm{e}^{\int q dt}dt\right).$$

The equation

$$\tilde{u}_{\tilde{t}} + \phi(\tilde{t})\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{u}^2\tilde{u}_{\tilde{x}} + c_1\tilde{u}\tilde{u}_{\tilde{x}} = 0$$

is connected with equation (3) under the mapping

$$\begin{aligned}
\tilde{t} &= \int f\mathrm{e}^{-\int(q+2h)dt}dt,\\
\tilde{x} &= \mathrm{e}^{-\int q dt}x + \int \frac{1}{4f}\mathrm{e}^{-\int(q+2h)dt}\left[\left(k^2 - 4fp\right)\mathrm{e}^{2\int h dt} - c_1^2 f^2\right]dt,\\
\tilde{u} &= \mathrm{e}^{\int h dt}u + \frac{k}{2f}\mathrm{e}^{\int h dt} - \frac{c_1}{2},
\end{aligned}$$

where the invariant equation $A_1 = 0$ must hold and

$$\phi = \frac{g}{f}\mathrm{e}^{2\int(h-q)dt}.$$

The equation

$$\tilde{u}_{\tilde{t}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + \phi(\tilde{t})\tilde{u}^2\tilde{u}_{\tilde{x}} + c_1\tilde{u}\tilde{u}_{\tilde{x}} = 0$$

is connected with equation (3) where $h = 2q$ under the mapping

$$\begin{aligned}
\tilde{t} &= \int g\mathrm{e}^{-3\int q dt}dt,\\
\tilde{x} &= \mathrm{e}^{-\int q dt}x + \int \frac{1}{4f}\mathrm{e}^{-\int q dt}\left(k^2 - c_1^2 g^2 - 4fp\right)dt,\\
\tilde{u} &= \mathrm{e}^{2\int q dt}u + \frac{k - c_1 g}{2f}\mathrm{e}^{2\int q dt},
\end{aligned}$$

provided $A_1 - c_1 A_2 = 0$ and

$$\phi = \frac{f}{g}\mathrm{e}^{-2\int q dt}.$$

Here $c_1 \neq 0$, otherwise the result is the same as in the previous case with $c_1 = 0$ and $h = 2q$.

5. Operators of invariant differentiation

Here we find an operator ofinvariant differentiation that transform each invariant of equation (3) into invariants of higher-order of the same equation. Since arbitrary elements are functions of t we look for an operator of invariant differentiation of the form

$$\mathcal{D} = \psi D_t,$$

where $\psi = \psi(t, x, u, f, g, \dots, l, f_t, \dots, l_t, \dots)$ and can be found by solving the differential equations

$$X_A^{(n)}(\psi) = A_t\psi, \quad X_B^{(n)}(\psi) = 0, \quad X_\Gamma^{(n)}(\psi) = 0, \quad X_\Theta^{(n)}(\psi) = 0, \quad X_\Psi^{(n)}(\psi) = 0.$$

For zero order we find $\psi = 0$, for first order

$$\psi = \frac{fg}{A_2} H\big(I^{(1)}\big),$$

and for second order

$$\psi = \frac{fg}{A_2} H\big(I^{(1)}, I_1^{(2)}, I_2^{(2)}\big).$$

Since the function H is arbitrary, we can take it, without of generality, $H = 1$. Hence,

$$\mathcal{D} = \frac{fg}{A_2} D_t.$$

Now, if we apply the invariant differentiation to $I^{(1)}$ we obtain

$$\mathcal{D}(I^{(1)}) = \frac{fg}{A_2} D_t\big(I^{(1)}\big) = I_2^{(2)} + \tfrac{7}{6} I^{(1)} - \tfrac{4}{3} I^{(1)} I_1^{(2)}$$

and therefore,

$$I_2^{(2)} = \mathcal{D}\big(I^{(1)}\big) - \tfrac{7}{6} I^{(1)} + \tfrac{4}{3} I^{(1)} I_1^{(2)}.$$

This means that we have one new differential invariant of second order, while the second can be obtained with the application of the invariant differentiation to the differential invariant of first order. Further calculations showed that the class of equations (3) admits two differential invariants of third order and two of fourth order. It appears that the class (3) has a basis of two differential invariants: $\big\{I^{(1)}, I_1^{(2)}\big\}$. Any other differential invariant of higher order can be obtained with the employment of the invariant differentiation. However this result needs to be proved and consequently, it will be a task for the near future.

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