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A Bernasconi model for constructing ground-state spin systems and optimal binary sequences

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Abstract. A Bernasconi model for a one-dimensional chain of quantum particles is considered. It is shown that searching for the ground state of such a quantum system is equivalent to searching for optimal binary sequences with minimum energy levels. The second criteria for the optimality of binary sequences with low levels of aperiodic autocorrelation is the minimax criterion, in which the peak level of side lobe (PSL) must be minimal. A review and new results regarding the construction of such binary sequences up to length N=82 are presented.

1. Introduction

Classical lattice systems allowing us to describe interatomic potential using finite numbers of values in points defined by the lattice nodes are widely used in statistical physics. The simplest of these is the one-dimensional Ising model [1]. Within such a model, the spin can assume one of two values in each lattice node with number n: $s_n = \pm 1$. Let us consider a one-dimensional spin lattice $\vec{S} = (s_0,\ s_1, \ldots, s_{N-1})$ consisting of N particles with equal distances between neighboring particles.

The energy of such a quantum system consists of pairwise exchange interaction between neighboring atom spins and the interaction of the spins and external magnetic field:

$$E\left(\vec{S}\right) = -\sum_{n,\tau} J_{n\tau} s_n s_\tau - \sum_n h_n s_n \ , \qquad (1)$$

where indices n and τ denote the number of lattice nodes, h_n is the strength of the external magnetic field in the nth node, and $J_{n\tau}$ is the energy of interaction for spin located in nodes n and τ. With external magnetic field $h = 0$, any energy level is doubly degenerate, since the energy of interaction does not change upon the rotation of all spins. In the simplest case, the energy of interaction can be considered identical for all pairs of neighboring atoms, i.e. $J_{n\tau} = J$. As a result, we obtain the following model for the quantum system's energy:

$$E\left(\vec{S}\right) = -J\sum_{n,\tau} s_n s_\tau \ , \tag{2}$$

In [2], Bernasconi pointed out the connection between the problem of searching for low-energy states of a simplified one-dimensional Ising model (2) and that of constructing binary sequences $\vec{S} = (s_0,\ s_1,...,s_{N-1})$ of N length with low levels of autocorrelation.

Aperiodic ACF $\vec{C} = (c_0, c_1,...,c_{N-1})$ of the binary sequence can be written in the form

$$c_\tau = \sum_{n=0}^{N-1-\tau} s_n \cdot s_{n+\tau} \ ,\ \tau = 0,1,...,N-1 . \tag{3}$$

In Bernasconi's model, the Hamiltonian of the quantum system of the simplest one-dimensional Ising model is equal to the energy of the sidelobes of the ACF

$$H\left(\vec{S}\right) = \sum_{\tau=1}^{N-1} c_\tau^2 \ . \tag{4}$$

The problem of constructing the simplest quantum system in the form of a one-dimensional Ising model in thus reduced to one of constructing binary sequences with lowest energy of the sidelobes. There are two criteria for the optimality of binary sequences with low levels of aperiodic autocorrelation. The first is the minimax criterion. The peak sidelobe PSL

$$PSL\left(\vec{C}\right) = \max_{1 \le \tau \le N-1} \left|c_\tau\right| \tag{4}$$

must be minimal

$$MPS = \min_{S} PSL . \tag{4}$$

The second is the MF (merit factor) coefficient characterizing the ratio of the main frame energy to that of the sidelobes of ACF

$$MF\left(\vec{C}\right) = \frac{N^2}{2\sum_{\tau=1}^{N-1} c_\tau^2} . \tag{4}$$

The optimum binary sequences of a given length N has the highest value of the MF coefficient.

Binary sequences that are the optimal according to these criteria are generally different, but for an entire set of lengths they are identical. In this paper we present as known as a new results in the area of constructing optimal MF and MPS sequences. There is known analytical technique to construct sequences with lowest aperiodic autocorrelation, and exhaustive searches have to made in order to find the lowest autocorrelation binary sequence (LABS) for a given length. Many authors have put considerable computational effort to finding binary sequences with lowest or small PSL and the highest or big MF. It should be noted that LABS is problem 005 in the CSPLIB library.

2. Related work

Let us mention in short known results to computer search of such binary sequences. There are two search strategies of finding binary sequences of desired length and optimal aperiodic autocorrelations: global and local methods. Local search methods require relatively short time but have shortcomings.

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Although they find “reasonable” answers, they can-not guarantee optimality. Also local search methods usually based on initial codes, then determine next codes by using an intermediate criterion. So, if optimum sequences are required, the only solution is global search, for an example, exhaustive search.

The main idea [3] to use an exhaustive search for optimal MF sequences is based on a branch and bound algorithm. Symmetry breaking procedures for identifying equivalent sequences allow the search space to be reduced to approximately one-eighth with the runtime complexity $O\left(1{,}85^N\right)$. Let us introduce the main results of global search for optimal MF binary sequences

- for $2 \le N \le 6$ in 1965 by Lunelly [4];
- for $7 \le N \le 19$ in 1966 by Swinnerton-Dyer, as presented by Littlewood [5];
- for $N \le 32$ by Turyn, as presented by Golay [6] in 1982;
- for $N \le 48$ in1996 by Mertens [3];
- for $N \le 60$ in 2004 (the current record) by Mertens and Bauke [7]

A similar approach has been applied to exhaustive search for optimal MPS sequences. Last modification [8] of such method allows to achieve the runtime complexities $O\left(1{,}42^N\right)$ for $PSL = 2$, $O\left(1{,}57^N\right)$ for $PSL = 3$ and $O\left(1{,}79^N\right)$ for $PSL = 4$. The main results of global search for optimal MPS binary sequences are

- for $N \le 40$ in 1975 by Lidner [9];
- for $N \le 48$ in 1990 by Cohen et al. [10];
- for $N \le 61$ in 1997 by Elders-Boll et all [11];
- for $N = 64$ in 2005 by Coxson and Russo [12];
- for $N \le 68$ in 2012 by Leukhin and Potekhin [13];
- for $N \le 74$ in 2013 by Leukhin and Potekhin [8];
- for $N \le 80$ in 2014 (the current record) by Leukhin and Potekhin [14].

This paper adds to available knowledge for record length of binary MPS sequences and provides numbers of non-equivalent sequences for lengths $N = 81$ and $N = 82$. Also sample binary MPS sequences with highest value of merit factor MF are shown for lengths 2 to 82 and optimal or best known MF sequences for comparing.

3. Results of exhaustive search for binary *MPS* sequences

Using our modification of branch and bound algorithm [8] we are able to find all non-equivalent classes of binary MPS sequences for each length N up to $N = 82$. The examples of binary sequences in hexadecimal format for each length $N \in [2;82]$ are presented in table 1:

- the lengths of sequences are presented in the first column shows;
- the values of PSL of optimal MPS sequences are presented in the second column;
- the values of highest merit factor MF of optimal MPS sequences are presented in the third column;
- the optimal MPS sequences with highest merit factor are shown in the forth column;
- the values of PSL of optimal MF sequences (for length $N \in [2;60]$) or best known MF sequences (for length $N \in [2;60]$) are presented in the fifth column;

- the values of highest merit factor *MF* of optimal or best known *MF* sequences are presented in the sixth column;
- the optimal *MF* sequences or best known *MF* sequences are shown in the last column.

Table 1. Results of an exhaustive search of binary optimal MPS sequences and binary optimal MF or best known MF sequences.

Len	PSL	MF	Optimal MPS	PSL	MF	Optimal or best MF
2	1	2	0	1	2	0
3	1	4.5	3	1	4.5	3
4	1	4	2	1	4	2
5	1	6.25	02	1	6.25	02
6	2	2.571	0B	2	2.571	0B
7	1	8.167	0D	1	8.167	0D
8	2	4	16	2	4	16
9	2	3.375	029	2	3.375	029
10	2	3.846	076	2	3.846	076
11	1	12.1	0ED	1	12.1	0ED
12	2	7.2	0A6	2	7.2	0A6
13	1	14.083	00CA	1	14.083	00CA
14	2	5.158	019A	2	5.158	019A
15	2	4.891	0329	3	7.5	7CCA
16	2	4.571	1DDA	3	5.333	D2EE
17	2	4.516	0192B	2	4.516	0192B
18	2	6.48	0168C	2	6.48	0168C
19	2	4.878	07112	3	6.224	7A864
20	2	5.263	04D4E	3	7.692	FA2C6
21	2	6.485	005D39	3	8.481	180CA9
22	3	6.205	013538	3	6.205	013538
23	3	5.628	084BA3	3	5.628	084BA3
24	3	8	31FAB6	3	8	31FAB6
25	2	7.102	031FAB6	3	8.681	1C7F526
26	3	7.511	07015B2	3	7.511	07015B2
27	3	9.851	0F1112D	3	9.851	0F1112D
28	2	7.84	1E2225B	2	7.84	1E2225B
29	3	6.782	031FD5B2	3	6.782	031FD5B2
30	3	7.627	03F6D5CE	3	7.627	03F6D5CE
31	3	7.172	00E326A5	3	7.172	00E326A5
32	3	7.111	01E5AACC	4	8	FEB51CD9
33	3	8.508	003CB5599	3	8.508	003CB5599
34	3	8.892	0CC01E5AA	3	8.892	0CC01E5AA
35	4	7.562	0CC01E5AA	4	8.39	7F64D4AE3
36	3	6.894	3314A083E	4	7.902	E073AEDAD
37	3	6.985	006C94A8E7	4	7.959	1FE1E5AA66
38	3	8.299	003C34AA66	3	8.299	003C34AA66
39	3	6.391	13350BEF3C	5	7.682	7F92D8F1AA
40	3	7.407	2223DC3A5A	3	7.407	2223DC3A5A
41	3	7.504	038EA520364	5	7.782	1C3AA66012D
42	3	8.733	04447B874B4	3	8.733	04447B874B4
43	3	6.748	005B2ACCE1C	5	8.482	765B0153862
44	3	6.286	202E2714B96	4	7.934	C1F30422535

Table 1. Continued

Len	PSL	MF	Optimal MPS	PSL	MF	Optimal or best MF
45	3	6.575	02AF0CC6DBF6	5	8.581	1FE4B6270E55
46	3	6.491	03C0CF7B6556	4	8.076	3FCE1D896CAA
47	3	7.126	069A7E851988	4	8.181	61B0DDD10A50
48	3	6.128	24AC8847B87C	4	8.229	C3CC6D1B5402
49	4	8.827	05E859E984451	4	8.827	05E859E984451
50	4	8.17	07837FB996B2A	4	8.17	07837FB996B2A
51	3	7.517	0E3F88C89524B	5	8.5	63C6A1117E4B6
52	4	8.145	50AE3808C8DB6	4	8.145	50AE3808C8DB6
53	4	7.89	07C0CFBDB4CD56	5	8.262	19FFC3CCD2D559
54	4	7.327	116E1DF7D2C6E6	5	8.331	300661C5D66A52
55	4	7.451	1658A2BC0A133B	7	8.845	7FC9392F1B1CAA
56	4	8.167	0C790164F6752A	4	8.167	0C790164F6752A
57	4	7.963	01B4DE3455B93BF	7	8.641	1C6303AB012B26D
58	4	8.538	008D89574E1349E	4	8.538	008D89574E1349E
59	4	8.328	1CAD63EFF126A2E	5	8.49	61B4DE3455B93BF
60	4	8.108	119D01522ED3C34	6	8.257	CD33A79757B5F80
61	4	7.563	0024BA568EB83731	6	8.232	1DEFCE255D6C1F92
62	4	8.179	000C67247C59568B	4	8.179	000C67247C59568B
63	4	9.587	1B3412F0501539CE	5	9.587	1B3412F0501539CE
64	4	9.846	26C9FD5F5A1D798C	4	9.846	26C9FD5F5A1D798C
65	4	8.252	04015762C784EC369	5	8.802	0319EB85AFABF9364
66	4	7.751	03FEF2CCB0B8CAC54	6	8.475	0F07995533FF996B4
67	4	7.766	073C2FADC44255264	7	9.313	000F631E169362D55
68	4	8.438	562B8CA48E0C9027E	5	9.248	FFD0B564E4D74798E
69	4	7.988	0292582AC6A767CC03	6	8.688	0002F4A9B1B28671
70	4	7.313	01C2FFD4AF33356596	6	8.305	0F61EC20332A2C5A74
71	4	8.105	12493BE76A5EE2A3F1	5	9.165	1EC3D840665458B4E9
72	4	7.2	27C8D6E165A71577FE	5	8.64	33CCD84D05FDE1EBD5
73	4	8.327	012DE781C9167577AB7	6	8.651	01FC677F7A4A2CB972A
74	4	7.039	00ABFA66C560E3094C2	6	8.029	06193EFB2B3DA91F42E
75	4	7.878	0E0038AEB50B59C99B6	6	8.549	00A0AF0E350EE646DB2
76	4	7.113	2CD864E4AA90B8073DE	5	8.647	01415E1C6A1DCC8DB64
77	4	6.959	066B7BDB752AA6F80E3C	11	8.281	00901E934C8CF385AB8A
78	4	7.548	0C4852361E77C0574BAC	5	8.767	01C771D006C9293B51AB
79	4	7.308	0028AE35C3A59AC4ED89	6	9.205	7F36491D815A531AA871
80	4	6.349	01A4F07798EA85AE6C48	7	9.091	FFE81E89A8D1C665A9A5
81	4	7.594	076375955A2F0C69004F9	6	8.819	07A8E3FAA5F7968836676
82	4	6.554	3CB25D380CE3B7765695F	5	8.918	003F70E1112351D4ECB69

Size of set of non-equivalent MPS sequences is equal to 2 for the length $N = 81$ and is equal to 1 for the length $N = 82$.

4. Conclusion

The problem of constructing the ground states of a quantum system in the form of a sequence of pairwise interacting spins with length N was considered. According to the Bernasconi model the problem can be reduced to one of constructing binary sequences with minimum energy levels of the sidelobes of aperiodic ACF. This paper presents some improvements to previously known results of an exhaustive search for minimum peak sidelobe sequences. The examples of optimal MPS binary sequences up to length $N = 82$ with highest merit factor, the examples of optimal MF binary sequences up to length $N = 60$ and the examples of best known MF binary sequences for the lengths from $N \in [61;82]$ are shown.

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