

# The Poincaré algebras $p(1, 1)$ and $p(1, 2)$ : realizations and deformations

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**Abstract.** All inequivalent realizations of the Poincaré algebras  $p(1, 1)$  and  $p(1, 2)$  acting in spaces of not more than three variables are constructed. First-order deformations of  $p(1, 1)$  and  $p(1, 2)$  are proposed and the generic realizations for the initial and deformed Poincaré algebras are presented.

## 1. Introduction

Realizations of Lie algebras by vector fields are widely applicable in mathematics and physics, indeed, representations by first-order differential operators are effectively used in Lie group theory of differential equations: reduction, integration, calculation of differential invariants, etc.; in group classification of partial differential equations; in the theory of differential invariants; in general relativity and other physical problems such as classification of gravity fields of a general form under the motion groups and groups of conformal transformations, or quantization based on Noether symmetries, see also [1–5] and [6] for review.

In this work we apply an algebraic approach proposed in [7] and the classification of subalgebras obtained in [8] for the explicit construction of vector fields of the lowest Poincaré algebra  $p(1, 1)$ , the six-dimensional Poincaré algebra  $p(1, 2)$  and their deformations. Realizations of different Poincaré algebras were widely studied (see, e.g., [9–13]), but only covariant realizations were considered and a number of realizations are not represented explicitly as far as some of the coefficients require the solution of partial differential equations, that can not be presented in quadratures. Along with the realizations we also propose three one-parametric deformations of  $p(1, 1)$  and  $p(1, 2)$  and construct their generic realizations by the same method. The results of this paper is a part of the forthcoming general classification of the realizations of Poincaré algebras  $p(1, 2)$  and  $p(1, 3)$  [14].

Let  $\mathfrak{g}$  be an  $n$ -dimensional Lie algebra over a field  $\mathbb{R}$  or  $\mathbb{C}$ . We denote an open subset of  $\mathbb{R}^m$  as  $M$  and the Lie algebra of vector fields on it as  $\text{Vect}(M)$ , the vector fields are taken in a form of linear first-order differential operators with analytical coefficients and the Lie product of vector fields is given by their commutator. The group of all automorphisms of  $\mathfrak{g}$  is denoted by  $\text{Aut}(\mathfrak{g})$  and the notion of realization of a Lie algebra is defined as follows (see [6] for details).

A *realization* of a Lie algebra  $\mathfrak{g}$  in vector fields on  $M$  is a homomorphism  $R: \mathfrak{g} \rightarrow \text{Vect}(M)$ . The realization is *faithful* if  $\ker R = \{0\}$  and *unfaithful* otherwise. If the action of the local Lie group that corresponds to a fixed realization is transitive on  $M$ , then this realization is called *generic*.



Realizations of Lie algebras are constructed up to  $\text{Aut}(\mathfrak{g})$ -equivalence and nondegenerate change of coordinates on  $M$ . Note that for any generic realization the dimension of  $M$  coincides with the dimension of  $\mathfrak{g}$  that is  $m = n$ .

Below we propose the practical scheme for construction of left-invariant vector fields extracted from [7]. The coefficients  $\xi_k^i(x)$  of the generic realization  $\Xi_i = \sum_{k=1}^n \xi_k^i(x) \frac{\partial}{\partial x_k}$ ,  $i = 1, 2, \dots, n$ , can be recovered from the left-invariant differential one-forms  $\Omega^i = \sum_{l=1}^n \omega_l^i(x) dx_l$  using the duality of the left-invariant vector fields and differential one-forms  $\omega_l^i(x) \xi_k^i(x) = \delta_k^l$  and the coefficients  $\omega_l^i(x)$  of the differential one-forms are constructed as follows:

$$\omega_l^i(x) = \left( A^{(1)}(x^1) A^{(2)}(x^2) \cdots A^{(i-1)}(x^{i-1}) \right)_i^l, \quad i = 2, 3, \dots, n, \quad l = 1, 2, \dots, n, \quad \omega_1^1 = \delta_1^1,$$

where the matrices  $A^{(p)}$ ,  $p = 1, 2, \dots, n$ , are the exponential solutions of the system

$$\dot{A}^{(p)}(t) = -\text{ad}_{e_p} A^{(p)}(t), \quad A^{(p)}(0) = I.$$

All the rest of realizations of a fixed Lie algebra are constructed by means of projection of the generic realization using the known set of  $\text{Aut}(\mathfrak{g})$ -inequivalent subalgebras and the following rule.

Let  $\mathfrak{h} = \langle e_{m+1}, \dots, e_n \rangle$  be a subalgebra of  $\mathfrak{g} = \langle e_1, \dots, e_n \rangle$  with a complementary space  $\{e_1, \dots, e_m\}$ , then, using the above approach and the shortcut  $\partial_i = \frac{\partial}{\partial x_i}$ , we will obtain the basis elements in the form

$$e_i = \xi_i^1(x_1, x_2, \dots, x_m) \partial_1 + \xi_i^2(x_1, x_2, \dots, x_m) \partial_2 + \cdots + \xi_i^m(x_1, x_2, \dots, x_m) \partial_m + \xi_i^{m+1}(x_1, x_2, \dots, x_n) \partial_{m+1} + \xi_i^{m+2}(x_1, x_2, \dots, x_n) \partial_{m+2} + \cdots + \xi_i^n(x_1, x_2, \dots, x_n) \partial_n.$$

The realization projected on the coordinates  $x_1, x_2, \dots, x_m$  is well defined and has the form

$$\text{pr}_{\mathfrak{h}}(e_i) = \xi_i^1(x_1, x_2, \dots, x_m) \partial_1 + \xi_i^2(x_1, x_2, \dots, x_m) \partial_2 + \cdots + \xi_i^m(x_1, x_2, \dots, x_m) \partial_m.$$

The number of necessary variables  $m = \dim(\mathfrak{g}) - \dim(\mathfrak{h})$  coincides with the codimension of  $\mathfrak{h}$  and the projected realization  $R(\mathfrak{h})$  is called *realization corresponding to the subalgebra  $\mathfrak{h}$* . In particular this means that the generic realization corresponds to zero subalgebra  $\mathfrak{h}_0 = \langle 0 \rangle$  and  $R(\mathfrak{h}_0)$  is always realized in  $n = \dim(\mathfrak{g})$  variables.

The structure of realizations constructed by means of the algebraic method reminds a tree diagram, namely: a realization corresponding to a subalgebra  $\mathfrak{h}_1$  can be constructed by means of projection from a realization corresponding to a subalgebra  $\mathfrak{h}_2$  if  $\mathfrak{h}_2 \subset \mathfrak{h}_1$ .

Note that all inequivalent realizations of a fixed Lie algebra can be obtained by the above method, as far as any realization corresponds to a quotient group  $G/H$  that acts effectively on some subspace  $M$ , where  $H$  is a subgroup that corresponds to some subalgebra  $\mathfrak{h}$  [7].

## 2. Realizations of the Poincaré algebra $p(1, 1)$

The ‘smallest’ Poincaré algebra  $p(1, 1)$  is supposed to act in the two-dimensional space-time  $\mathbb{R}^{1,1}$  and has dimension three:  $p(1, 1) = \langle P_0, P_1, J_{01} \rangle$ . Its basis elements satisfy the commutation relations

$$[P_0, P_1] = 0, \quad [P_0, J_{01}] = P_1, \quad [P_1, J_{01}] = P_0. \quad (1)$$

Only covariant realizations of the Lie algebra  $p(1, 1)$  in space of three variables were constructed in [9]. All inequivalent faithful realizations of  $p(1, 1)$  were also obtained by the direct

**Table 1.** Realizations of  $p(1, 1)$ 

Complementary vector space	Subalgebra	Realization
$\{P_0, P_1, J_{01}\}$	$\langle 0 \rangle$	$P_0 = \partial_1, P_1 = \partial_2, J_{01} = x_2 \partial_1 + x_1 \partial_2 + \partial_3$
$\{J_{01}, P_1\}$	$\langle P_0 \rangle$	$P_0 = -\sinh x_1 \partial_2, P_1 = \cosh x_1 \partial_2, J_{01} = \partial_1$
$\{P_0, P_1\}$	$\langle J_{01} \rangle$	$P_0 = \partial_1, P_1 = \partial_2, J_{01} = x_2 \partial_1 + x_1 \partial_2$
$\{P_0, J_{01}\}$	$\langle P_0 + P_1 \rangle$	$P_0 = \partial_1, P_1 = -\partial_1, J_{01} = -x_1 \partial_1 + \partial_2$
$\{P_0, J_{01}\}$	$\langle P_0 - P_1 \rangle$	$P_0 = \partial_1, P_1 = \partial_1, J_{01} = x_1 \partial_1 + \partial_2$
$\{J_{01}\}$	$\langle P_0, P_1 \rangle$	$P_0 = 0, P_1 = 0, J_{01} = \partial_1$
$\{P_0\}$	$\langle P_0 + P_1, J_{01} \rangle$	$P_0 = \partial_1, P_1 = -\partial_1, J_{01} = -x_1 \partial_1$
$\{P_0\}$	$\langle P_0 - P_1, J_{01} \rangle$	$P_0 = \partial_1, P_1 = \partial_1, J_{01} = x_1 \partial_1$

method in [6] for the isomorphic Lie algebra  $A_{3,4}^{-1}$  with the commutation relations ( $e_1 = P_0 + P_1$ ,  $e_2 = P_0 - P_1$ ,  $e_3 = J_{01}$ )

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = -e_2. \quad (2)$$

We list realizations of  $p(1, 1)$  together with the corresponding subalgebras and bases of complementary vector spaces in Table 1.

The subalgebras that contain one of ideals  $\langle P_0 \pm P_1 \rangle$  of  $p(1, 1)$  lead to unfaithful realizations which are new and were not constructed in [6]. The rest of realizations are equivalent to the faithful realizations obtained for  $A_{3,4}^{-1}$ .

We purposely do not fix independent (say  $t, x$ ) and dependent (say  $u, w$ ) variables in the presented realizations. One can choose them in any order convenient for the certain application. For example, the first realization from Table 1 can be considered as

$$\begin{aligned} &\{P_0 = \partial_t, P_1 = \partial_x, J_{01} = x \partial_t + t \partial_x + \partial_u\}, \quad \text{for } u = u(t, x) \quad \text{or} \\ &\{P_0 = \partial_u, P_1 = \partial_w, J_{01} = w \partial_u + u \partial_w + \partial_t\}, \quad \text{where } u = u(t) \text{ and } w = w(t). \end{aligned}$$

### 3. Realizations of the Poincaré algebra $p(1, 2)$

The Poincaré algebra

$$p(1, 2) = \langle P_0, P_1, P_2, J_{01}, J_{02}, J_{12} \rangle$$

is six-dimensional and usually considered in (1+2)-dimensional Minkowski space with the basis elements that satisfy the nonzero commutation relations

$$\begin{aligned} [P_0, J_{01}] &= P_1, & [P_1, J_{01}] &= P_0, & [P_2, J_{02}] &= P_0, \\ [P_0, J_{02}] &= P_2, & [P_1, J_{12}] &= -P_2, & [P_2, J_{12}] &= P_1, \\ [J_{01}, J_{02}] &= -J_{12}, & [J_{02}, J_{12}] &= J_{01}, & [J_{01}, J_{12}] &= -J_{02}. \end{aligned}$$

Since we are interested in the generic realization and realizations in not more than three variables, we need the subalgebras of  $p(1, 2)$  of the dimensions three, four and five only. Using the subalgebras of the Poincaré algebra obtained in [8] and applying the algorithm described in the first section we obtained the desired realizations. The result is arranged in tables together with the subalgebras and bases of the complementary vector spaces used in calculations.

**Table 2.** Realizations of  $p(1, 2)$  in three variables

Complementary vector space	Subalgebra	Realization
$\{J_{01}, J_{02}, J_{12}\}$	$\langle P_0, P_1, P_2 \rangle$	$P_0 = 0, P_1 = 0, P_2 = 0,$ $J_{01} = \partial_1,$ $J_{02} = -\sinh x_1 \tanh x_2 \partial_1 + \cosh x_1 \partial_2 - \frac{\sinh x_1}{\cosh x_2} \partial_3,$ $J_{12} = \cosh x_1 \tanh x_2 \partial_1 - \sinh x_1 \partial_2 + \frac{\cosh x_1}{\cosh x_2} \partial_3$
$\{P_0 - P_2, J_{01} + J_{12}, J_{02}\}$	$\langle P_0 + P_2, P_1,$ $\beta(P_0 - P_2) +$ $J_{01} - J_{12} \rangle,$ $\beta \in \{0, 1\}$	$P_0 = \frac{1}{2}(1 + x_2^2)\partial_1, P_1 = -x_2\partial_1,$ $P_2 = \frac{1}{2}(-1 + x_2^2)\partial_1,$ $J_{01} = (-x_1x_2 - \frac{\beta}{2}e^{-2x_3})\partial_1 + \frac{1}{2}(1 - x_2^2)\partial_2 + x_2\partial_3,$ $J_{02} = -x_1\partial_1 - x_2\partial_2 + \partial_3,$ $J_{12} = (x_1x_2 + \frac{\beta}{2}e^{-2x_3})\partial_1 + \frac{1}{2}(1 + x_2^2)\partial_2 - x_2\partial_3$
$\{P_0, P_1, P_2\}$	$\langle J_{01}, J_{02}, J_{12} \rangle$	$P_0 = \partial_1, P_1 = \partial_2, P_2 = \partial_3,$ $J_{01} = x_2\partial_1 + x_1\partial_2, J_{02} = x_3\partial_1 + x_1\partial_3,$ $J_{12} = x_3\partial_2 - x_2\partial_3$
$\{P_0 - P_2, J_{01}, J_{12}\}$	$\langle P_0 + P_2,$ $P_1, J_{02} \rangle$	$P_0 = (\sinh x_1 \sin x_3 + \cosh x_1)\partial_2,$ $P_1 = (-\sinh x_1 - \cosh x_1 \sin x_3)\partial_2,$ $P_2 = -\cos x_3\partial_2, J_{01} = \partial_1,$ $J_{02} = -\cosh x_1 \tan x_3\partial_1 - \frac{\cosh x_1}{\cos x_3} \partial_2 - \sinh x_1\partial_3,$ $J_{12} = \sinh x_1 \tan x_3\partial_1 + \frac{\sinh x_1}{\cos x_3} \partial_2 + \cosh x_1\partial_3$
$\{J_{02}, J_{01}, P_0\}$	$\langle \alpha P_0 + J_{12},$ $P_1, P_2 \rangle,$ $\alpha \geq 0$	$P_0 = \cosh x_1 \cosh x_2\partial_3, P_1 = -\sinh x_2\partial_3,$ $P_2 = -\sinh x_1 \cosh x_2\partial_3,$ $J_{01} = -\sinh x_1 \tanh x_2\partial_1 + \cosh x_1\partial_2 - \alpha \frac{\sinh x_1}{\cosh x_2} \partial_3,$ $J_{02} = \partial_1,$ $J_{12} = -\cosh x_1 \tanh x_2\partial_1 + \sinh x_1\partial_2 - \alpha \frac{\cosh x_1}{\cosh x_2} \partial_3$
$\{J_{12}, J_{01}, P_1\}$	$\langle \alpha P_1 + J_{02},$ $P_0, P_2 \rangle,$ $\alpha \geq 0$	$P_0 = -\sinh x_2\partial_3, P_1 = \cos x_1 \cosh x_2\partial_3,$ $P_2 = -\sin x_1 \cosh x_2\partial_3,$ $J_{01} = -\sin x_1 \tanh x_2\partial_1 + \cos x_1\partial_2 - \alpha \frac{\sin x_1 \cosh(2x_1)}{2 \cosh x_2} \partial_3,$ $J_{02} = -\cos x_1 \tanh x_2\partial_1 - \sin x_1\partial_2 - \alpha \frac{\cos x_1 \cosh(2x_1)}{2 \cosh x_2} \partial_3,$ $J_{12} = \partial_1$
$\{P_0 - P_2, P_1, J_{01} + J_{12}\}$	$\langle P_0 + P_2,$ $\alpha P_1 + J_{02},$ $J_{01} - J_{12} \rangle,$ $\alpha \geq 0$	$P_0 = (1 - x_3^2)\partial_1 - x_3\partial_2, P_1 = \partial_2, P_2 = -(1 + x_3^2)\partial_1 - x_3\partial_2,$ $J_{01} = (x_2 - x_3^2(x_2 + 2\alpha))\partial_1 + (x_1 - x_3(x_2 + \alpha))\partial_2 + (1 - x_3^2)\partial_3,$ $J_{02} = -2(x_1 + \alpha x_3)\partial_1 - \alpha\partial_2 - 2x_3\partial_3,$ $J_{12} = (x_2 + x_3^2(x_2 + 2\alpha))\partial_1 - (x_1 - x_3(x_2 + \alpha))\partial_2 + (1 + x_3^2)\partial_3$

Generic realization of  $p(1, 2)$  has the form

$$\begin{aligned}
 P_0 &= \partial_1, \quad P_1 = \partial_2, \quad P_2 = \partial_3, \quad J_{01} = x_2\partial_1 + x_1\partial_2 + \partial_4, \\
 J_{02} &= x_3\partial_1 + x_1\partial_3 - \sinh x_4 \tanh x_5\partial_4 + \cosh x_4\partial_5 - \frac{\sinh x_4}{\cosh x_5}\partial_6, \\
 J_{12} &= x_3\partial_2 - x_2\partial_3 + \cosh x_4 \tanh x_5\partial_4 - \sinh x_4\partial_5 + \frac{\cosh x_4}{\cosh x_5}\partial_6.
 \end{aligned}$$

Comparing the realizations of  $p(1, 2)$  obtained in this section with the realizations obtained in [13] we conclude that the result of [13] is a partial case of the results obtained in this paper as far as only covariant realizations (i.e. the operators  $P_0, P_1$  and  $P_3$  are represented by the commuting shifts only) were considered and the sets of dependent and independent variables were fixed before the calculations.

**Table 3.** Realizations of  $p(1, 2)$  in two variables

Complementary vector space	Subalgebra	Realization
$\{J_{01} + J_{12}, J_{02}\}$	$\langle P_0, P_1, P_2, J_{01} - J_{12} \rangle$	$P_0 = 0, P_1 = 0, P_2 = 0,$ $J_{01} = \frac{1}{2}(1 - x_1^2)\partial_1 + x_1\partial_2,$ $J_{02} = -x_1\partial_1 + \partial_2, J_{12} = \frac{1}{2}(1 + x_1^2)\partial_1 - x_1\partial_2$
$\{J_{01}, J_{12}\}$	$\langle P_0, P_1, P_2, J_{02} \rangle$	$P_0 = 0, P_1 = 0, P_2 = 0,$ $J_{01} = \partial_1, J_{02} = -\cosh x_1 \tan x_2 \partial_1 - \sinh x_1 \partial_2,$ $J_{12} = \sinh x_1 \tan x_2 \partial_1 + \cosh x_1 \partial_2$
$\{J_{01}, J_{02}\}$	$\langle P_0, P_1, P_2, J_{12} \rangle$	$P_0 = 0, P_1 = 0, P_2 = 0, J_{01} = \partial_1,$ $J_{02} = -\sinh x_1 \tanh x_2 \partial_1 + \cosh x_1 \partial_2,$ $J_{12} = \cosh x_1 \tanh x_2 \partial_1 - \sinh x_1 \partial_2$
$\{P_0 - P_2, J_{01} + J_{12}\}$	$\langle P_0 + P_2, P_1, J_{01} - J_{12}, J_{02} \rangle$	$P_0 = \frac{1}{2}(1 + x_2^2)\partial_1, P_1 = -x_2\partial_1,$ $P_2 = -\frac{1}{2}(1 - x_2^2)\partial_1, J_{01} = -x_1x_2\partial_1 + \frac{1}{2}(1 - x_2^2)\partial_2,$ $J_{02} = -x_1\partial_1 - x_2\partial_2, J_{12} = x_1x_2\partial_1 + \frac{1}{2}(1 + x_2^2)\partial_2$

**Table 4.** Realization of  $p(1, 2)$  in one variable

Complementary vector space	Subalgebra	Realization
$\{J_{01} + J_{12}\}$	$\langle P_0, P_1, P_2, J_{02}, J_{01} - J_{12} \rangle$	$P_0 = 0, P_1 = 0, P_2 = 0, J_{01} = \frac{1}{2}(1 - x_1^2)\partial_1,$ $J_{02} = -x_1\partial_1, J_{12} = \frac{1}{2}(1 + x_1^2)\partial_1$

#### 4. Deformations of $p(1, 1)$ and $p(1, 2)$

Deformations of Lie algebras appear in mathematical theories when it is necessary to classify non-isomorphic structures or to implement an analytical structure on the variety of such objects. In physics the deformations are used for quantization or for integration of theories with different symmetries into one model, etc., see also [15] for review.

Here we use the notion of one-parametric deformation proposed by Gerstenhaber [16]. Let  $\mathfrak{g} = (V, [\cdot, \cdot])$  be a Lie algebra over  $V$ , then its *deformation* is the one-parametric family of Lie algebras  $\mathfrak{g}_t = (V, [\cdot, \cdot]_t)$ ,  $t \in \mathbb{R}$ , where  $[x, y]_t = [x, y] + t\varphi_1(x, y) + t^2\varphi_2(x, y) + \dots$ , and  $\varphi_i: V \times V \rightarrow V$  are bilinear, antisymmetric and satisfy the Jacobi identity.

Construction of all possible deformations of a given Lie algebra requires the study of respective cohomology algebra, in particular, for the existence of infinitesimal deformation it is necessary that  $\mathcal{H}^2(\mathfrak{g}, \mathfrak{g}) \neq 0$ . However for the Poincaré Lie algebra  $p(1, 1)$  we can construct one of its infinitesimal deformations directly from the classification of three-dimensional Lie algebras using the isomorphism  $p(1, 1) \sim A_{3,4}^{-1}$ :

$$[P_0, J_{01}]_{t'} = P_1 + t'(P_0 - P_1), \quad [P_1, J_{01}]_{t'} = P_0 + t'(P_1 - P_0), \quad t' \in \mathbb{R}.$$

Another deformation of the Poincaré algebra can be constructed using the fact that there exists a contraction from the de Sitter algebras to the Poincaré one

$$[P_0, J_{01}]_t = P_1, \quad [P_1, J_{01}]_t = P_0, \quad [P_0, P_1]_t = tJ_{01}, \quad t \in \mathbb{R}.$$

The same scheme works for all higher-dimensional Poincaré algebras, in particular for  $p(1, 2)_t$ :

$$\begin{aligned} [P_0, J_{01}]_t &= P_1, & [P_1, J_{01}]_t &= P_0, & [P_2, J_{02}]_t &= P_0, \\ [P_0, J_{02}]_t &= P_2, & [P_1, J_{12}]_t &= -P_2, & [P_2, J_{12}]_t &= P_1, \\ [P_1, P_2]_t &= tJ_{12}, & [P_1, P_2]_t &= tJ_{01}, & [P_0, P_2]_t &= tJ_{02}, \\ [J_{01}, J_{02}]_t &= -J_{12}, & [J_{02}, J_{12}]_t &= J_{01}, & [J_{01}, J_{12}]_t &= -J_{02}. \end{aligned}$$

Using the same method as in previous sections we construct the generic realizations  $R(p(1, 1)_{t'})$  and  $R(p(1, 1)_t)$  of the above deformations, before each realization we present the complementary vector space that was used for the calculation of this realization:

$$\begin{aligned} \{P_0, P_1, J_{01}\}, \quad R(p(1, 1)_{t'}): \quad P_0 &= \partial_1, \quad P_1 = \partial_2, \\ J_{01} &= (t'x_1 + (1 - t')x_2)\partial_1 + ((1 - t')x_1 + t'x_2)\partial_2 + \partial_3; \\ \{P_0 + P_1, P_0 - P_1, J_{01}\}, \quad R(p(1, 1)_t): \quad P_0 &= \frac{1}{2}(1 - tx_1^2)\partial_1 + (\frac{1}{2} + tx_1x_2)\partial_2 - tx_1\partial_3, \\ P_1 &= \frac{1}{2}(1 + tx_1^2)\partial_1 - (\frac{1}{2} + tx_1x_2)\partial_2 + tx_1\partial_3, \\ J_{01} &= x_1\partial_1 - x_2\partial_2 + \partial_3. \end{aligned}$$

To construct the generic realization of the deformed algebra  $p(1, 2)_t$  we use the complementary space to the empty subalgebra with the basis  $\{P_0 + P_2, P_0 - P_2, P_1, J_{01} + J_{12}, J_{01} - J_{12}, J_{02}\}$  and obtain the following result:

$$\begin{aligned} P_0 &= -\frac{1}{2}B\partial_1 + C\partial_2 + tx_1x_4\partial_4 - tx_1\partial_5, \\ P_1 &= -Tx_1F\partial_1 - (1 + 2tx_1x_2)Tx_2F\partial_2 + (1 + 2tx_1x_2)\partial_3 + \\ &\quad + \frac{1}{E}((2x_2C - 2tx_1x_2x_4TD - x_1x_4^2)\partial_4 + (2tx_1x_4 + tx_2TD)\partial_5 + (tx_1e^{-x_5})\partial_6), \\ P_2 &= \frac{1}{2}A\partial_1 - C\partial_2 - tx_1x_4\partial_4 + tx_1\partial_5, \\ J_{01} &= \frac{AF}{2T}\partial_1 + \frac{(1 + 2tx_2(x_1 + x_2A))F}{2T}\partial_2 + (x_1 + x_2A)\partial_3 - \frac{A(x_4^2 - tx_2^2) - 2C}{2E}\partial_4 + \\ &\quad + x_2x_4TAF\partial_4 + \frac{A(x_2TD + x_4)}{E}\partial_5 + \frac{Ae^{-x_5}}{2E}\partial_6, \\ J_{02} &= x_1\partial_1 - x_2\partial_2 - x_4\partial_4 + \partial_5, \\ J_{12} &= \frac{BF}{2T}\partial_1 + \frac{(1 + 2tx_2(x_1 + x_2B))F}{2T}\partial_2 + (x_1 + x_2B)\partial_3 - \frac{B(x_4^2 - tx_2^2) - 2C}{2E}\partial_4 + \\ &\quad + x_2x_4TBF\partial_4 + \frac{(x_2TD + x_4)B}{E}\partial_5 + \frac{Be^{-x_5}}{2E}\partial_6. \end{aligned}$$

To improve the presentation of the above realization we used the shortcuts  $A = tx_1^2 + 1$ ,  $B = tx_1^2 - 1$ ,  $C = tx_1x_2 + \frac{1}{2}$ ,  $D = \sinh((-t)^{1/2}x_3)$ ,  $E = \cosh((-t)^{1/2}x_3)$ ,  $F = \tanh((-t)^{1/2}x_3)$ ,  $T = (-t)^{1/2}$ .

Note, that each of the last three realizations represent not only one Lie algebra, but the family of non-isomorphic Lie algebras at the same time. As soon as the set of dependent and independent variables will be fixed all the obtained realizations can be used for the construction of different types of equations and systems invariant with respect to the underlying symmetry groups.

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## References

- [1] Olver P 2000 *Applications of Lie Groups to Differential Equations* (New York: Springer)
- [2] Fushchych W and Nikitin A 1994 *Symmetries of Equations of Quantum Mechanics* (New York: Allerton Press Inc.)
- [3] Hernández Heredero R and Olver P 1996 Classification of invariant wave equations *J. Math. Phys.* **37** 6419
- [4] Fushchych W, Tsyfra I and Boyko V 1994 Nonlinear representations for Poincaré and Galilei algebras and nonlinear equations for electromagnetic field *J. Nonlinear Math. Phys.* **1** 210 (*Preprint* arXiv:math-ph/0207024)
- [5] Fushchych V and Zhdanov R 1997 *Symmetries and Exact Solutions of Nonlinear Dirac Equations* (Kyiv: Mathematical Ukraina Publisher) (*Preprint* arXiv:math-ph/0609052)
- [6] Popovych R, Boyko V, Nesterenko M and Lutfullin M 2003 Realizations of real low-dimensional Lie algebras *J. Phys. A* **36** 7337 (*Preprint* arXiv:math-ph/0301029)
- [7] Magazev A, Mikheyev V, Shirokov I 2013 Evaluation of composition function along with invariant vector fields on Lie group using structural constants of corresponding Lie algebra (*Preprint* arXiv:1312.0362)
- [8] Fushchych W, Barannyk L and Barannyk A 1991 *Subgroup Analysis of the Galilei and Poincaré Groups and Reduction of Nonlinear Equations* (Kiev: Naukova Dumka) [in Russian]
- [9] Rideau G and Winternitz P 1990 Nonlinear equations invariant under the Poincaré, similitude and conformal groups in two-dimensional space-time *J. Math. Phys.* **31** 1095
- [10] Rideau G and Winternitz P 1993 Evolution equations invariant under two-dimensional space-time Schrödinger group *J. Math. Phys.* **34** 558
- [11] Fushchych W, Zhdanov R and Lahno V 1994 On linear and nonlinear representations of the generalized Poincaré groups in the class of Lie vector fields *J. Nonlinear Math. Phys.* **1** 295
- [12] Yehorchenko I 1992 Nonlinear representation of the Poincaré algebra and invariant equations, in *Symmetry Analysis of Equations of Mathematical Physics*, ed W I Fushchych (Kyiv: Institute of Mathematics) p 62
- [13] Zhdanov R, Lahno V and Fushchych W 2000 On new realizations of the Poincaré groups  $P(1,2)$  and  $P(2,2)$  *Comm. Math. Phys.* **212** 535
- [14] Boyko V, Nesterenko M and Popovych R Realizations of Poincaré algebras (in preparation)
- [15] Nesterenko M and Popovych R 2006 Contractions of Low-dimensional Lie Algebras *J. Math. Phys.* **47** 123515 (*Preprint* arXiv:math-ph/0608018)
- [16] Gerstenhaber M 1964 On the deformation of rings and algebras *Ann. Math.* **79** 59