Application of group analysis to the spatially homogeneous and isotropic Boltzmann equation with source using its Fourier image

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Abstract. Group analysis of the spatially homogeneous and molecular energy dependent Boltzmann equations with source term is carried out. The Fourier transform of the Boltzmann equation with respect to the molecular velocity variable is considered. The correspondent determining equation of the admitted Lie group is reduced to a partial differential equation for the admitted source. The latter equation is analyzed by an algebraic method. A complete group classification of the Fourier transform of the Boltzmann equation with respect to a source function is given. The representation of invariant solutions and corresponding reduced equations for all obtained source functions are also presented.

1. Introduction

The classical Boltzmann equation which plays a central role in kinetic gas theory with mathematical point of view is a uniform integro-differential equation. At the same time, there is frequently the need to include additional source terms into it. In particular, the Boltzmann equation with autonomous sources, independent of the distribution function, are of interest for the kinetic description of initiation of high threshold processes by “hot” particles, for reacting gas flows over catalytic surfaces, and some others. A first attempt of application of group analysis for studying the Boltzmann equation with such a source term was done in [1], where by using the approach of [2], the spatially homogeneous and isotropic in velocity space Boltzmann equation with the Maxwell molecular model was reduced to the equation for a generating function of power moments. The complete Lie symmetry group of the equation for the moment generating function was recently found in [3]. However, the transformation of the invariant solutions thus obtained into the corresponding solutions of the original Boltzmann equation is a very complicated task. An example of typical difficulties can be found in [2], where this transition obstacle was overcome for obtaining the BKW-solution.

A more promising approach is to consider the Fourier image of the homogeneous and isotropic Boltzmann equation [4, 5]. The extensive group analysis of this equation without a source term was carried out in [6]. In the present paper these results are generalized for the non-uniform case.

The paper is organized as follows. After introducing the equation studied, the determining equation of an admitted Lie group is derived. The solution of the determining equation is reduced to the study of a partial differential equation for an admitted source function with some undefined constants.

For further study of the reduced equation an equivalence Lie group of considered integro-differential equation is obtained. It is shown that actions of the obtained equivalence transformations are similar to the automorphisms of the Lie algebra obtained in [6]. This allows us to use a method of constructing an optimal system of subalgebras for the group classification of the source function. These studies are followed by consideration of all possible inequivalent representations of invariant solutions and corresponding reduced equations of the Fourier image equation.

2. The equation studied

The Fourier image of the spatially homogeneous and isotropic Boltzmann equation with a source term has the form [4]:

$$\varphi_t(x,t) + \varphi(x,t)\varphi(0,t) = \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\,\mathrm{d}s + \hat{q}(x,t). \tag{1}$$

Here the function $\varphi(x,t)$ is related with the Fourier transform $\tilde{\varphi}(k,t)$ of the distribution function $f(v,t)$, isotropic in the 3D-space of molecular velocities by the formula

$$\varphi(x,t) \equiv \varphi(k^2/2,t) = \tilde{\varphi}(k,t) = \frac{4\pi}{k}\int_0^\infty v\sin(kv)f(v,t)\,\mathrm{d}v.$$

Similarly, the transform of the isotropic source function $q(v,t)$ is

$$\hat{q}(x,t) \equiv \hat{q}(k^2/2,t) = \tilde{\hat{q}}(k,t) = \frac{4\pi}{k}\int_0^\infty v\sin(kv)q(v,t)\,\mathrm{d}v.$$

The inverse Fourier transform of $\tilde{\varphi}(k,t)$ gives the distribution function

$$f(v,t) = \frac{4\pi}{v}\int_0^\infty k\sin(kv)\tilde{\varphi}(k,t)\,\mathrm{d}k.$$

In the process of solving the determining equation, we use the property that for any smooth function $\varphi_0(x)$ there exists a solution of the Cauchy problem of equation (1) with the initial data

$$\varphi(x,t_0) = \varphi_0(x).$$

3. Determining equation

The classical group analysis method cannot be applied to the integro-differential equation (1). One needs to use the method for equations with nonlocal, in particular, integral terms developed in [7–9]. A generator of the admitted Lie group is sought in the form[1]

$$X = \xi(x,t,\varphi)\partial_x + \eta(x,t,\varphi)\partial_t + \zeta(x,t,\varphi)\partial_\varphi.$$

According to the algorithm, the determining equation for equation (1) is

$$D_t\psi(x,t) + \psi(0,t)\varphi(x,t) + \psi(x,t)\varphi(0,t) - 2\int_0^1 \varphi(x(1-s),t)\psi(xs,t)\,\mathrm{d}s = 0, \tag{2}$$

[1] We follow here notations accepted in [10].

where D_t is the total derivative with respect to t, and the function $\psi(x,t)$ is

$$\psi(x,t) = \zeta(x,t,\varphi(x,t)) - \xi(x,t,\varphi(x,t))\varphi_x(x,t) - \eta(x,t,\varphi(x,t))\varphi_t(x,t).$$

The determining equation (2) has to be satisfied for any solution of equation (1). This allows us to exclude the derivatives φ_t, φ_{xt} and φ_{tt} from the determining equation. In fact, differentiating (1) and substituting φ_t found from (1), we obtain

$$\begin{aligned}
\varphi_{xt}(x,t) =& -\varphi_x(x,t)\varphi(0,t) + 2\int_0^1 s\varphi_x(xs,t)\varphi(x(1-s),t)\,\mathrm{d}s + \hat{q}_x(x,t),\\
\varphi_{tt}(x,t) =& \varphi(x,t)\varphi^2(0,t) - 3\varphi(0,t)\int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\,\mathrm{d}s\\
&+ 2\int_0^1\int_0^1 \varphi(x(1-s),t)\varphi(xss',t)\varphi(xs(1-s'),t)\,\mathrm{d}s'\mathrm{d}s\\
&- \hat{q}(0,t)\varphi(x,t) - \hat{q}(x,t)\varphi(0,t) + 2\int_0^1 \varphi(x(1-s),t)\hat{q}(xs,t)\,\mathrm{d}s + \hat{q}_t(x,t).
\end{aligned}$$

Analysing the determining equation (2) with a method similar to that developed in [9] for the case $\hat{q}=0$, we get that admitted generators have the form

$$X = c_0X_0 + c_1X_1 + c_2X_2 + c_3X_3,$$

where

$$X_0 = x\partial_x,\quad X_1 = x\varphi\partial_\varphi,\quad X_2 = \varphi\partial_\varphi - t\partial_t,\quad X_3 = \partial_t, \tag{3}$$

the constants c_0, c_1, c_2, c_3 and the function $\hat{q}$ satisfy the classifying equation

$$(c_2t - c_3)\hat{q}_t - c_0x\hat{q}_x + (c_1x + 2c_2)\hat{q} = 0. \tag{4}$$

In order to find proper values of the constants c_0, c_1, c_2, c_3 and the function $\hat{q}$ we apply an algebraic method [11].

Note that the case with the function $\hat{q}=0$ was completely studied in [6], where it was shown that the admitted Lie algebra is four-dimensional and which is spanned by the generators X_0, X_1, X_2 and X_3.

3.1. Equivalence transformations

To carry out a group classification one needs to know equivalence transformations of equation (1). For convenience we introduce the operator L defined by

$$L\varphi = \varphi_t(x,t) + \varphi(x,t)\varphi(0,t) - \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\,\mathrm{d}s. \tag{5}$$

Considering the transformations of $L\varphi$ corresponding to the generators X_0, X_1, X_2 and X_3, some equivalence transformations can be obtained.

The transformations corresponding to the generator $X_0 = x\partial_x$ map a function $\varphi(x,t)$ into the function $\bar{\varphi}(\bar{x},\bar{t}) = \varphi(\bar{x}e^{-a},\bar{t})$, where a is the group parameter. The transformed expression of (5) becomes

$$\begin{aligned}
\bar{L}\bar{\varphi} &= \bar{\varphi}_{\bar{t}}(\bar{x},\bar{t}) + \bar{\varphi}(\bar{x},\bar{t})\bar{\varphi}(0,\bar{t}) - \int_0^1 \bar{\varphi}(\bar{x}s,\bar{t})\bar{\varphi}(\bar{x}(1-s),\bar{t})\,\mathrm{d}s\\
&= \varphi_{\bar{t}}(\bar{x}e^{-a},\bar{t}) + \varphi(\bar{x}e^{-a},\bar{t})\varphi(0,\bar{t}) - \int_0^1 \varphi(\bar{x}e^{-a}s,\bar{t})\varphi(\bar{x}e^{-a}(1-s),\bar{t})\,\mathrm{d}s\\
&= \varphi_t(x,t) + \varphi(x,t)\varphi(0,t) - \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\,\mathrm{d}s\\
&= L\varphi.
\end{aligned}$$

This means that

$$T_a^0\colon\quad \bar{x} = xe^a,\quad \bar{t} = t,\quad \bar{\varphi} = \varphi,\quad \bar{\hat{q}} = \hat{q}$$

are the equivalence transformations of equation (1). Similarly, the transformations corresponding to the generator $X_3 = \partial_t$ induce the equivalence Lie group

$$T_a^3\colon\quad \bar{x} = x,\quad \bar{t} = t + a,\quad \bar{\varphi} = \varphi,\quad \bar{\hat{q}} = \hat{q}.$$

The transformations corresponding to the generator $X_2 = \varphi\partial_\varphi - t\partial_t$ map a function $\varphi(x,t)$ to the function $\bar{\varphi}(\bar{x},\bar{t}) = e^a\varphi(\bar{x},\bar{t}e^a)$, which gives

$$\begin{aligned}
\bar{L}\bar{\varphi} &= \bar{\varphi}_{\bar{t}}(\bar{x},\bar{t}) + \bar{\varphi}(\bar{x},\bar{t})\bar{\varphi}(0,\bar{t}) - \int_0^1 \bar{\varphi}(\bar{x}s,\bar{t})\bar{\varphi}(\bar{x}(1-s),\bar{t})\,\mathrm{d}s \\
&= e^a\varphi_{\bar{t}}(\bar{x},\bar{t}e^a) + e^a\varphi(\bar{x},\bar{t}e^a)e^a\varphi(0,\bar{t}e^a) - \int_0^1 e^{2a}\varphi(\bar{x}s,\bar{t}e^a)\varphi(\bar{x}(1-s),\bar{t}e^a)\,\mathrm{d}s \\
&= e^{2a}\varphi_t(\bar{x},\bar{t}e^a) + e^{2a}\varphi(\bar{x},\bar{t}e^a)\varphi(0,\bar{t}e^a) - e^{2a}\int_0^1 \varphi(\bar{x}s,\bar{t}e^a)\varphi(\bar{x}(1-s),\bar{t}e^a)\,\mathrm{d}s \\
&= e^{2a}\left(\varphi_t(x,t) + \varphi(x,t)\varphi(0,t) - \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\,\mathrm{d}s\right) \\
&= e^{2a}L\varphi.
\end{aligned}$$

Hence we conclude that the transformations

$$T_a^2\colon\quad \bar{x} = x,\quad \bar{t} = te^{-a},\quad \bar{\varphi} = \varphi e^a,\quad \bar{\hat{q}} = \hat{q}e^{2a}$$

compose another equivalence Lie group of equation (1).

The transformations corresponding to the generator $X_1 = x\varphi\partial_\varphi$ map a function $\varphi(x,t)$ to the function $\bar{\varphi}(\bar{x},\bar{t}) = e^{\bar{x}a}\varphi(\bar{x},\bar{t})$. Thus,

$$\begin{aligned}
\bar{L}\bar{\varphi} &= \bar{\varphi}_{\bar{t}}(\bar{x},\bar{t}) + \bar{\varphi}(\bar{x},\bar{t})\bar{\varphi}(0,\bar{t}) - \int_0^1 \bar{\varphi}(\bar{x}s,\bar{t})\bar{\varphi}(\bar{x}(1-s),\bar{t})\,\mathrm{d}s \\
&= e^{\bar{x}a}\varphi_{\bar{t}}(\bar{x},\bar{t}) + e^{\bar{x}a}\varphi(\bar{x},\bar{t})\varphi(0,\bar{t}) - \int_0^1 e^{\bar{x}as}\varphi(\bar{x}s,\bar{t})e^{\bar{x}(1-s)a}\varphi(\bar{x}(1-s),\bar{t})\,\mathrm{d}s \\
&= e^{\bar{x}a}\varphi_t(\bar{x},\bar{t}) + e^{\bar{x}a}\varphi(\bar{x},\bar{t})\varphi(0,\bar{t}) - e^{\bar{x}a}\int_0^1 \varphi(\bar{x}s,\bar{t})\varphi(\bar{x}(1-s),\bar{t})\,\mathrm{d}s \\
&= e^{xa}\left(\varphi_t(x,t) + \varphi(x,t)\varphi(0,t) - \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\,\mathrm{d}s\right) \\
&= e^{xa}L\varphi,
\end{aligned}$$

which gives the equivalence Lie group constituted by the transformations

$$T_a^1\colon\quad \bar{x} = x,\quad \bar{t} = t,\quad \bar{\varphi} = \varphi e^{xa},\quad \bar{\hat{q}} = \hat{q}e^{xa}.$$

Summarizing these calculations one can conclude that the generators

$$X_0^{\mathrm{e}} = x\partial_x,\quad X_1^{\mathrm{e}} = x\varphi\partial_\varphi + x\hat{q}\partial_{\hat{q}},\quad X_2^{\mathrm{e}} = \varphi\partial_\varphi - t\partial_t + x\hat{q}\partial_{\hat{q}},\quad X_3^{\mathrm{e}} = \partial_t \tag{6}$$

define an equivalence Lie algebra of equation (1).

Note also that the transformation

$$E:\quad \bar{t} = -t,\quad \bar{\varphi} = -\varphi \tag{7}$$

does not change equation (1), i.e., this is a discrete equivalence transformation being an involution.

Let us study the change of a generator $X = x_0X_0 + x_1X_1 + x_2X_2 + x_3X_3$ under the transformations corresponding to these equivalence transformations. After the change defined by an equivalence transformation one gets the generator

$$X = \hat{x}_0\hat{X}_0 + \hat{x}_1\hat{X}_1 + \hat{x}_2\hat{X}_2 + \hat{x}_3\hat{X}_3, \tag{8}$$

where

$$\hat{X}_0 = \bar{x}\partial_{\bar{x}}, \quad \hat{X}_1 = \bar{x}\bar{\varphi}\partial_{\bar{\varphi}}, \quad \hat{X}_2 = \bar{\varphi}\partial_{\bar{\varphi}} - \bar{t}\partial_{\bar{t}}, \quad \hat{X}_3 = \partial_{\bar{t}}.$$

The corresponding transformations of the basis generators are

$$\begin{aligned}
&T_a^0: \quad X_0 = \hat{X}_0, \quad X_1 = e^{-a}\hat{X}_1, \quad X_2 = \hat{X}_2, \quad X_3 = \hat{X}_3;\\
&T_a^1: \quad X_0 = \hat{X}_0 + a\hat{X}_1, \quad X_1 = \hat{X}_1, \quad X_2 = \hat{X}_2, \quad X_3 = \hat{X}_3;\\
&T_a^2: \quad X_0 = \hat{X}_0, \quad X_1 = \hat{X}_1, \quad X_2 = \hat{X}_2, \quad X_3 = e^{-a}\hat{X}_3;\\
&T_a^3: \quad X_0 = \hat{X}_0, \quad X_1 = \hat{X}_1, \quad X_2 = \hat{X}_2 + a\hat{X}_3, \quad X_3 = \hat{X}_3.
\end{aligned}$$

or coordinates of the generator X are changed as follows

$$\begin{aligned}
T_a^0: &\quad \hat{x}_1 = x_1e^{-a},\\
T_a^1: &\quad \hat{x}_1 = x_1 + ax_0,\\
T_a^2: &\quad \hat{x}_3 = x_3e^{-a},\\
T_a^3: &\quad \hat{x}_3 = x_3 + ax_2,
\end{aligned}$$

where only changeable coordinates are presented.

3.2. Algebraic approach for analyzing equation (4)

Group classification of equation (1) is carried out up to the equivalence transformations considered in the previous section. The method for classifying the source function $\hat{q}$ is similar to the method which was used for classifying equations for moment generating function in [3]. First of all we note that actions of the equivalence transformations T_a^i, $i = 0, \dots, 3$, are equivalent to inner automorphisms of the Lie algebra L_4 spanned by the generators X_0, X_1, X_2 and X_3.

The table of commutators of these generators is

	X_0	X_1	X_2	X_3
X_0	0	X_1	0	0
X_1	$-X_1$	0	0	0
X_2	0	0	0	$-X_3$
X_3	0	0	X_3	0

Using the table of commutators, the inner automorphisms are obtained as follows

$$\begin{aligned}
A_0: &\quad \hat{x}_1 = x_1e^{a},\\
A_1: &\quad \hat{x}_1 = x_1 + ax_0,\\
A_2: &\quad \hat{x}_3 = x_3e^{a},\\
A_3: &\quad \hat{x}_3 = x_3 + ax_2,
\end{aligned}$$

where only changeable coordinates are presented.

Thus we conclude that the actions of equivalence transformations coincides with the actions of inner automorphisms. Because of this property we use an optimal system of subalgebras of the algebra L_4 for classifying equation (1).

The commutator table of the Lie algebra L_4 coincides with the commutator table considered in [3], where group classification of the equation for a moment generating function was studied. The difference in constructing an optimal system here consists of the set of involutions: in the present case the involution corresponding to $\hat{x}_1 = -x_1$ is absent comparing with [3]. The optimal system of subalgebras of the Lie algebra L_4 is presented in Table 1, where γ is an arbitrary constant.

Table 1. Optimal system of subalgebras of L_4

No.	Basis	No.	Basis
1.	$X_0,\ X_1,\ X_2,\ X_3$	13.	$X_0 + X_3,\ X_1$
2.	$\gamma X_0 + X_2,\ X_1,\ X_3$	14.	$X_1,\ X_3$
3.	$X_0,\ X_1,\ X_3$	15.	$X_2,\ X_3$
4.	$X_0,\ X_1,\ X_2$	16.	$X_0,\ X_1$
5.	$X_0,\ X_2,\ X_3$	17.	$\gamma X_0 + X_2$
6.	$X_2,\ X_3$	18.	$X_1 + X_2$
7.	$X_2 + X_0,\ X_1 +\ X_3$	19.	$X_1 - X_2$
8.	$\gamma X_2 + 2X_0,\ X_3$	20.	$X_0 + X_3$
9.	$X_1 + X_2,\ X_3$	21.	$X_1 + X_3$
10.	$X_1 - X_2,\ X_3$	22.	X_0
11.	$X_0,\ X_2$	23.	X_1
12.	$\gamma X_0 + X_2,\ X_1$	24.	X_3

Using the optimal system of subalgebras for group classification of equation (1), the function $\hat{q}(t,x)$ is obtained by substituting, into equation (4), the constants c_i corresponding to the basis generators of a given subalgebra of the optimal system of subalgebras, and solving the obtained system of equations. For example, one can consider one-dimensional subalgebra $\{X_1 - X_2\}$ (see Case 19 of Table 1). For this case there is a single equation for the function $\hat{q}(t,x)$,

$$t\hat{q}_t - x\hat{q}_x + 2\hat{q} = 0.$$

The general solution of this equation is $\hat{q} = t^{x-2}\Phi(x)$, where $\Phi(x)$ is an arbitrary function.

As another example, the two-dimensional Lie subalgebra $\{\gamma X_2 + 2X_0,\ X_3\}$ (see Case 8 of Table 1) is considered. For this case there are two sets of the coefficients c_i, $i = 0, \dots, 3$:

$$\begin{aligned} \gamma X_2 + 2X_0 :&\quad c_0 = 2, \quad c_1 = 0, \quad c_2 = \gamma, \quad c_3 = 0; \\ X_3 :&\quad c_0 = 0, \quad c_1 = 0, \quad c_2 = 0, \quad c_3 = 1. \end{aligned}$$

These sets define a system of two equations for $\hat{q}(t,x)$. Substituting both sets of c_i into equation (4), we obtain

$$\gamma\left(\frac{1}{2}t\hat{q}_t + \hat{q}\right) - x\hat{q}_x = 0, \qquad \hat{q}_t = 0.$$

The general solution of these equations is $\hat{q} = \beta x^\gamma$, where β is a constant.

The complete group classification obtained by calculations of this kind is presented in Table 2.

Table 2. Group classification of equation (1)

No.	$\hat{q}(t,x)$	Generators
1.	0	$X_0,\ X_1,\ X_2\ X_3$
2.	$\beta x^2 e^{tx}$	$X_2 + X_0,\ X_1 + X_3$
3.	βx^{γ}	$\gamma X_2 + 2X_0,\ X_3$
4.	βt^{-2}	$X_0,\ X_2$
5.	$t^{-2}\Phi(xt^{\gamma})$	$\gamma X_0 + X_2$
6.	$t^{-(x+2)}\Phi(x)$	$X_1 + X_2$
7.	$t^{x-2}\Phi(x)$	$X_1 - X_2$
8.	$\Phi(xe^{-t})$	$X_0 + X_3$
9.	$e^{xt}\Phi(x)$	$X_1 + X_3$
10.	$\Phi(t)$	X_0
11.	$\Phi(x)$	X_3

Here β and γ are arbitrary constants, and the function Φ is an arbitrary function of its argument. The other subalgebras of Table 1, which are omitted in Table 2, give inconsistent systems for $\hat{q}(t,x)$.

4. Reduced equations and invariant solutions

In this section, for each obtained function $\hat{q}$ we consider the admitted Lie algebra. Using an optimal system of subalgebras of these Lie algebras, we derive a form of invariant solutions and corresponding reduced equations. Similarly to differential equations, the reduced equations for finding invariant solutions have fewer number of the independent variables than equation (1). There are some trivial cases where invariant solutions are obtained in an explicit form.

4.1. The function $\hat{q} = \beta x^2 e^{xt}$

For the source function $\hat{q} = \beta x^2 e^{tx}$ the admitted Lie algebra of equation (1) is $\{X_2{+}X_0,\ X_1{+}X_3\}$. An optimal system of subalgebras of this Lie algebra consists of the subalgebras

$$\{X_2 + X_0\}, \quad \{X_1 + X_3\}, \quad \{X_2 + X_0,\ X_1 + X_3\}.$$

A representation of invariant solutions corresponding to the subalgebra $\{X_2 + X_0\}$ is $\varphi = t^{-1}r(z)$, where $z = xt$. The substitution of this representation into equation (1) gives

$$zr'(z) - r(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))\,\mathrm{d}s = \beta z^2 e^z. \tag{9}$$

Equation (9) is an equation with the single independent variable z.

A representation of invariant solutions corresponding to the subalgebra $\{X_1 + X_3\}$ is $\varphi = e^{xt}r(x)$. Substituting this representation into equation (1), we obtain the reduced equation

$$r(x)(x + r(0)) - \int_0^1 r(xs)r(x(1-s))\,\mathrm{d}s = \beta x^2.$$

The subalgebra $\{X_2 + X_0,\ X_1 + X_3\}$ gives invariant solutions in the form $\varphi = Cxe^{xt}$, where C is a constant. After substituting this representation into equation (1), we get an algebraic equation for the constant C,

$$C^2 - 6C + 6\beta = 0.$$

If $\beta \le \frac{3}{2}$, then $C = 3 \pm \sqrt{9 - 6\beta}$.

4.2. The source function $\hat{q} = \beta x^\gamma$

An optimal system of subalgebras is constituted by the subalgebras $\{\gamma X_2 + 2X_0,\, X_3\}$, $\{X_3\}$ and either $\{\gamma X_2 + 2X_0\}$ for $\gamma \neq 0$ or $\{X_3 + \alpha X_0\}$ for $\gamma = 0$.

A representation of invariant solutions corresponding to the subalgebra $\{\gamma X_2 + 2X_0,\, X_3\}$ is $\varphi = Cx^{\frac{\gamma}{2}}$, where the constant C satisfies the algebraic reduced equation

$$C^2 B + \beta = 0 \qquad \text{with} \qquad B := \mathrm{B}\left(\frac{\gamma}{2}, \frac{\gamma}{2}\right) = \int_0^1 s^{\frac{\gamma}{2}}(1-s)^{\frac{\gamma}{2}}\,\mathrm{d}s,$$

and B is the beta function [12].

A representation of invariant solutions corresponding to the subalgebra $\{X_3\}$ is $\varphi = r(x)$, and the associated reduced equation is

$$r(x)r(0) - \int_0^1 r(xs)r(x(1-s))\,\mathrm{d}s = \beta x^\gamma.$$

For $\gamma \neq 0$ and the subalgebra $\{\gamma X_2 + 2X_0\}$, invariant solutions are represented as $\varphi = t^{-1}r(z)$, where $z = t^2 x^\gamma$. Substituting this representation into equation (1), we have the reduced equation

$$2zr'(z) - r(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))\,\mathrm{d}s = \beta z$$

For $\gamma = 0$ and the subalgebra $\{X_3 + \alpha X_0\}$, invariant solutions takes in the form $\varphi = r(z)$, where $z = xe^{-\alpha t}$, and the function r is a solution of the reduced equation

$$-\alpha zr'(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))\,\mathrm{d}s = \beta.$$

4.3. The source function $\hat{q} = \beta t^{-2}$

The admitted Lie algebra of equation (1) with the source function $\hat{q} = \beta t^{-2}$ is $\{X_0,\, X_2\}$, whose optimal system of subalgebras consists of the subalgebras $\{X_0,\, X_2\}$, $X_2 + \alpha X_0\}$ and $\{X_0\}$.

The single invariant solution corresponding to the subalgebra $\{X_0,\, X_2\}$ is $\varphi = -\beta t^{-1}$.

For the subalgebra $\{X_2 + \alpha X_0\}$, a representation of invariant solutions is $\varphi = t^{-1}r(z)$, where $z = xt^\alpha$. Substituting this representation into equation (1), we get the reduced equation

$$\alpha zr'(z) - r(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))\,\mathrm{d}s = \beta.$$

A representation of invariant solutions corresponding to the subalgebra $\{X_0\}$ is $\varphi = r(t)$, which gives the the solution $\varphi = -\beta t^{-1} + C$, where C is a constant.

4.4. The source function $\hat{q} = t^{-2}\Phi(xt^\gamma)$

In this case the admitted Lie algebra is $\{\gamma X_0 + X_2\}$. Invariant solutions have the representation $\varphi = t^{-1}r(z)$, where $z = xt^\gamma$. Substituting this representation into equation (1), we obtain the reduced equation

$$\gamma zr'(z) - r(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))\,\mathrm{d}s = \Phi(z).$$

For the other one-dimensional algebras of Table 2 we just present the final results including representations of invariant solutions and reduced equations.

Case 6: $\varphi = t^{-(x+1)}r(x), \quad -(x+1)r(x) + r(x)r(0) - \int_0^1 r(xs)r(x(1-s))\,\mathrm{d}s = \Phi(x).$

Case 7: $\varphi = t^{x-1}r(x), \quad (x-1)r(x) + r(x)r(0) - \int_0^1 r(xs)r(x(1-s))\,\mathrm{d}s = \Phi(x).$

Case 8: $\varphi = r(z)$, where $z = xe^{-t}, \quad -zr'(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))\,\mathrm{d}s = \Phi(z).$

In particular, for BKW-solution $r = 6e^z(1-z)$ which gives that $\Phi = 0$.

Case 9: $\varphi = e^{xt}r(x), \quad xr(x) + r(x)r(0) - \int_0^1 r(xs)r(x(1-s))\,\mathrm{d}s = \Phi(x).$

Case 10: $\varphi = \int \Phi(t)\, dt.$

Case 11: $\varphi = r(x), \quad r(x)r(0) - \int_0^1 r(xs)r(x(1-s))\,\mathrm{d}s = \Phi(x).$

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