

Laplace–Runge–Lentz vectors for arbitrary spin and arbitrary dimension

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Abstract. Laplace–Runge–Lentz (LRL) vector is a cornerstone of celestial mechanics. It also plays an important role in quantum mechanics, being an integral of motion for the Hydrogen atom and some other systems. However, the majority of models of non-relativistic systems admitting LRL vector ignore the spin of orbital particles. In this survey a new collection of QM systems admitting LRL vector with spin is presented. It includes $2d$ and $3d$ systems with arbitrary spin, as well as systems of arbitrary dimension with spins 0, $1/2$, and 1. All these systems are superintegrable and can be solved exactly. They emulate neutral particles with non-trivial multipole momenta (in particular, the neutron) interacting with a central external field.

1. Introduction

Some quantum mechanical systems are characterized by very attractive global properties: *exact solvability*, *super integrability*, and (or) *supersymmetry*.

Exact solvability means that all energy levels can be calculated algebraically; and the corresponding wave functions are polynomials multiplied by an overall gauge factor.

Integrability of a d -dimensional system presupposes the existence of $d-1$ integrals commuting with the Hamiltonian and amongst each other. If the system admits at least one more integral of motion (not necessary commuting with the other ones) it is called superintegrable. The system is called maximally superintegrable, if it admits $2d-1$ integrals of motion including the Hamiltonian. Moreover, d of them should commute with each other.

Supersymmetry in quantum mechanics has various faces, but we will discuss only one of them, namely, the shape invariance. That is a specific symmetry of the Hamiltonian with respect to the Darboux transform.

Some systems, like the Hydrogen atom or isotropic harmonic oscillator, are both superintegrable, supersymmetric and exactly solvable. Notice that there exists a tight coupling between superintegrability and supersymmetry [1]. A more contemporary discussion of such coupling can be found in [2–4]. However, we will see that this connection is missing for the case of more general systems with arbitrary spins.

Superintegrability and (or) supersymmetry usually cause the exact solvability of a QM system. However, the exact solvability even of maximally superintegrable systems never was proved rigorously and can be considered only as a suggestion, see, e.g., [5].

The classification of superintegrable models of quantum mechanics which was started with papers [6, 7] has an interesting and inspiring history. We will not discuss it (see survey [8]),



but restrict ourselves to contemporary results obtained for superintegrable system with spin. Moreover, just the systems which admit analogues of the Laplace–Runge–Lenz (LRL) vector will be considered.

The LRL vector is a corner stone of celestial mechanics. It has a great value also in quantum mechanics. In particular, using this vector, Pauli found the spectrum of the Hydrogen atom before the Schrödinger equation was discovered [9].

Till the year 2013 there were only few known examples of $3d$ quantum mechanical systems with spin admitting the LRL vector. They are the dyon with gyromagnetic ratio $g = 4$, interacting with a magnetic monopole field plus a Coulomb plus a fine-tuned inverse-square potential [10], and a neutral particle with a non-trivial dipole momentum [11]. In addition, a system with spin-orbit interaction and a special inverse-square potential with *fixed* coupling constant was presented in [12], but it is equivalent to the direct sum of two scalar Hydrogen like systems [11]. All the mentioned systems include particles with spin $1/2$.

However in 2013 and 2014 a number of new systems with spin admitting the LRL vector have been presented in papers [13–15]. In fact, the complete description of an extended class of such systems has been made, which includes systems with arbitrary spin and arbitrary dimension. Let us mention that the two dimension systems with arbitrary spin had been classified a bit earlier in papers [16] and [17].

In this presentation the results of papers [11, 13–17] are summarized. In other words a survey of the contemporary results concerning the quantum mechanical systems admitting the LRL vector with arbitrary spin in arbitrary dimension space is presented. In addition, supersymmetric aspects of the these systems are discussed using the classification of shape invariant matrix potentials presented in papers [18] and [19].

Mathematically, the subject of our analysis are systems of coupled Schrödinger equations of the following form

$$H\psi = E\psi, \quad (1)$$

where

$$H = -\frac{\Delta_d}{2m} + V(\mathbf{x}). \quad (2)$$

Here ψ is a *multicomponent* wave function, $\mathbf{x} = (x_1, x_2, \dots, x_d)$, Δ_d is Laplace operator in d -dimensional Euclidean space, and $V(\mathbf{x})$ is a *matrix potential*. Physically, just equations of generic form (1) for $d = 3$ are requested to construct models of neutral particles which have nontrivial dipole moments. A perfect example of such particle is the neutron.

Symmetries of systems (1) including two or three equations have been investigated in [20] and [21]. However, it was done only for diagonal potentials $V(\mathbf{x})$ depending on one spatial variable. The systems classified in the present paper include hermitian matrices $V(\mathbf{x})$ of general form and arbitrary dimension.

2. Planar systems

2.1. Pron'ko–Stroganov model

The first (historically) model with Hamiltonian (2) including spin, which admits an analogue of the LRL vector, was proposed by Pron'ko and Stroganov [22]. This $2d$ model is based on the following Hamiltonian

$$\mathcal{H} = \frac{p_1^2 + p_2^2}{2m} + \frac{\mu(\mathbf{n})}{r}, \quad (3)$$

where

$$\mu(\mathbf{n}) = \lambda \frac{\sigma_1 x_2 - \sigma_2 x_1}{r}, \quad r = \sqrt{x_1^2 + x_2^2}, \quad \mathbf{n} = \frac{\mathbf{x}}{r}, \quad p_1 = -i \frac{\partial}{\partial x_1}, \quad p_2 = -i \frac{\partial}{\partial x_2}, \quad (4)$$

and describes a neutral spinor with a non-trivial dipole momentum, interacting with the magnetic field generated by a constant straight line currents. In (4) σ_1 and σ_2 are Pauli matrices and λ is the integrated coupling constant.

Let us note that making the unitary transformation $\mathcal{H} \rightarrow U\mathcal{H}U^\dagger$ with $U = \frac{1}{\sqrt{2}}(1 - i\sigma_3)$ it is possible to reduce matrix $\mu(\mathbf{n})$ and potential $V = \frac{\mu(\mathbf{n})}{r}$ to the following equivalent forms:

$$\mu(\mathbf{n}) \rightarrow U\mu(\mathbf{n})U^\dagger = \sigma_1 n_1 + \sigma_2 n_2, \quad V \rightarrow \hat{V} = \frac{\sigma_1 x_1 + \sigma_2 x_2}{r^2}. \quad (5)$$

Hamiltonian (3) commutes with the total orbital momentum

$$J_3 = x_1 p_2 - x_2 p_1 + S_3, \quad (6)$$

where $S_3 = \frac{1}{2}\sigma_3$. There are two more integrals of motion for (3), namely:

$$\begin{aligned} K_1 &= \frac{1}{2}(J_3 p_1 + p_1 J_3) + \frac{m}{r}\mu(\mathbf{n})x_2, \\ K_2 &= \frac{1}{2}(J_3 p_2 + p_2 J_3) - \frac{m}{r}\mu(\mathbf{n})x_1. \end{aligned} \quad (7)$$

Operators (7) commute with Hamiltonian \mathcal{H} and satisfy the following relations:

$$[J_3, K_1] = iK_2, \quad [J_3, K_2] = -iK_1, \quad [K_1, K_2] = -2imJ_3\mathcal{H}.$$

Changing Hamiltonian \mathcal{H} by its eigenvalue E we come to the Lie algebra isomorphic to $\mathfrak{so}(3)$ provided $E < 0$, to $e(2)$ provided $E = 0$, and to $\mathfrak{so}(1, 2)$ if E is positive.

Thus there are three integrals of motion for a system with two degrees of freedom, but only two of them are functionally independent. It means that the Pron'ko–Stroganov system is maximally superintegrable. Moreover, operators K_1 and K_2 can be treated as components of the two-dimensional analogue of the LRL vector.

Using these symmetries it is possible to find eigenvalues \mathcal{E} for coupled states algebraically. First we rescale operators K_1 and K_2 and introduce the following terms J_1 and J_2 :

$$J_1 = \frac{K_1}{\sqrt{-2mE}}, \quad J_2 = \frac{K_2}{\sqrt{-2mE}}, \quad (8)$$

which, together with J_3 , realize a representation of algebra $\mathfrak{so}(3)$. This representation is irreducible if wave function ψ satisfies the following condition:

$$C\psi \equiv (J_1^2 + J_2^2 + J_3^2)\psi = n(n+1)\psi, \quad (9)$$

where n are half integers. In addition, ψ is supposed to be an eigenvector of operator J_3 commuting with \mathcal{H} and C :

$$J_3\psi = k\psi, \quad k = \pm\frac{1}{2}, \pm\frac{3}{2}, \dots, \pm n. \quad (10)$$

Substituting (6) and (7), (8) into (9) and using (10) we obtain the energy spectrum in the following form:

$$E = -\frac{m\lambda^2}{2N^2}, \quad (11)$$

where $N = 2n + 2k + 1 = 1, 2, \dots$

The spectrum (11) and the corresponding eigenvectors can be found in another way, using tools of supersymmetric quantum mechanics. Indeed, introducing the polar coordinates and expanding solutions via eigenfunctions of J we reduce the related equation (1) to the following eigenvalue problem for radial functions:

$$\mathcal{H}_\kappa \psi_\kappa \equiv \left(-\frac{\partial^2}{\partial r^2} + \kappa(\kappa - \sigma_3) \frac{1}{r^2} + \sigma_1 \frac{\lambda}{r} \right) \psi_\kappa = \mathcal{E} \psi_\kappa,$$

where $\mathcal{E} = 2mE$.

The effective potential

$$V_\kappa(r) = \kappa(\kappa - \sigma_3) \frac{1}{r^2} + \sigma_1 \frac{\lambda}{r}$$

is *shape invariant*. To prove this statement we represent V_κ as

$$V_k(x) = -\frac{\partial W_k}{\partial r} + W_k^2 - \frac{\lambda^2}{(2\kappa + 1)^2} \quad (12)$$

and solve this Riccati equation for W_κ . We obtain

$$W_\kappa = \frac{1}{2r} \sigma_3 - \frac{\lambda}{2\kappa + 1} \sigma_1 - \frac{2\kappa + 1}{2r}. \quad (13)$$

Using (13) we can define the superpartner for potential V_κ

$$V_k^+ = \frac{\partial W_k}{\partial r} + W_k^2 = (\kappa + 1)(\kappa + 1 - \sigma_3) \frac{1}{r^2} + \sigma_1 \frac{\lambda}{r} + \frac{\lambda^2}{(2\kappa + 1)^2}$$

which appears to be shape invariant, i.e.,

$$V_\kappa^+ = V_{\kappa+1} + C_\kappa, \quad C_\kappa = \frac{\lambda^2}{(2\kappa + 1)^2} - \frac{\lambda^2}{(2\kappa + 3)^2}.$$

Thus the Pron'ko–Stroganov system is shape invariant and can be solved using the standard technique of supersymmetric quantum mechanics, see Section 2.4.

2.2. 2d systems with arbitrary spin

The Pron'ko–Stroganov model describes a particle with spin 1/2. Let us show how it can be generalized to the case of arbitrary spin.

By definition a system with arbitrary spin should be invariant w.r.t. the rotations, i.e., commute with the total orbital momentum (6), where S_3 is a matrix of spin s , which can be chosen in the diagonal form:

$$S_3 = \text{diag}(s, s - 1, s - 2, \dots, -s). \quad (14)$$

We will search for Hamiltonians of generic form (3), which commute with J_3 and have two additional integrals of motion given by equation (7), where $\mu(\mathbf{n})$ are unknown matrices of dimension $(2s + 1) \times (2s + 1)$.

Operators (7) commute with Hamiltonian (3) provided matrix $\mu(\mathbf{n})$ satisfies the following equations [17]:

$$[\mu_s(\mathbf{n}), J_3] = 0, \quad (15)$$

$$\mu_s(\mathbf{n}) S_z + S_z \mu_s(\mathbf{n}) = 0. \quad (16)$$

The condition (15) is satisfied iff $\mu_s(\mathbf{n})$ is a function of $\mathbf{S} \cdot \mathbf{n} = S_1 n_1 + S_2 n_2$, $\mathbf{S} \times \mathbf{n} = S_1 n_2 - S_2 n_1$ and S_3 , where S_1 and S_2 are spin matrices realizing (together with S_3 (14)) irreducible representation $D(s)$ of algebra $\mathfrak{so}(3)$. First we consider matrices $\mu_s(\mathbf{n})$ which are polynomials in $\mathbf{S} \cdot \mathbf{n}$ and expand them via projectors:

$$\mu_s(\mathbf{n}) = \sum_{\nu=-s}^s c_\nu \Lambda_\nu, \quad \Lambda_\nu = \prod_{\nu' \neq \nu} \frac{\mathbf{S} \cdot \mathbf{n} - \nu'}{\nu - \nu'}. \quad (17)$$

Matrix (17) satisfies condition (15) by construction. Substituting (17) into (16) and using the identities

$$\begin{aligned} S_3 \mu_s(\mathbf{n}) + \mu_s(\mathbf{n}) S_3 &= 2S_3 \mu_s(\mathbf{n}) + [\mu_s(\mathbf{n}), S_3], \\ [\Lambda_\nu, S_3] &= \frac{1}{2} S_3 (2\Lambda_\nu - \Lambda_{\nu+1} - \Lambda_{\nu-1}) + \frac{i}{2} (n_1 S_2 - n_2 S_1) (\Lambda_{\nu+1} - \Lambda_{\nu-1}) \end{aligned} \quad (18)$$

we obtain [17]:

$$\mu_s(\mathbf{n}) = \lambda \sum_{\nu} (-1)^{[\nu]} \Lambda_\nu, \quad (19)$$

where $[\nu]$ is the entire part of ν , $\nu = s, s-1, \dots, -s$.

Formula (19) is defined for arbitrary spin s and gives a particular solution of equations (15). It is possible to show [17] that the most general solution can be represented as:

$$\tilde{\mu}_s(\mathbf{n}) = \sum_{\nu \geq 0} \lambda_\nu (\Lambda_\nu + \Lambda_{-\nu}) \mu_s(\mathbf{n}), \quad (20)$$

where $\mu_s(\mathbf{n})$ is matrix (19), λ_ν are arbitrary real parameters, and summation is imposed over all non negative values of $\nu = s, s-1, \dots$.

Thus the Hamiltonian for superintegrable model of arbitrary spin can be represented in the following form:

$$\mathcal{H}_s = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{r} \tilde{\mu}_s(\mathbf{n}), \quad (21)$$

where $\hat{\mu}_s(\mathbf{n})$ is the matrix defined by equations (19) and (20). Hamiltonian (21) admits integrals of motion given by equations (6) and (7) with $\mu(\mathbf{n}) \rightarrow \tilde{\mu}_s(\mathbf{n})$. In the particular case $s = 1/2$ operator (21) is equivalent to the Pron'ko–Stroganov Hamiltonian (3), (4).

2.3. 2d system with spin 1

Consider one more particular case of the presented systems which corresponds to spin 1. The entries of the related spin matrices can be chosen in the form:

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (22)$$

while matrix $\tilde{\mu}_s(\mathbf{n})$ is reduced to the following one:

$$\tilde{\mu}_s(\mathbf{n}) = \mu_1(\mathbf{n}) = \nu(2(\mathbf{S} \times \mathbf{n})^2 - 1) + \lambda(2(\mathbf{S} \cdot \mathbf{n})^2 - 1),$$

where ν and λ are arbitrary parameters.

Let us consider in more detail the case $\nu = 0$, $\lambda = \omega^2 > 0$. The corresponding Hamiltonian (21) takes the following form:

$$\mathcal{H}_1 = \frac{p_1^2 + p_2^2}{2m} + \omega^2 \left(\frac{2(\mathbf{S} \cdot \mathbf{x})^2}{r^3} - \frac{1}{r} \right) \quad (23)$$

and can be rewritten as

$$\mathcal{H}_1 = \frac{p_1^2 + p_2^2}{2m} + \omega^2 Q_{ab} \frac{\partial E_a}{\partial x_b} + \frac{\omega^2}{3} \operatorname{div} \mathbf{E}, \quad (24)$$

where

$$Q_{ab} = S_a S_b + S_b S_a - \frac{2}{3} s(s+1) \delta_{ab}$$

is the quadruple interaction tensor, and

$$\mathbf{E} = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0 \right). \quad (25)$$

Operator (24) can be interpreted as a Hamiltonian of spin-one particle, which includes a quadruple and Darwin interaction with the external electric field (25).

2.4. Shape invariance and exact solutions

Thus we present a countable set of quantum mechanical systems with arbitrary spins, any of which admits three integrals of motion given by relations (6), (7) and (17). All these systems are superintegrable.

Let us show that the presented systems are also supersymmetric. Consider the eigenvalue problem (1) for Hamiltonian (1), (19). We introduce polar coordinates and expand solutions via eigenvectors of operator J_3 (6) where S_3 is matrix (14). These eigenvectors can be represented as:

$$\psi_\kappa = \frac{1}{\sqrt{r}} \exp(i(\kappa - S_z)\theta) \Phi_\kappa(r), \quad \Phi_\kappa(r) = \operatorname{column}(\phi_s, \phi_{s-1}, \dots, \phi_{-s}).$$

As a result we obtain the following equation for radial functions Φ_κ :

$$\hat{\mathcal{H}}_\kappa \Phi_\kappa \equiv \left(-\frac{\partial^2}{\partial r^2} + V_\kappa \right) \Phi_\kappa = \mathcal{E} \Phi_\kappa, \quad (26)$$

where $\mathcal{E} = 2mE$ and

$$V_\kappa = \left((k - S_z)^2 - \frac{1}{4} \right) \frac{1}{r^2} + \tilde{\mu}_s \frac{1}{r}, \quad \tilde{\mu}_s = 2m\mu_s(\mathbf{n})|_{n_y=0}.$$

Hamiltonian $\hat{\mathcal{H}}_\kappa$ and matrix S_3^2 commute with each other and so have a mutual system of eigenfunctions. The eigenvalue problem (26) can be decoupled to the following equations:

$$\hat{\mathcal{H}}_{\kappa,\nu} \psi_{\kappa,\nu} \equiv \left(-\frac{\partial^2}{\partial r^2} + V_\kappa \right) \psi_{\kappa,\nu} = \mathcal{E} \psi_{\kappa,\nu}, \quad (27)$$

where $\psi_{\kappa,\nu} = \begin{pmatrix} \phi_\nu \\ \phi_{-\nu} \end{pmatrix}$ are two-component functions, and

$$V_{\kappa,\nu} = \frac{(\kappa - \nu\sigma_3)^2 - \frac{1}{4}}{r^2} + \frac{\tilde{\lambda}}{r} \sigma_1, \quad \tilde{\lambda} = 2m\lambda, \quad \nu = s, s-1, \dots, \neq 0. \quad (28)$$

For $\nu = 0$ the corresponding reduced potential is one-dimensional:

$$V_{\kappa,0} = \frac{\kappa^2 - \frac{1}{4}}{r^2} + \frac{\tilde{\lambda}}{r}. \quad (29)$$

Potentials (28) were discussed in paper [18], where parameter ν was denoted as $\mu + 1/2$. These potentials are shape invariant and can be expressed via superpotentials as follows:

$$V_{\kappa,\nu} = W_{\kappa,\nu}^2 - W'_{\kappa,\nu} + c_{\kappa}, \quad (30)$$

where

$$W_{\kappa,\nu} = \frac{\nu}{r}\sigma_3 - \frac{\tilde{\lambda}}{2\kappa+1}\sigma_1 - \frac{2\kappa+1}{2r}, \quad c_{\kappa} = -\frac{\tilde{\lambda}^2}{(2\kappa+1)^2}. \quad (31)$$

One-dimensional potential (29) is shape invariant too. It can be represented in the standard form (30) with

$$W_{\kappa,0} = -\frac{\tilde{\lambda}}{2\kappa+1} - \frac{2\kappa+1}{2r}. \quad (32)$$

Alternatively, potentials (30) can be expressed via another superpotential:

$$V_{\kappa,\nu} = W_{\nu,\kappa}^2 - W'_{\nu,\kappa} + c_{\nu}, \quad c_{\nu} = -\frac{\tilde{\lambda}^2}{(2\nu+1)^2}, \quad (33)$$

where $W_{\nu,\kappa}$ is the superpotential obtained from $W_{\kappa,\nu}$ by the change $\nu \rightarrow \kappa, \kappa \rightarrow \nu$. In other words, potential (28) appears to be shape invariant w.r.t. the shifts of two parameters, i.e., κ and ν . This is a particular case of the dual shape invariance phenomena discovered in [18].

Thus equations (27) are shape invariant and can be solved exactly using tools of SUSY quantum mechanics. The ground state vectors $\psi_{\kappa,\nu}^0 = \text{column}(\phi_{\kappa,\nu}^0, \phi_{\kappa,-\nu}^0)$, $\nu = s, s-1, s-2, \dots, \nu > 0$ should solve the first order equation,

$$\left(\frac{\partial}{\partial x} + W_{\kappa,\nu} \right) \psi_{\kappa,\nu}^0 = 0,$$

and have the following form [17]:

$$\begin{aligned} \phi_{\kappa,\nu}^0 &= d_{\nu} r^{\kappa+1} K_{\nu+\frac{1}{2}} \left(\frac{\tilde{\lambda}r}{2\kappa+1} \right), \quad \kappa \geq \nu, \\ \phi_{\kappa,-\nu}^0 &= d_{\nu} (-1)^{\nu-\frac{1}{2}} r^{\kappa+1} K_{\nu-\frac{1}{2}} \left(\frac{\tilde{\lambda}r}{2\kappa+1} \right), \end{aligned} \quad (34)$$

for superpotential (31), and

$$\begin{aligned} \phi_{\kappa,\nu}^0 &= d_{\nu} r^{\nu+1} K_{\kappa+\frac{1}{2}} \left(\frac{\tilde{\lambda}r}{2\nu+1} \right), \quad 0 \leq \kappa < \nu, \\ \phi_{\kappa,-\nu}^0 &= d_{\nu} (-1)^{\kappa-\frac{1}{2}} r^{\nu+1} K_{\kappa-\frac{1}{2}} \left(\frac{\tilde{\lambda}r}{2\nu+1} \right) \end{aligned} \quad (35)$$

for the alternative potential $W_{\nu,\kappa}$. Here $K(\cdot)$ are modified Bessel functions, and d_{ν} are integration constants. Solution for $\nu = 0$, i.e., the component ϕ_0^0 , is given by the following equation:

$$\phi_{\kappa,0}^0 = d u^{\kappa+\frac{1}{2}} \exp(-\tilde{\lambda}r), \quad \kappa = 0, 1, 2, \dots$$

Vectors for excited states and the corresponding energy levels are given by the following formulae

$$\psi_{\kappa,\nu}^n(r) = a_{\kappa,\nu}^+ a_{\kappa+1,\nu}^+ \cdots a_{\kappa+n-1,\nu}^+ \psi_{\kappa+n,\nu}^0(r) \quad (36)$$

and

$$\mathcal{E}_n = -\frac{\tilde{\lambda}^2}{(2\kappa + 2n + 1)^2}, \quad (37)$$

where $a_{\kappa,\nu}^+ = -\frac{\partial}{\partial r} + W_{\kappa,\nu}$. In accordance with the analysis presented in [18] all such vectors are square integrable and vanish at $r = 0$.

In complete analogy with the above we can solve the eigenvalue problem for the more general Hamiltonian (21) including $[s+1]$ arbitrary parameters $\tilde{\lambda}_\nu$. Actually, to this effect it is sufficient to change $\tilde{\lambda} \rightarrow \tilde{\lambda}_\nu$ in formulae (28), (31)–(35) and (37).

The presented solutions are valid for models with arbitrary spin s , including the particular cases $s = 1/2$ and $s = 1$ discussed in Sections 2.1, 2.2 and 2.3.

3. Superintegrable systems in 3d space

3.1. Runge–Lenz vector for Hydrogen atom

Let us start with the most prominent system admitting the LRL vector, i.e., with the Hydrogen atom. This system is specified by the following Hamiltonian:

$$H = \frac{p^2}{2m} + V(x), \quad (38)$$

where $p^2 = p_1^2 + p_2^2 + p_3^2$, $p_1 = -i\frac{\partial}{\partial x_1}$, $V = -\frac{\alpha}{x}$, $x = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $\alpha = e^2 > 0$.

Hamiltonian (38) commutes with generators of group $SO(3)$ which are components of the orbital momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$. There is also the additional vector constant of motion:

$$\mathbf{K} = \frac{1}{2m}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) + \mathbf{x}V \quad (39)$$

whose components satisfy the following commutation relations:

$$[L_a, L_b] = i\varepsilon_{abc}L_c, \quad [K_a, L_b] = i\varepsilon_{abc}K_c, \quad [K_a, K_b] = -\frac{2i}{m}\varepsilon_{abc}L_cH. \quad (40)$$

Changing H in (40) by its eigenvalue E we obtain the Lie algebra isomorphic to $\mathfrak{so}(4)$ provided $E < 0$ or to $\mathfrak{so}(1, 3)$ if E is positive.

This symmetry causes the maximal superintegrability of the Hydrogen atom. Moreover, using the presented integrals of motion it is possible to find the energy spectrum of the Hydrogen atom algebraically. It has been done by Pauli as long as in 1926 [9].

3.2. Runge–Lenz vector for 3d systems with arbitrary spin

The non-relativistic model of the Hydrogen atom ignores the spin of the orbital electron. To introduce a spin it is necessary to change the angular momentum \mathbf{L} by the total angular momentum

$$\mathbf{L} \rightarrow \mathbf{J} = \mathbf{L} + \mathbf{S}, \quad (41)$$

where \mathbf{S} is a spin vector whose components satisfy

$$[S_a, S_b] = i\varepsilon_{abc}S_c, \quad S_1^2 + S_2^2 + S_3^2 = s(s+1)I.$$

By analogy with (39) we will search for the LRL vector with spin in the following form:

$$\hat{\mathbf{K}} = \frac{1}{2m}(\mathbf{p} \times \mathbf{J} - \mathbf{J} \times \mathbf{p}) + \mathbf{x}\hat{V}. \quad (42)$$

By definition vector (42) should commute with the Hamiltonian

$$H = \frac{p^2}{2m} + \hat{V}.$$

It is the case iff potential \hat{V} satisfies the following conditions:

$$[\hat{V}, \mathbf{J}] = \mathbf{0}, \quad (43)$$

$$\mathbf{x} \cdot \nabla \hat{V} + \hat{V} = 0, \quad (44)$$

$$\mathbf{S} \times \nabla \hat{V} - \nabla \hat{V} \times \mathbf{S} = 0, \quad (45)$$

where \cdot and \times are the symbols of the scalar and vector products and ∇ is the gradient vector.

In accordance with (43) and (44) \hat{V} should be a rotational scalar of dimension $\frac{1}{x}$,

$$\hat{V} = \lambda \frac{\mu_s(\mathbf{n})}{x}, \quad (46)$$

where $\mu_s(\mathbf{n})$ is a scalar and dimensionless matrix which, without loss of generality, can be represented in form form (17) with $\mathbf{S} \cdot \mathbf{n} = S_1 n_1 + S_2 n_2 + S_3 n_3$ and $n_a = \frac{x_a}{x}$. Substituting (17) into (45) and using the 3d analogue of relation (18) we come to an algebraic equation for coefficients c_ν which is easy solvable. As a result we obtain the following expressions for potentials \hat{V} [13]:

$$\hat{V} = \frac{\lambda}{x} \Lambda_0 \quad \text{for integer spins,} \quad (47)$$

$$\hat{V} = \frac{\lambda}{x} \sum_{\nu=-s}^s \frac{1}{\nu} \Lambda_\nu \quad \text{for half integer spins.} \quad (48)$$

Thus we find 3d systems with arbitrary spin which admit the LRL vector. The most important systems which correspond to $s = 1/2$ and $s = 1$ are considered in more detail in the following sections.

3.3. 3d system with spin 1/2

Let $s = 1/2$ then, in accordance with (46) and (48), the potential is reduced to the following form

$$\hat{V} = \lambda \frac{\boldsymbol{\sigma} \cdot \mathbf{x}}{x^2} \Rightarrow H = \frac{p^2}{2m} + \lambda \frac{\boldsymbol{\sigma} \cdot \mathbf{x}}{x^2}. \quad (49)$$

This Hamiltonian admits two vector integrals of motion given by equations (41) and (42) which satisfy the same commutation relations (40) as in the case of the Hydrogen atom. Thus, like in the Hydrogen atom, the bound states of our system has a hidden symmetry w.r.t. algebra $\mathfrak{so}(4)$. This symmetry makes it possible to find the spectrum of Hamiltonian H (49) algebraically [11]. Moreover, the admissible eigenvalues of E for bound states are [11]

$$E = -\frac{m\lambda^2}{2N^2} \quad (50)$$

where $N = 2n + 2j + 1/2$, $n = 0, 1, 2, \dots$. In contrast with the Hydrogen atom, the main quantum number N should be half integer.

The considered system with spin $1/2$ inherits one more nice property of the Hydrogen atom. It is supersymmetric (shape invariant) and can be integrated with using tools of SUSY quantum mechanics. Indeed, the corresponding radial equations are

$$\mathcal{H}_j \Phi_j \equiv \left(-\frac{\partial^2}{\partial r^2} + V_j \right) \Phi_j = \mathcal{E} \Phi_j,$$

where V_j is the matrix potential

$$V_j = \left(j(j+1) + \frac{1}{4} - \sigma_3 \left(j + \frac{1}{2} \right) \right) \frac{1}{r^2} - \sigma_1 \frac{\omega}{r}, \quad \omega = 2m\lambda.$$

This potential belongs to the list of shape invariant matrix potentials found in [18]. It can be expressed via the superpotential (compare with (12)):

$$V_j(x) = -\frac{\partial W_j}{\partial x} + W_j^2 - \frac{\lambda^2}{(2j+1)^2}, \quad (51)$$

where

$$W_j = \frac{1}{2x} \sigma_3 - \frac{\lambda}{2j+1} \sigma_1 - \frac{2j+1}{2x}.$$

Thus to find state vectors corresponding to eigenvalues (50) we again can use tools of SUSY quantum mechanics. The ground state vector Φ_j^0 looks as follows

$$\Phi_j^0 = C_j x^{j+\frac{3}{2}} \begin{pmatrix} K_1 \left(\frac{\lambda x}{2(j+1)} \right) \\ K_0 \left(\frac{\lambda x}{2(j+1)} \right) \end{pmatrix} \quad (52)$$

with normalization constants C_j given by the following formulae

$$C_j = 2^{-2(j+\frac{1}{2})} \left(G_{00}^{22} \left(1 |_{j,j+1}^{-1,0} \right) + G_{00}^{22} \left(1 |_{j+1,j+1}^{0,0} \right) \right)^{-1},$$

where $G_{00}^{22}(\cdot|\cdot)$ are the Meijer functions. The n^{th} excited state can be calculated in complete analogy with (36).

Comparing (49) with (5), and (50) with (11) we see that the $3d$ system with spin $1/2$ is a rather straightforward generalization of the Pron'ko–Stroganov system. However, in contrast with the $2d$ case, hamiltonian (49) does not describe interaction of a fermion with the magnetic field. On the contrary, potential $V(\mathbf{x})$ represents a Pauli type interaction with the external vector field

$$\mathbf{E} \sim \frac{\mathbf{x}}{x^2} \quad (53)$$

which can be interpreted as the electric field which can be realized experimentally at least on finite interval $a < x < b, a > 0$ [11]. Notice that functions (53) also solve equations of axion electrodynamics [23,24].

3.4. $3d$ system with spin 1

Consider now a model for vector particle. The corresponding potential is given by equation (47) for $s = 1$, i.e.,

$$\hat{V} = \frac{\alpha}{x} (1 - (\mathbf{S} \cdot \mathbf{n})^2),$$

where \mathbf{S} is the spin vector whose components are given by equation (22). The related eigenvalue problem takes the form

$$H\psi \equiv \left(\frac{p^2}{2m} + \frac{\alpha}{x}(1 - (\mathbf{S} \cdot \mathbf{n})^2) \right) \psi = E\psi. \quad (54)$$

The eigenvectors $\psi = \psi(\mathbf{x})$ are supposed to be normalizable and vanishing at $\mathbf{x} = 0$.

This rather complicated system of three coupled Schrödinger equations can be integrated in closed form thanks to the sufficient number of its integrals of motion. The admissible energy levels for bound states are given by equation (50) where $N = n + j + 1$, $n = 0, 1, \dots$ [13]. The corresponding eigenvectors was found in the following form

$$\psi = \frac{1}{r} \sum_{j,\kappa,\lambda} \psi_{j,\kappa,\lambda}(r) \Omega_{j,\kappa,\lambda}^s(\varphi, \theta),$$

where $r = \sqrt{-2mEx}$, $s = 1$, $j = 0, 1, \dots$, $\kappa = -j, -j+1, \dots, j$, $\lambda = 1, 0, -1$, and $\Omega_{j,\kappa,\lambda}^s$ are spherical harmonics for spin 1. In addition, the radial wave functions take the form [13]:

$$\begin{aligned} \psi_{j,\kappa,-1} &= C_{jn} \sqrt{j} r^{j+2} \exp(-r) ((j+n)(j+1) \mathcal{F}(-n, 2j+2, 2r) - n \mathcal{F}(1-n, 2j+3, 2r)), \\ \psi_{j,\kappa,+1} &= C_{jn} \sqrt{j+1} r^{j+1} \exp(-r) (nr \mathcal{F}(1-n, 2j+3, 2r) \\ &\quad - (j+1)(2j+1+(n+j)r) \mathcal{F}(-n, 2j+2, 2r)), \quad \psi_{j,\kappa,0} = 0, \end{aligned}$$

where $\mathcal{F}(\cdot, \cdot, \cdot, 2r)$ are confluent hypergeometric functions ${}_1F_1$, and C_{jn} are normalization constants.

In contrast to $s = 1/2$ case, $3d$ Hamiltonian (54) is by no means a straightforward generalization of the $2d$ Hamiltonian for $s = 1$ discussed in Section 2.3. In particular, as distinct from operator (23) Hamiltonian (54) is not shape invariant.

4. Superintegrable systems in any dimension

4.1. General analysis

Consider a generic d -dimensional stationary Schrödinger equation with a matrix potential given by equations (1) and (2).

We suppose that Hamiltonian H is invariant with respect to the rotation group $\text{SO}(d)$ whose generators can be chosen in the standard form

$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \quad (55)$$

where indices μ and ν run over the values $1, 2, \dots, d$, and $S_{\mu\nu}$ are matrices satisfying the familiar $\mathfrak{so}(d)$ commutation relations:

$$[S_{\mu\nu}, S_{\lambda\sigma}] = i(\delta_{\mu\lambda} S_{\nu\sigma} + \delta_{\nu\sigma} S_{\mu\lambda} - \delta_{\mu\sigma} S_{\nu\lambda} - \delta_{\nu\lambda} S_{\mu\sigma}),$$

where $\delta_{\mu\lambda}$ is the Kronecker symbol. By definition the matrix potential $V(\mathbf{x})$ should commute with generators (55):

$$[V, J_{\mu\nu}] = 0. \quad (56)$$

We search for equations (1) admitting integrals of motion:

$$K_\mu = \frac{1}{2m} (p_\nu J_{\mu\nu} + J_{\mu\nu} p_\nu) + x_\mu V. \quad (57)$$

Operators K_μ commute with H iff

$$x_\nu \nabla_\nu V + V = 0, \quad (58)$$

$$S_{\mu\nu} \nabla_\nu V + \nabla_\nu V S_{\mu\nu} = 0, \quad (59)$$

where $\nabla_\nu = \frac{\partial}{\partial x_\nu}$ and summation from 1 to d is imposed over the repeating index ν .

If conditions (56), (58) and (59) are fulfilled then operators $J_{\mu\nu}$ and K_μ form a basis of algebra $\mathfrak{so}(d+1)$, $\mathfrak{so}(1, d)$ or $\mathfrak{e}(d)$ for $E < 0$, $E > 0$ and $E = 0$ respectively. In other words, all systems whose potentials satisfy conditions (56), (58) and (59) admit a hidden symmetry of Fock type. The corresponding integral of motion (57) is an analogue of the LRL vector for d -dimensional space.

4.2. Scalar systems

Let matrices $S_{\mu\nu}$ be trivial. The corresponding potential should satisfy

$$[\hat{V}, L_{\mu\nu}] = 0, \quad x_\nu \nabla_\nu \hat{V} = -\hat{V},$$

where $L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$. The general solution of these equations is the d -dimensional Coulomb potential [15]

$$V = -\frac{\alpha}{r}, \quad (60)$$

where α is a constant.

The corresponding equation (1) is superintegrable and admits a d -dimensional analogue of the LRL vector, given by formula (57) with $J_{\mu\nu} = L_{\mu\nu}$.

Thus we recover the known result [25] concerning the generalization of the LRL vector in d dimensions. Moreover, in [15] a formal proof is presented that the only scalar potential which is compatible with the d -dimensional LRL vector is the Coulomb potential.

To solve equations (1) we use the hyper-spherical variables connected with the Cartesian variables via the following relations:

$$\begin{aligned} x_d &= r \cos \theta_{d-1}, \\ x_{d-1} &= r \sin \theta_{d-1} \cos \theta_{d-2}, \\ x_{d-2} &= r \sin \theta_{d-1} \sin \theta_{d-2} \cos \theta_{d-3}, \\ &\dots\dots\dots \\ x_2 &= r \sin \theta_{d-1} \sin \theta_{d-2} \dots \sin \theta_2 \cos \theta_1, \\ x_1 &= r \sin \theta_{d-1} \sin \theta_{d-2} \dots \sin \theta_2 \sin \theta_1. \end{aligned} \quad (61)$$

Expanding wave function ψ via hyper-spherical harmonics Y_λ^l

$$\Psi = r^{\frac{1-d}{2}} \psi_{l\lambda} Y_\lambda^l \quad (62)$$

and substituting (60), (61) and (62) into (1), we come to the following equation for radial functions $\psi_{l\lambda}$:

$$H_\mu \psi_{l\lambda}(r) \equiv \left(-\frac{\partial^2}{\partial r^2} + \frac{\mu(\mu+1)}{r^2} + \hat{V} \right) \psi_{l\lambda}(r) = \epsilon \psi_{l\lambda}(r), \quad (63)$$

where $\hat{V} = 2mV = -\frac{2m\alpha}{r}$, $\epsilon = 2mE$ and

$$\mu = l + \frac{d-3}{2}, \quad l = 0, 1, 2, \dots$$

Up to the admissible values of μ equation (63) coincides with the radial equation for the $3d$ Hydrogen atom. Negative eigenvalues ϵ and the corresponding values of E are given by the following formulae

$$\epsilon = -\frac{m^2\alpha^2}{N^2}, \quad E = -\frac{m\alpha^2}{2N^2}, \quad (64)$$

where

$$N = n + l + \frac{d-1}{2}, \quad n = 0, 1, 2, \dots$$

For $d = 3$ equation (64) is reduced to the familiar Balmer formula.

4.3. d -dimensional systems with $s = 1/2$

Consider the case when matrices $S_{\mu\nu}$ realize irreducible representation $D(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ of algebra $\mathfrak{so}(d)$ for even d or representation $D(\frac{1}{2}, \frac{1}{2}, \dots, -\frac{1}{2})$ for odd d . Making reduction of these representations on subalgebra $\mathfrak{so}(3)$ we obtain a direct sum of representations $D(\frac{1}{2})$. Thus the corresponding equations (1) present a model of a particle with spin $\frac{1}{2}$.

The considered matrices $S_{\mu\nu}$ admit the following uniform representation

$$S_{\mu\nu} = \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \quad (65)$$

where γ_μ are basis elements of the Clifford algebra satisfying the following relations

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}.$$

The dimension of irreducible matrices (65) is equal to $2^{\lfloor \frac{d}{2} \rfloor}$, where $\lfloor \frac{d}{2} \rfloor$ is the integer part of $\frac{d}{2}$.

The general solution of relations (56), (58) and (59) where $S_{\mu\nu}$ are matrices (65) are given by the following formula [15]

$$\hat{V} = \frac{\alpha}{r}\gamma_\nu n_\nu, \quad (66)$$

where $n_\nu = \frac{x_\nu}{r}$.

The system with potential (66) is invariant with respect to group $SO(d)$ and admits the generalized LRL vector (57). In addition, it is shape invariant and so can be solved algebraically [15].

To separate variables in equation (1) with potential (66) we use hyper-spherical variables (61) and expand solutions as

$$\Psi(\mathbf{r}) = r^{\frac{d-1}{2}}\psi_{j\varrho\lambda}(r)\Omega_{\varrho\lambda}^j, \quad (67)$$

where $\Omega_{\varrho\lambda}^j$ are hyper-spherical spinors satisfying the conditions

$$\begin{aligned} \frac{1}{2}J_{\mu\nu}J_{\mu\nu}\Omega_{\varrho\lambda}^j &= \left(j(j+d-2) + \frac{1}{8}(d-2)(d-3)\right)\Omega_{\varrho\lambda}^j, \\ D\Omega_{\varrho\lambda}^j &\equiv \left(\frac{1}{2}\gamma_\mu\gamma_\nu L_{\mu\nu} + \frac{d-1}{2}\right)\Omega_{\varrho\lambda}^j = \varrho\Omega_{\varrho\lambda}^j \end{aligned}$$

with $j = \frac{1}{2}, \frac{3}{2}, \dots$, and $\varrho = \pm(j + \frac{d-2}{2})$.

Substituting (66) and (67) into (1) we come to the following equation for radial wave functions

$$H_\varrho\phi \equiv \left(-\frac{\partial^2}{\partial r^2} + \frac{\varrho^2 + \sigma_3\varrho}{r^2} + \frac{\omega}{r}\sigma_1\right)\phi = \varepsilon\phi, \quad \phi = \begin{pmatrix} \psi_{j|\varrho|\lambda}(r) \\ \psi_{j-|\varrho|\lambda}(r) \end{pmatrix}, \quad (68)$$

where $\omega = 2m\alpha$.

Hamiltonian \mathcal{H}_j includes matrix potential which again belongs to the list of shape invariant potentials found in [18]. Thus equation (68) can be solved in the way used in Section 3.3. To have this effect it is sufficient simply to change $j \rightarrow \rho - \frac{1}{2}$ in equations (50), (51) and (52).

4.4. Systems with spin 1

Let us present a bosonic d -dimensional system admitting generalized LRL vector (57). We suppose the corresponding matrices $S_{\mu\nu}$ are irreducible and realize representation $D(1, 0, 0, \dots, 0)$ of algebra $\mathfrak{so}(d)$, where the symbols in brackets are the Gelfand–Tsetlin numbers. Up to equivalence, their entries $(S_{\mu\nu})_{ab}$ can be represented in the following form:

$$(S_{\mu\nu})_{ab} = i(\delta_{\mu a}\delta_{\nu b} - \delta_{\nu a}\delta_{\mu b}), \quad \mu, \nu, a, b = 1, 2, \dots, d, \quad d > 2. \quad (69)$$

The general solution of the corresponding equations (56), (58) and (59) has the following form [15]

$$V = \frac{\alpha}{(d-2)r} ((d-1)(d-4) + 2S_{\mu\nu}n_\nu S_{\mu\lambda}n_\lambda), \quad d \neq 2 \quad (70)$$

and so the potential includes the following entries

$$V_{\mu\nu} = \frac{\alpha}{2r} ((d-3)\delta_{\mu\nu} + 2n_\mu n_\nu).$$

Hamiltonian (2) with potentials (70) commutes with the total orbital momentum (55) and the LRL vector (57), where $S_{\mu\nu}$ are matrices (69). It means that the corresponding Schrödinger equations (1) possess symmetries of Fock type. They are exactly solvable for arbitrary d . We will not reproduce the cumbersome procedure of construction of the exact solutions which is presented in [15], but give the explicit formulae for the energy spectrum and the corresponding eigenvalues:

$$E = -\frac{m\alpha^2(d-1)^2}{2(2n+2l+d-1)^2}, \quad n = 0, 1, 2, \dots, \quad l = 0, 1, 2, \dots,$$

and $\Psi = \text{column}(\Psi_1, \Psi_2, \dots, \Psi_d)$ with

$$\Psi_\mu = \varphi_{l\lambda}^1(r) \frac{x_\mu}{r} Y_\lambda^l + \varphi_{l\lambda}^2(r) r \nabla_\mu Y_\lambda^l,$$

where Y_λ^l are hyper-spherical harmonics, and the radial functions are:

$$\begin{aligned} \varphi_{l\lambda}^2 &= C_{ln} z^{l+d-1} \exp(-z) \mathcal{F}(-n, l+d-1, 2z), \\ \varphi_{l\lambda}^1 &= C_{ln} \left(z^{l+d-1} \exp(-z) \left((l+d+(k-1)z) \mathcal{F}(-n, l+d-1, 2z) \right. \right. \\ &\quad \left. \left. - \frac{2kz}{l+d-1} \mathcal{F}(-n+1, l+d, 2z) \right) \right), \end{aligned}$$

where C_{ln} are integration constants, $z = \frac{m\alpha r}{k}$, $k = \frac{2n+2l+d-1}{d-1}$, and $\mathcal{F}(-n, l+d-1, 2z)$ are confluent hypergeometric functions ${}_1F_1$. The presented solutions are normalizable and vanish at $z = 0$.

5. Discussion

The main goal of this presentation is to extend the field of known superintegrable quantum mechanical systems admitting LRL vector. Originally this vector was specified for three-dimensional space. Then it was generalized to the case of d dimensions [25], but it was done only for scalar QM systems.

The first example of LRL vector with spin was proposed apparently in paper [22], where a $2d$ superintegrable system was discovered. LRL vectors with spin in three-dimensional space were discussed in [11]. Then these results were extended to the case of arbitrary spin [13] and arbitrary dimension [15].

The d -dimensional systems with spin admitting LRL keep all basic properties of the systems with $d = 3$. They are superintegrable, and their extended symmetries present effective tools for finding exact solutions.

The discussed systems with spin $1/2$ and arbitrary dimension are supersymmetric, i.e., the corresponding radial equations are shape invariant. Moreover, they belong to the list of shape invariant systems classified in [18] and [19].

The systems with spin $s > 1/2$ and $d = 2$ are also supersymmetric. However, for $d > 2$ they are neither supersymmetric nor shape invariant. Nevertheless, these systems are superintegrable and exactly solvable, see Section 4.4 and paper [15] for their exact solutions for arbitrary d . Exact solutions for $d = 2$ and $d = 3$ are given in the present paper in more details.

Thus the tight connection between superintegrability and shape invariance discussed in [1–4] is missing in cases when $s > 1/2$ and simultaneously $d > 2$.

To find superintegrable systems with arbitrary spin and dimension we restrict ourselves to a priori specified forms (42) and (57) of the generalized LRL vectors. However, at least for $3d$ systems with spin $1/2$ this restriction is not essential, and potential (49) can be obtained requiring only for the rotation invariance of equation (1) [26].

The discussed superintegrable systems with $d = 2$ and $d = 3$ have a clear physical meaning and describe neutral particles with non-trivial multipole momenta interacting with the external electromagnetic fields. A perfect example of such particle is the neutron. For more detailed physical interpretation see papers [11], [13] and [17].

Let us mention that there are known examples of *relativistic* (super)integrable systems with spin, see, e.g., [11, 26, 27]. The complete classification of such systems at least for spin $1/2$ is a rather inspiring challenge.

One more challenge is an extension of the results of papers [11, 13, 14] to the case of non-stationary Schrödinger equations, like it was done in [28] for scalar systems.

Acknowledgments

The author is grateful for the hospitality and financial support provided by the University of Cyprus.

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