

# Consistent quantization of massless fields of any spin and the generalized Maxwell's equations

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**Abstract.** A simplified formalism of first quantized massless fields of any spin is presented. The angular momentum basis for particles of zero mass and finite spin  $s$  of the  $D^{(s-1/2,1/2)}$  representation of the Lorentz group is used to describe the wavefunctions. The advantage of the formalism is that by equating to zero the  $s-1$  components of the wavefunctions, the  $2s-1$  subsidiary conditions (needed to eliminate the non-forward and non-backward helicities) are automatically satisfied. Probability currents and Lagrangians are derived allowing a first quantized formalism. A simple procedure is derived for connecting the wavefunctions with potentials and gauge conditions. The spin 1 case is of particular interest and is described with the  $D^{(1/2,1/2)}$  vector representation of the well known self-dual representation of the Maxwell's equations. This representation allows us to generalize Maxwell's equations by adding the  $E_0$  and  $B_0$  components to the electric and magnetic four-vectors. Restrictions on their existence are discussed.

Key words: wave equations, massless particles, any spin, generalized Maxwell's equations

## 1. Introduction

In a previous paper [2] consistent quantization of massless fields of any spin was developed from first principles. Probability currents and Lagrangians were derived allowing a first quantized formalism. A simple procedure was derived for connecting the wavefunctions with potentials and gauge conditions. Here we will present a simplified version of this formalism and elaborate an extension of the self-dual representation of Maxwell's equations. This representation allows us to generalize Maxwell's equations by adding non-zero time like components  $E_0$  and  $B_0$  to the electric and magnetic four-vectors. The existence of these components allows the formation of scalar electromagnetic waves. We will examine some results.

In our previous paper [2] we started our formalism using Dirac's [4] derivation of equations for massless particles with spin  $s$ , which in the ordinary vector notation are,

$$\frac{1}{s} \left[ s\hat{p}_0 I^{(2s+1)} + S_x \hat{p}_x + S_y \hat{p}_y + S_z \hat{p}_z \right] \psi = \left[ \frac{\hat{E}}{c} I^{(2s+1)} + \frac{\mathbf{S}}{s} \cdot \hat{\mathbf{p}} \right] \psi = 0, \quad (1)$$

$$\left[ s\hat{p}_x I^{(s)} + S_x \hat{p}_0 - iS_y \hat{p}_z + iS_z \hat{p}_y \right] \psi = 0, \quad (2)$$



$$\left[ s\hat{p}_y I^{(s)} + S_y \hat{p}_0 - iS_z \hat{p}_x + iS_x \hat{p}_z \right] \psi = 0, \quad (3)$$

$$\left[ s\hat{p}_z I^{(s)} + S_z \hat{p}_0 - iS_x \hat{p}_y + iS_y \hat{p}_x \right] \psi = 0, \quad (4)$$

where  $\psi$  is a  $(2s+1)$  component wave function and  $S_n$  are the  $(2s+1) \times (2s+1)$  spin matrices which satisfy,

$$[S_x, S_y] = iS_z, \quad [S_z, S_x] = iS_y, \quad [S_y, S_z] = iS_x, \quad S_x^2 + S_y^2 + S_z^2 = s(s+1)I^{(2s+1)}. \quad (5)$$

In the above, the  $\hat{p}_n$  are the momentum operators,  $\hat{p}_0 = \hat{E}/c$ ,  $\hat{E}$  the energy operator, and  $I^{(2s+1)}$  is a  $(2s+1) \times (2s+1)$  unit matrix. Eqs. (1-4) were analyzed extensively by Bacry [5], who derived them using Wigner's condition [6] on the Pauli-Lubanski vector  $W^\mu$  for massless fields,

$$W^\mu = s\hat{p}^\mu, \quad \mu = x_0, x, y, z, \quad (6)$$

$$W^\mu = \frac{-i}{2} \epsilon^{\mu\nu\rho\lambda} S_{\rho\lambda} \hat{p}_\nu, \quad S_{\rho\lambda} = \begin{pmatrix} 0 & S_x & S_y & S_z \\ -S_x & 0 & -iS_z & iS_y \\ -S_y & iS_z & 0 & -iS_x \\ -S_z & -iS_y & iS_x & 0 \end{pmatrix} \equiv (\mathbf{S}, i\mathbf{S}). \quad (7)$$

Dirac suggested using Eq. (1) as the basic helicity equation and substitute from it the  $\hat{p}_0\psi$  into the other 3 equations (2-4). Free massless particles may have only two helicity projections (forward and backward along the momentum vector). Using the Dirac procedure one obtains the  $2s+1$  component Eq. (1) and  $2s-1$  independent subsidiary conditions which reduce the number of helicity projections to 2,

$$\left[ \hat{E} I^{(2s+1)} + \frac{c}{s} \mathbf{S} \cdot \hat{\mathbf{p}} \right] \psi^{(2s+1)} = 0, \quad (8)$$

$$(\mathbf{\Pi} \cdot \hat{\mathbf{p}}) \psi^{(2s+1)} = 0, \quad (9)$$

where the components of  $\mathbf{\Pi}$  are  $(2s-1) \times (2s+1)$  matrices of the subsidiary conditions. These two sets of equations can be combined into  $(2s+1) + (2s-1) = 4s$  equations [1],

$$\left[ \hat{E} I^{(4s)} + c\mathbf{\Gamma}^{(4s)} \cdot \hat{\mathbf{p}} \right] \Phi^{(4s)} = 0, \quad (10)$$

equivalent to the former equations, provided that the rows of the  $\mathbf{\Pi}$  matrices are normalized so that the eigenvalues of  $\mathbf{\Gamma}^{(4s)} \cdot \hat{\mathbf{p}}$  are  $\pm p$  (two helicities) and,

$$\left( \hat{E} I^{(4s)} - c\mathbf{\Gamma}^{(4s)} \cdot \hat{\mathbf{p}} \right) \left( \hat{E} I^{(4s)} + c\mathbf{\Gamma}^{(4s)} \cdot \hat{\mathbf{p}} \right) = \left( \hat{E}^2 - c^2 \hat{\mathbf{p}}^2 \right) I^{(4s)}, \quad (11)$$

i.e. they factorize the d'Alembertian. The wave function  $\Phi^{(4s)}$  is of the form [1] (with  $2s-1$  zeros),

$$\Phi^{(4s)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \psi_{2s+1}^{(2s+1)} \\ \vdots \\ \psi_{-2s-1}^{(2s+1)} \end{pmatrix}, \quad (12)$$

and the components of  $\mathbf{\Gamma}^{(4s)}$  are  $4s \times 4s$  matrices, which form a (reducible) representation of the algebra of the Pauli matrices,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (13)$$

## 2. Wave equations for free massless fields

We shall use the following relations [3],

$$W_\mu W^\mu = -s(s+1) \hat{p}_\mu \hat{p}^\mu I^{(s)}, \quad W_\mu \hat{p}^\mu = 0, \quad (14)$$

with the aim of obtaining the Wigner condition Eq. (6) and simultaneously to factorize the d'Alembertian (in the form  $\hat{p}_\mu \hat{p}^\mu$ ). One can easily see that,

$$(W_\mu - s\hat{p}_\mu)(W^\mu + s\hat{p}^\mu) = -s(2s+1) \hat{p}_\mu \hat{p}^\mu I^{(s)}, \quad (15)$$

or

$$(W_\mu + s\hat{p}_\mu)(W^\mu - s\hat{p}^\mu) = -s(2s+1) \hat{p}_\mu \hat{p}^\mu I^{(s)}. \quad (16)$$

From Eqs. (15-16) one can see that if  $\psi$ , the wavefunction, satisfies,

$$(W^\mu + s\hat{p}^\mu) \psi = 0, \quad \text{or} \quad (W^\mu - s\hat{p}^\mu) \psi = 0, \quad \mu = 0, 1, 2, 3, \quad (17)$$

then the basic massless particle requirement,

$$\hat{p}_\mu \hat{p}^\mu I^{(s)} \psi = 0, \quad (18)$$

will be satisfied. Explicitly we have,

$$W^0 = W_0 = -S_x \hat{p}_x - S_y \hat{p}_y - S_z \hat{p}_z, \quad (19)$$

$$-W^1 = W_x = -S_x \hat{p}_0 - iS_y \hat{p}_z + iS_z \hat{p}_y, \quad (20)$$

$$-W^2 = W_y = -S_y \hat{p}_0 - iS_z \hat{p}_x + iS_x \hat{p}_z, \quad (21)$$

$$-W^3 = W_z = -S_z \hat{p}_0 - iS_x \hat{p}_y + iS_y \hat{p}_x. \quad (22)$$

Now the main problem is to choose the basis of a space on which the above operators will act. In Ref. [2] we have shown that the angular momentum basis of the  $D^{(s-1/2, 1/2)}$  representation of the Lorentz group is the proper one to use. This basis is sum of the bases of spins  $s$  and spin  $s - 1$  with

$$(2s + 1) + (2s - 1) = 4s \text{ components.}$$

In this basis the wavefunction is,

$$\Phi^{(4s)} = \begin{pmatrix} \psi_{s-1}^{(2s-1)} \\ \vdots \\ \psi_{-s+1}^{(2s-1)} \\ \psi_s^{(2s+1)} \\ \vdots \\ \psi_{-s}^{(2s+1)} \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \psi_s^{(2s+1)} \\ \vdots \\ \psi_{-s}^{(2s+1)} \end{pmatrix}. \quad (23)$$

The advantage of the formalism, as we have shown in Ref. [2], is that by equating to zero the spin  $(s - 1)$  components of the wavefunction  $\psi_\mu^{(2s-1)}$ , the  $2s - 1$  subsidiary conditions (needed to eliminate the non-forward and non-backward helicities, according to Wigner's analysis [6] are automatically satisfied. So we are left with equations Eq. (17) with  $\psi = \Phi^{(4s)}$ , but not all equations are needed, the equations with  $\mu = 1, 2, 3$  are the equations Eqs. (2 - 4) from which the subsidiary conditions were derived. Now they are not needed and instead of Eqs. (17) we have,

$$(s\hat{p}^0 + W^0) \Phi^{(4s)} = 0, \quad \text{or} \quad (s\hat{p}^0 - W^0) \Phi^{(4s)} = 0, \quad (24)$$

which are the generalized helicity equations having solutions only in the forward and backward direction. Note that in Eq. (24)  $W^0$  is a  $4s \times 4s$  matrix. The explicit form of Eq. (24) is given in Refs. [1, 2] with the result,

$$\left[ \hat{E}\Gamma_0^{(4s)} - c\mathbf{\Gamma}^{(4s)} \cdot \hat{\mathbf{p}} \right] \Phi^{(4s)} = 0, \quad \text{or} \quad \left[ \hat{E}\Gamma_0^{(4s)} + c\mathbf{\Gamma}^{(4s)} \cdot \hat{\mathbf{p}} \right] \Phi^{(4s)} = 0, \quad (25)$$

where  $\Gamma_0^{(4s)} = I^{(4s)}$  is the  $4s \times 4s$  unit matrix,  $\hat{E} = \hat{p}^0/c$ , and the  $\Gamma$  matrices have the form (in the angular momentum basis),

$$\Gamma_k^{(4s)} = \frac{1}{s} \begin{pmatrix} (-1)^{k+s-1} S_k^{(2s-1)} & \Pi_k \\ \Pi_k^H & (-1)^{k+s} S_k^{(2s+1)} \end{pmatrix}, \quad k = 1, 2, 3, \quad (26)$$

where  $S_k^{(2s-1)}$  are the spin  $(s-1)$  matrices of Eq. (5),  $S_k^{(2s+1)}$  are the spin  $(s)$  matrices of Eq. (5),  $\Pi_k$  is the  $(2s-1) \times (2s+1)$  normalized matrix of the subsidiary conditions and  $\Pi_k^H$  is the Hermitian conjugate of  $\Pi_k$ . The matrix elements of  $\Pi_k$  can be evaluated directly from the normalized subsidiary conditions

$$\psi_\mu^{(2s-1)} = \sqrt{s(2s+1)} \sum_{m_1, m_2} \langle 1m_1; sm_2 | (1s)s-1, \mu \rangle p_{m_1} \psi_{m_2}^{(2s+1)} = 0, \quad (27)$$

where the expansion is in terms of Clebsch-Gordan coefficients and the factor  $\sqrt{s(2s+1)}$  comes from the normalization factor of the r.h.s. of Eq. (15).

For example, for spin 1, one obtains,

$$\Pi_x = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix}, \quad \Pi_y = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}, \quad \Pi_z = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}, \quad (28)$$

In a Cartesian basis, the  $S_k^{(3)}$  are

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (29)$$

$$\Gamma_k^{(4)} = \begin{pmatrix} 0 & \Pi_k \\ \Pi_k^H & S_k^{(3)} \end{pmatrix}, \quad k = 1, 2, 3, \quad \Gamma_0^{(4)} = \text{unit matrix}, \quad (30)$$

For spin 2, one obtains,

$$\Pi_x = \begin{pmatrix} \sqrt{3} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\sqrt{3} \end{pmatrix}, \quad (31)$$

$$\Pi_y = i \begin{pmatrix} -\sqrt{3} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\sqrt{3} \end{pmatrix}, \quad (32)$$

$$\Pi_z = \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}. \quad (33)$$

$$\Gamma_k^{(8)} = \frac{1}{2} \begin{pmatrix} (-1)^{k+1} S_k^{(3)} & \Pi_k \\ \Pi_k^H & (-1)^k S_k^{(5)} \end{pmatrix}, \quad k = 1, 2, 3, \quad (34)$$

More details can be found in Refs. [1, 2].

### 3. First quantization

As is in the case of the Klein-Gordon or Dirac equations, we substitute in Eqs. (25)

$$\hat{E} \Rightarrow i\hbar \frac{\partial}{\partial t}, \quad \hat{\mathbf{p}} \Rightarrow -i\hbar \nabla, \quad (35)$$

with a result

$$i\hbar \left( \Gamma_0^{(4s)} \frac{\partial}{\partial t} + c \mathbf{\Gamma}^{(4s)} \cdot \nabla \right) \Phi^{(4s)} = 0. \quad (36)$$

Similarly to the Schroedinger and Dirac equations we define the Hamiltonian  $\mathcal{H}$  from Eq. (36),

$$i\hbar \frac{\partial}{\partial t} \Phi^{(4s)} = -i\hbar c \mathbf{\Gamma}^{(4s)} \cdot \nabla \Phi^{(4s)} = \mathcal{H} \Phi^{(4s)}, \quad (37)$$

and find a conserved probability current (the superscript  $H$  denotes the Hermitian conjugate),

$$\partial_t \left( \left( \Phi^{(4s)} \right)^H \Phi^{(4s)} \right) + \frac{c}{s} \nabla \cdot \left( \left( \Phi^{(4s)} \right)^H \tilde{\mathbf{S}} \Phi^{(4s)} \right) = 0. \quad (38)$$

Thus the probability density (which should be normalized) is,

$$\rho = \left( \Phi^{(4s)} \right)^H \Phi^{(4s)}, \quad \int \int \int dx dy dz \rho = 1. \quad (39)$$

In the above the density is non-negative and the scalar product is positive definite. Having this result we can find a Lagrangian density,

$$\mathcal{L} = i\hbar \left( \Phi^{(4s)} \right)^H \left( \Gamma_0^{4s} \frac{\partial}{\partial t} + c \mathbf{\Gamma}^{(4s)} \cdot \nabla \right) \Phi^{(4s)}, \quad (40)$$

and using the definition of the energy momentum tensor  $T^{\mu\nu}$ , we find [1],

$$\int \int \int dx dy dz T^{00} = \int \int \int dx dy dz \left( \Phi^{(4s)} \right)^H \mathcal{H} \Phi^{(4s)} = \langle \mathcal{H} \rangle, \quad (41)$$

$$\int \int \int dx dy dz T^{0k} = \int \int \int dx dy dz \left( \Phi^{(4s)} \right)^H (c \hat{p}_k) \Phi^{(4s)} = \langle c \hat{p}_k \rangle, \quad k = 1, 2, 3, \quad (42)$$

i.e. consistent with the expectation values of energy and momentum.

#### 4. Wave equations for free fields

From the Lagrangian Eq. (40) we obtain the two equations Eqs. (25), for particles of spin  $s$  ( $4s$  is the dimension of the matrices),

$$i\hbar \left( \Gamma_0^{(4s)} \frac{\partial}{\partial t} + c \mathbf{\Gamma}^{(4s)} \cdot \nabla \right) \Phi^{(4s)} = 0, \quad (43)$$

$$i\hbar \left( \Gamma_0^{(4s)} \frac{\partial}{\partial t} - c \mathbf{\Gamma}^{(4s)} \cdot \nabla \right) \Phi^{(4s)} = 0. \quad (44)$$

The  $\Gamma$  matrices form a reducible representation of the Pauli matrices Eq. (13). For the spin  $\frac{1}{2}$  Eqs. (43-44) coincide with the massless neutrino equations. The  $\Gamma$  matrices have the same eigenvalues as the Pauli matrices  $\pm 1$  ( $2s$  degenerate). With the built-in subsidiary conditions only two solutions of each of the equations Eqs. (43-44) are possible. Thus Eqs. (43-44) are restricted to forward or backward helicity equations for free massless particles. As the energy operator is proportional to the helicity the two solutions only differ in sign of the energy, i.e. one solution corresponds to positive energy and the second to negative energy. One can also note that the solutions of Eq. (43) are also solutions of Eq. (44) with the opposite sign of the helicity. Therefore Eq. (43) has positive energy solution in the helicity's forward direction; its negative energy solution in the helicity's backward direction is the positive energy solution of Eq. (44) for the helicity's backward direction.

In Summary Eq. (43) is the equation of the helicity in the forward direction and Eq. (44) is the equation for the helicity in the backward direction, if the physical requirement of positive energy is imposed.

*4.0.1. Examples* For spin 1, the photon with helicity  $\pm 1$  and total momentum  $P = \sqrt{p_x^2 + p_y^2 + p_z^2}$ , the normalized solution of Eq. (43) (in Cartesian basis) for positive energy and forward helicity is

$$\Phi^{(4)} = \frac{1}{\sqrt{2P}} \begin{pmatrix} 0 \\ -\frac{p_x p_z - i p_y P}{\sqrt{p_x^2 + p_y^2}} \\ -\frac{i p_x P + p_y p_z}{\sqrt{p_x^2 + p_y^2}} \\ \sqrt{p_x^2 + p_y^2} \end{pmatrix} \exp\left(\frac{1}{i\hbar}(Et - \mathbf{p} \cdot \mathbf{x})\right), \quad (45)$$

and the normalized solution of Eq. (44) (in Cartesian basis) for positive energy and backward helicity is

$$\Phi^{(4)} = \frac{1}{\sqrt{2}P} \begin{pmatrix} 0 \\ -\frac{ip_y P + p_x p_z}{\sqrt{p_x^2 + p_y^2}} \\ -\frac{p_y p_z - ip_x P}{\sqrt{p_x^2 + p_y^2}} \\ \sqrt{p_x^2 + p_y^2} \end{pmatrix} \exp\left(\frac{-1}{i\hbar}(Et - \mathbf{p} \cdot \mathbf{x})\right). \quad (46)$$

For spin 2, the graviton with helicity  $\pm 2$  and total momentum  $P = \sqrt{p_x^2 + p_y^2 + p_z^2}$ , the normalized solution of Eq. (43) (in angular momentum basis) for positive energy and forward helicity is

$$\Phi^{(8)} = \frac{(p_x^2 + p_y^2)^2}{4P^4} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \left( \frac{p_z - P}{p_x - ip_y} \right)^2 \\ \left( \frac{p_z - P}{p_x - ip_y} \right) \\ \sqrt{\frac{3}{2}} \\ \frac{p_x - ip_y}{p_z - P} \\ \frac{1}{2} \left( \frac{p_x - ip_y}{p_z - P} \right)^2 \end{pmatrix} \exp\left(\frac{1}{i\hbar}(Et - \mathbf{p} \cdot \mathbf{x})\right), \quad (47)$$

and the normalized solution of Eq. (44) (in angular momentum basis) for positive energy and backward helicity is

$$\Phi^{(8)} = \frac{(p_x^2 + p_y^2)^2}{4P^4} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \left( \frac{p_z + P}{p_x - ip_y} \right)^2 \\ \left( \frac{p_z + P}{p_x - ip_y} \right) \\ \sqrt{\frac{3}{2}} \\ \frac{p_x - ip_y}{p_z + P} \\ \frac{1}{2} \left( \frac{p_x - ip_y}{p_z + P} \right)^2 \end{pmatrix} \exp\left(\frac{-1}{i\hbar}(Et - \mathbf{p} \cdot \mathbf{x})\right). \quad (48)$$

## 5. Generalized Maxwell's equations

### 5.1. Maxwell's equations

Maxwell's equations [9] in Gaussian system units are,

$$\nabla \cdot \mathbf{E} = \alpha j_0, \quad (49)$$

$$\nabla \times \mathbf{B} - \partial_0 \mathbf{E} = \alpha \mathbf{j}, \quad (50)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (51)$$

$$\nabla \times \mathbf{E} + \partial_0 \mathbf{B} = 0, \quad (52)$$

where  $\alpha = \frac{4\pi}{c}$ ,  $j_0 = c\rho$ ,  $\partial_0 = \frac{1}{c} \frac{\partial}{\partial t}$ .

Minkowski [8] found the covariant form of Maxwell's equations,

$$\partial_\mu F^{\mu\nu} = \alpha j^\nu, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0, \quad \nu = 0, 1, 2, 3, \quad (53)$$

where the antisymmetric tensor  $F^{\mu\nu}$  and its dual  $\tilde{F}^{\mu\nu}$  are defined via the electric and magnetic fields  $\mathbf{E}$  and  $\mathbf{B}$  respectively as,

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \quad (54)$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}, \quad (55)$$

where  $\epsilon^{\mu\nu\alpha\beta}$  is the totally antisymmetric tensor ( $\epsilon^{0123} = 1$ ,  $\epsilon^{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta}$ ). The sum of the  $F^{\mu\nu}$  and  $i\tilde{F}^{\mu\nu}$  is the self-dual antisymmetric tensor, which depends only on the combination

$$\Psi = (\mathbf{E} + i\mathbf{B}), \quad (56)$$

$$F^{\mu\nu} + i\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -E_x - iB_x & -E_y - iB_y & -E_z - iB_z \\ E_x + iB_x & 0 & iE_z - B_z & B_y - iE_y \\ E_y + iB_y & B_z - iE_z & 0 & iE_x - B_x \\ E_z + iB_z & iE_y - B_y & B_x - iE_x & 0 \end{pmatrix}, \quad (57)$$

$$= \begin{pmatrix} 0 & -\Psi_x & -\Psi_y & -\Psi_z \\ \Psi_x & 0 & i\Psi_z & -i\Psi_y \\ \Psi_y & -i\Psi_z & 0 & i\Psi_x \\ \Psi_z & i\Psi_y & -i\Psi_x & 0 \end{pmatrix} \quad (58)$$

The self dual Maxwell's equations take the form [1],

$$\partial_\mu \left( F^{\mu\nu} + i\tilde{F}^{\mu\nu} \right) = - \left( \Gamma_\mu^{(4)} \right)^{\nu i} \partial_\mu \Psi_i = \alpha j^\nu, \quad (59)$$

where the  $\Gamma_\mu^{(4)}$  matrices are the spin 1 matrices defined by Eq. (30). Thus within the  $D(\frac{1}{2}, \frac{1}{2})$  representation of the Lorentz group, Maxwell's equation can be written as,

$$\left( \Gamma_0^{(4)} \partial_0 + \mathbf{\Gamma}^{(4)} \cdot \nabla \right) \Phi^{(4)} = -\alpha J, \quad \Phi^{(4)} = \begin{pmatrix} 0 \\ \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix}, \quad J = \begin{pmatrix} c\rho \\ j_x \\ j_y \\ j_z \end{pmatrix}. \quad (60)$$

Eq. (60) is the same as Eq. (59) which is Lorentz covariant, thus Eq. (60) is also Lorentz invariant. This invariance also applies to the zero component of  $\Phi^{(4)}$ , which was proven explicitly by Lomont [7]. For  $J = 0$ , the free Maxwell equations takes the form,

$$\left( \Gamma_0^{(4)} \partial_0 + \mathbf{\Gamma}^{(4)} \cdot \nabla \right) \Phi^{(4)} = 0, \quad (61)$$

which is exactly Eq. (43). The zero component is the perpendicularity condition for the photon equation and it ensures that the helicity can be only in the forward or backward direction. It also insures that the photon will be a particle of spin 1. This condition also remains for the inhomogenous Maxwell's equations Eq. (60). But in the presence of interactions it is not obvious that the spin 1 property should remain. Therefore we will now generalize the non-homogenous Maxwell equations and replace the zero component with  $\Psi_0 = E_0 + iB_0$ .

### 5.2. The generalized equations

The generalized equations are,

$$\left( \Gamma_0^{(4)} \partial_0 + \mathbf{\Gamma}^{(4)} \cdot \nabla \right) \Phi^{(4)} = -\alpha J, \quad \Phi^{(4)} = \begin{pmatrix} \Psi_0 \\ \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix}, \quad J = \begin{pmatrix} c\rho \\ j_x \\ j_y \\ j_z \end{pmatrix}. \quad (62)$$

from which one can extract the generalized Maxwell's equations in the form

$$\partial_0 E_0 + \nabla \cdot \mathbf{E} = \alpha j_0, \quad (63)$$

$$\partial_0 \mathbf{E} + \nabla E_0 - \nabla \times \mathbf{B} = -\alpha \mathbf{j}, \quad (64)$$

$$\partial_0 B_0 + \nabla \cdot \mathbf{B} = 0, \quad (65)$$

$$\nabla \times \mathbf{E} + \partial_0 \mathbf{B} + \nabla B_0 = 0. \quad (66)$$

Similar equations were derived by Dvoeglazov [10]. We will now look for new features of these equations.

### 5.3. Energy density

According to Eqs. (37) and (41) the energy density is proportional to,

$$\begin{pmatrix} \Psi_0^* & \Psi_x^* & \Psi_y^* & \Psi_z^* \end{pmatrix} (i\hbar c \partial_0) \begin{pmatrix} \Psi_0 \\ \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix} = i\hbar c (\Psi_0^* \partial_0 \Psi_0 + \mathbf{\Psi}^* \cdot (\partial_0 \mathbf{\Psi})) \quad (67)$$

$$= i\hbar c [(E_0 - iB_0) \partial_0 (E_0 + iB_0) + (\mathbf{E} - i\mathbf{B}) \cdot (\partial_0 (\mathbf{E} + i\mathbf{B}))], \quad (68)$$

which is different from the conventional form proportional to

$$\mathbf{\Psi}^* \cdot \mathbf{\Psi} = (\mathbf{E} - i\mathbf{B}) \cdot (\mathbf{E} + i\mathbf{B}) = \mathbf{E}^2 + \mathbf{B}^2$$

### 5.4. The potentials

Let us introduce the potentials,

$$A^{(4)} = \begin{pmatrix} A_0 \\ A_x \\ A_y \\ A_z \end{pmatrix}, \quad (69)$$

where the scalar potential  $V = cA_0$ . Their relation to the wavefunction  $\Phi^{(4)}$  can be obtained by requiring that in the Lorenz gauge they satisfy the wave equation,

$$\partial_\mu \partial^\mu A^{(4)} = \alpha J. \quad (70)$$

The  $\Gamma$  matrices, which are a representation of the Pauli matrices, will factorize the d'Alembertian the same way as the Pauli matrices factorize it, i.e.,

$$\left( \Gamma_0^{(4)} \partial_0 - \mathbf{\Gamma}^{(4)} \cdot \nabla \right) \left( \Gamma_0^{(4)} \partial_0 + \mathbf{\Gamma}^{(4)} \cdot \nabla \right) = \Gamma_0^{(4)} \partial_\mu \partial^\mu \quad (71)$$

In order to get Eq. (70), using Eqs. (62) and (71) the relation between the  $\Phi^{(4)}$  and  $A^{(4)}$  has to be

$$\Phi^{(4)} = \begin{pmatrix} E_0 + iB_0 \\ E_x + iB_x \\ E_y + iB_y \\ E_z + iB_z \end{pmatrix} = - \left( \Gamma_0^{(4)} \partial_0 - \mathbf{\Gamma}^{(4)} \cdot \nabla \right) \begin{pmatrix} A_0 \\ A_x \\ A_y \\ A_z \end{pmatrix} \quad (72)$$

$$= - \begin{pmatrix} \partial_0 A_0 + \partial_1 A_x + \partial_2 A_y + \partial_3 A_z \\ \partial_0 A_x + \partial_1 A_0 + i(\partial_2 A_z - \partial_3 A_y) \\ \partial_0 A_y + \partial_2 A_0 - i(\partial_1 A_z - \partial_3 A_x) \\ \partial_0 A_z + \partial_3 A_0 + i(\partial_1 A_y - \partial_2 A_x) \end{pmatrix}, \quad (73)$$

from which the relation between the fields and the potentials (in the Lorenz gauge) is,

$$E_0 = -\partial_0 A_0 - \nabla \cdot \mathbf{A}, \quad (74)$$

$$B_0 = 0, \quad (75)$$

$$\mathbf{E} = -\nabla A_0 - \partial_0 \mathbf{A}, \quad (76)$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (77)$$

Eqs. (74) and (75) are new relations of the generalized Maxwell's equations, but the Lorenz condition,

$$\partial_0 A_0 + \nabla \cdot \mathbf{A} = 0, \quad (78)$$

enforces the spin 0 component  $E_0$  to be zero, this result should be gauge invariant as the electric and magnetic fields are. Thus only when the Lorenz condition is not satisfied (while Eq. (70) is valid) the  $E_0$  can be different from zero. This result indicates that spin 1 is conserved in Maxwell's equations. In order to have  $E_0 \neq 0$  a spin changing interaction must be added to the 4-current, as the  $E_0$  component belongs to the spin 0 part of the four vector  $E_\mu$  which belongs to the  $D^{(\frac{1}{2}, \frac{1}{2})}$  representation of the Lorentz group with an angular momentum basis of spin 0 and spin 1.

### 5.5. Wave equations

Let us multiply Eq. (62) by  $\left( \Gamma_0^{(4)} \partial_0 - \mathbf{\Gamma}^{(4)} \cdot \nabla \right)$ ,

$$\Gamma_0^{(4)} \partial_\mu \partial^\mu \Phi^{(4)} = \left( \Gamma_0^{(4)} \partial_0 - \mathbf{\Gamma}^{(4)} \cdot \nabla \right) \left( \Gamma_0^{(4)} \partial_0 + \mathbf{\Gamma}^{(4)} \cdot \nabla \right) \Phi^{(4)} = -\alpha \left( \Gamma_0^{(4)} \partial_0 - \mathbf{\Gamma}^{(4)} \cdot \nabla \right) J, \quad (79)$$

from which we obtain,

$$\square E_0 = \alpha (\partial_0 j_0 + \nabla \cdot \mathbf{j}), \quad (80)$$

$$\square B_0 = 0, \quad (81)$$

$$\square \mathbf{E} = \alpha (\partial_0 \mathbf{j} + \nabla j_0), \quad (82)$$

$$\square \mathbf{B} = -\alpha \nabla \times \mathbf{j}, \quad (83)$$

where  $\square = \partial_\mu \partial^\mu$ .

Current conservation requires,

$$\partial_0 j_0 + \nabla \cdot \mathbf{j} = 0, \quad (84)$$

therefore when the current is conserved

$$\square E_0 = 0, \quad (85)$$

and only if the current is not conserved the possibility of  $\square E_0 \neq 0$  will exist. As indicated before, a spin changing interaction can induce the  $E_0$  component.

### 5.6. Scalar electrodynamic waves

There are claims that scalar electrodynamic waves have been detected [11]. As we have shown, within the framework of electrodynamics with conserved currents the scalar waves cannot exist. There exists a misconception that longitudinal waves are the scalar waves. Longitudinal waves can exist as solutions of Maxwell's equations when the perpendicularity condition  $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0$  is violated, for instance when  $\nabla \cdot \mathbf{E} = \alpha j_0$ , i.e. in the presence of a charge distribution. Therefore in Ref. [11] the longitudinal waves were actually vector waves. Other claims of scalar electromagnetic waves come from K. Meyl [12], which are based on a theory different from Maxwell's. A modification of Maxwell's equation to allow for current non-conservation was proposed by Hively and Giacosa [13] with the aim to experimentally detect scalar electromagnetic waves.

### 5.7. Summary and conclusions

A simplified formalism of first quantized massless fields of any spin is presented. Angular momentum basis for particles of zero mass and finite spin  $s$  of the  $D^{(s-1/2, 1/2)}$  representation of the Lorentz group is used to describe the wavefunctions. The advantage of the formalism is that by equating to zero the  $s-1$  components of the wavefunctions, the  $2s-1$  subsidiary conditions (needed to eliminate the non-forward and non-backward helicities) are automatically satisfied.

Probability currents and Lagrangians are derived allowing a first quantized formalism. A simple procedure is derived of connecting the wavefunctions with potentials and gauge conditions.

The spin 1 case is of particular interest and is described with the  $D^{(1/2,1/2)}$  vector representation of the well known self-dual representation of the Maxwell's equations Eq. (60). The wave function is the 4-vector

$$\Phi^{(4)} = \begin{pmatrix} 0 \\ \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix} = \begin{pmatrix} 0 \\ E_x + iB_x \\ E_y + iB_y \\ E_z + iB_z \end{pmatrix}. \quad (86)$$

The zero component remains unchanged under Lorentz transformations [7, 1]. It is well known that the vector  $D^{(1/2,1/2)}$  representation has an angular momentum basis of spin 1 plus spin 0. The meaning of it is that the wave function Eq. (86) contains only the spin 1 contribution. Thus Maxwell's equations contain an intrinsic spin 1. This representation allows to generalize Maxwell's equations by adding the  $E_0$  and  $B_0$  timelike components to create the electric and magnetic four-vectors,

$$\Phi^{(4)} = \begin{pmatrix} E_0 + iB_0 \\ E_x + iB_x \\ E_y + iB_y \\ E_z + iB_z \end{pmatrix}. \quad (87)$$

We found that as long as the electromagnetic current is conserved,  $E_0$  and  $B_0$  have to be zero. Thus scalar electromagnetic waves can be created only if charged conservation is violated, or if a spin changing interaction is added to the conserved current.

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