

# The generalized BLM approach to fix scale-dependence in QCD: the current status of investigations

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**Abstract.** I present a brief review of the generalized Brodsky-Lepage-McKenzie (BLM) approaches to fix the scale-dependence of the renormalization group (RG) invariant quantities in QCD. At first, these approaches are based on the expansions of the coefficients of the perturbative series for the RG-invariant quantities in the products of the coefficients  $\beta_i$  of the QCD  $\beta$ -function, which are evaluated in the  $\overline{\text{MS}}$ -like schemes. As a next step all  $\beta_i$ -dependent terms are absorbed into the BLM-type scale(s) of the powers of the QCD couplings. The difference between two existing formulations of the above mentioned generalizations based on the seBLM approach and the Principle of Maximal Conformality (PMC) are clarified in the case of the Bjorken polarized deep-inelastic scattering sum rule. Using the conformal symmetry-based relations for the non-singlet coefficient functions of the Adler D-function and of Bjorken polarized deep-inelastic scattering sum rules  $C_{\text{NS}}^{\text{BjP}}(a_s)$  the  $\beta_i$ -dependent structure of the NNLO approximation for  $C_{\text{NS}}^{\text{BjP}}(a_s)$  is predicted in QCD with  $n_{gl}$ -multiplet of gluino degrees of freedom, which appear in SUSY extension of QCD. The importance of performing the analytical calculation of the N<sup>3</sup>LO additional contributions of  $n_{gl}$  gluino multiplet to  $C_{\text{NS}}^{\text{BjP}}(a_s)$  for checking the presented in the report NNLO prediction and for the studies of the possibility to determine the discussed  $\{\beta\}$ -expansion pattern of this sum rule at the  $O(a_s^4)$ -level is emphasised.

## 1. Introduction

It is known that the results of perturbative calculations of the physical quantities, which obey the RG equations (for the development of the RG method see e.g. [1, 2, 3]), depend on the choice of the scale and scheme of the renormalization procedure. In the case of QCD this problem is of particular importance. Indeed, calculations of the multiloop corrections to the observable physical quantities and to the related RG- functions (namely,  $\beta$ -function and various anomalous dimensions) are usually performed in the class of minimal subtractions (MS) schemes, and in the  $\overline{\text{MS}}$ -scheme [4], in particular. In this case, an error of the comparison of theoretical results with experimental data is usually determined by varying the corresponding renormalization scale  $\mu^2$  within the concrete interval, say  $\mu^2/k \leq \mu^2 \leq k\mu^2$ , where  $k$  is the conventionally chosen number, i.e.  $k = 2 \div 4$  (this convention was recently used recently in [5]). As can be seen from this work and from the studies of heavy flavour contributions to DIS sum rules [6], this interval for  $k$  is indeed conventional. Say, the analysis of [6] motivates the choice of the following interval for  $\mu^2$ :  $m_q^2 \leq \mu^2 \leq (6.5m_q)^2$ , where  $m_q$  are the  $c$  and  $b$ -quark pole masses. Note, that in the process fitting the CCFR collaboration  $xF_3$  structure functions data for  $\nu N$  DIS [7] both ways of



fixation of scale variations were considered for estimating a theoretical error-bar of the extracted expression for  $\alpha_s^{\overline{\text{MS}}}(M_Z)$ . The first one was used in the case when the number of flavours was fixed as  $n_f = 4$ , while the second one was used to estimate the sensitivity of the fitted results to their transformation from  $n_f = 4$  numbers of flavours to the energy region with  $n_f = 5$  numbers of flavours.

Taking into account higher order perturbative corrections in the non-asymptotic QCD regime decreases, as a rule, theoretical errors of the analysed quantities and the extracted QCD parameters, which arise from the variations of scales within the conventionally chosen interval of values. In spite of this, there is quite understandable desire to formulate more concrete theoretical prescriptions for analysing scale-scheme dependence uncertainties using the RG-based language. Among the most applicable at present methods are the Principle of Minimal Sensitivity (PMS) [8], the Effective Charges (ECH) [9] and the Brodsky-Lepage-McKenzie (BLM) approaches [10]. The first two of them are based on the concepts of scheme-invariant quantities. Both ECH and PMS approaches are widely used in the concrete phenomenological studies (see e.g. [11, 12, 13, 14, 15]). In the process of these studies the gauge-invariant and vertex dependent schemes of defining the QCD coupling constant are usually used. These schemes include the original MS-scheme [16], its G-scheme [17] and the  $\overline{\text{MS}}$ -scheme [4] variants, and the number of other similar MS-like schemes, which are unified within the class of  $R_\delta$ -schemes [18]. Since all of them are related to the dimensional regularization [19] and are gauge-invariant, there are no problems with scale-scheme ambiguities of the BLM-approach, discussed [20, 21] in the case of various momentum subtraction (MOM) schemes, which in QCD depend on the gauge choice.

## 2. The generalizations of the BLM approach beyond the NLO: polarized Bjorken sum rule as a typical example

The first NNLO generalization of the BLM approach was formulated in [22]. It was shown that it is possible to absorb unambiguously all  $n_f$ -dependent terms from the NNLO corrections to the RG-invariant measurable quantities, evaluated in the  $\overline{\text{MS}}$ -scheme, by introducing the coupling dependence correction  $O(\alpha_s)$  to the BLM scale  $\mu_{BLM}^2$  fixed from the NLO approximations of the considered physical quantities. Note that this feature of the generalized BLM prescriptions was confirmed in the process of incomplete all-order extension of the BLM approach [23] aimed at resummation of the renormalon-type terms  $(\beta_0\alpha_s)^n$  to the BLM scales of the perturbative series for the  $\tau$ -hadronic width and of the relations between the pole and running heavy quark masses.

Now let us now consider modern approaches of formulating all-order generalizations of the BLM method using the N<sup>3</sup>LO approximation for the Bjorken sum rule of the polarized lepton-nucleon scattering as an example. This sum rule is defined as

$$S_{Bjp} = \int_0^1 g_1^{lp-ln}(x, Q^2) dx = \frac{g_A}{6} C^{Bjp}(a_s). \quad (1)$$

The function  $C^{Bjp}$  contains the non-singlet (NS) and singlet (SI) contributions  $C^{Bjp}(a_s) = C_{\text{NS}}^{\text{Bjp}}(a_s) + C_{\text{SI}}^{\text{Bjp}}(a_s)$ . The existence of the SI term at the  $O(\alpha_s^4)$  level was demonstrated in [24]. Its concrete analytical expression is not yet fixed by direct diagrammatic calculations. Since we are interested in applications of the generalized BLM approaches of [25], which are similar to the seBLM method [26], and of the PMC approach (see recent reviews [27, 28]), we will neglect this SI-type  $a_s^4$ -term and consider the expression for  $C_{\text{NS}}^{\text{Bjp}}(a_s)$ .

Both methods start with the application of the  $\{\beta\}$ -expansion of the perturbative  $\overline{\text{MS}}$ -scheme coefficients for  $C_{\text{NS}}^{\text{Bjp}}(a_s)$ . Within the seBLM-motivated approach of [25] this concrete structure of the  $\{\beta\}$ -expansion was obtained in [29] from the  $\overline{\text{MS}}$ -scheme generalizations of the

original Crewther relation [30] fixed at the NNLO in [31] and at the N<sup>3</sup>LO in [32] by using the  $\{\beta\}$ -expanded  $O(a_s^4)$  representation for the  $e^+e^-$  characteristic, namely, the  $C_D^{NS}(Q^2)$  function [26].

The expression for  $C_{NS}^{Bjp}(a_s)$  at the N<sup>3</sup>LO level

$$C_{NS}^{Bjp}(a_s) = 1 + \sum_i^4 c_i a_s^i \quad (2)$$

was calculated in [32] in the  $\overline{MS}$  scheme,  $a_s = \alpha_s/\pi$ . Using the  $\{\beta\}$ -expansion formalism we express the coefficients  $c_i$  as [29]

$$c_1 = c_1[0], \quad (3)$$

$$c_2 = \beta_0(n_f)c_2[1] + c_2[0], \quad (4)$$

$$c_3 = \beta_0^2(n_f)c_3[2] + \beta_1(n_f)c_3[0, 1] + \beta_0(n_f)c_3[1] + c_3[0], \quad (5)$$

$$c_4 = \beta_0^3(n_f)c_4[3] + \beta_1(n_f)\beta_0(n_f)c_4[1, 1] + \beta_2(n_f)c_4[0, 0, 1] + \beta_0^2c_4[2] + \beta_1(n_f)c_4[0, 1] + \beta_0(n_f)c_4[1] + c_4[0], \quad (6)$$

while certain values of the elements in the RHS are fixed [29, 25] following to generalized Crewther relation. Here  $n_f$  is the number of fermion flavours and  $\beta_i(n_f)$  are the coefficients of the QCD  $\beta$ -function in the MS-like schemes, which are defined as

$$\mu^2 \frac{da_s}{d\mu^2} = \beta(a_s) = - \sum_{i \geq 0} \beta_i(n_f) a_s^{i+2} \quad (7)$$

In the  $\overline{MS}$  -scheme with one scale  $\mu^2 = Q^2$  the analytical expressions for the NNLO  $\{\beta\}$ -approximation for  $C_{NS}^{Bjp}(a_s)$  contain the *underlined* terms in Eq.(5),(6). They are **absent** in a similar  $\{\beta\}$ -representation of the  $\overline{MS}$  perturbative expression for  $C_{NS}^{Bjp}(a_s)$  (see e.g. Eq.(160) review [27]). The reason for this is more technical than theoretical. As was shown in [26], at the NNLO it is possible to get the  $\{\beta\}$ -expansion for the  $e^+e^-$   $C_D^{NS}(a_s)$ -function with the concrete coefficients of the  $\{\beta\}$ -expanded pattern. It was possible to fix these terms only using an additional to  $n_f$  degree of freedom, namely, the number  $n_g$  of multiplets of massless gluino. Its analytical contribution to the NNLO correction of the  $C_D^{NS}(a_s)$ -function is known from the results of [32], while the additional contributions of  $n_{gl}$  to the  $\beta_0(n_f)$  and  $\beta_1(n_f)$ -functions of QCD with gluino are known from [34]. Since  $\beta_0(n_f, n_{gl})$ ,  $\beta_1(n_f, n_{gl})$  are liner in both  $n_f$  and  $n_{gl}$ , this allows us to separate the contributions of  $\beta_1$  and  $\beta_0$  to the NNLO correction of the  $D^{NS}(a_s)$ -function and obtain the  $\{\beta\}$ -expansion pattern of the  $O(a_s^3)$  coefficient with 4 terms similar to the ones entering in Eq.(5).

Note that without this additional information it is impossible to extract the contributions of  $\beta_1(n_f)$  and  $\beta_0(n_f)$  to the NNLO corrections of physical quantities from the ordinary  $n_f$ -expansion of these terms. Indeed, the NNLO correction contains three terms. In the case of polarized  $C_{NS}^{Bjp}(a_s)$  the  $O(a_s^3)$  coefficient has the following form:

$$c_3(n_f) = c_{3,0} + c_{3,1}n_f + c_{3,2}n_f^2. \quad (8)$$

To avoid rather delicate and complicated studies, the authors of [27, 28] prefer to neglect in the NNLO corrections of their initial  $\overline{MS}$  -expressions the term, proportional to the single power of

$\beta_0$ . The price for that is the corruption of the structure of the generalized Crewther relations in the  $\overline{\text{MS}}$ -scheme (see [25] and [35] for a more detailed discussion of this subject). Note that the generalized Crewther relations result from the fundamental properties of the conformal symmetry and its violation by the conformal anomaly in QCD (for the discussions see [36]).

Unfortunately, the absence of any information about the calculated effects of the manifestation of additional degrees of freedom (like the contributions of  $n_{gl}$ -multiplet of gluinos) in the analytical expression of the  $O(a_s^4)$  correction to  $C_D^{NS}$ , evaluated in [32] in the case of  $SU(N_c)$  colour gauge group, does not allow one to extract analytical expressions for the coefficients  $c_4[1, 1]$ ,  $c_4[0, 0, 1]$ ,  $c_4[2]$ ,  $c_4[0, 1]$ ,  $c_4[1]$  and  $c_4[0]$  using the ideas proposed in [26] (apart from its  $C_F^4$  contribution to  $c_4[0]$ , first determined in [37] from the application of the original Crewther relation [30], and the leading term of the  $\beta_0$ -expansion with the coefficient  $c_4[3]$ , which is known from the calculations of [31]). Like the coefficients  $c_3[2]$ ,  $c_3[0, 1]$ ,  $c_3[1]$  and  $c_3[0]$  analytically defined in [29], these still unknown terms are crucial for demonstrating the numerical difference between different generalizations of the BLM approach, namely between the PMC approach, discussed e.g. in [27],[28], and single-scale  $\{\beta\}$ -expanded generalization of the BLM approach, studied in [25]. In view of the lack of knowledge of the  $\{\beta\}$ -expanded expression of the coefficient  $c_4$  in Eq.(6), the single-scale analysis of the  $O(a_s^3)$ -approximation of Eq.(2) was considered only. Let us have a look at it from another point of view, namely, transforming it to the case of multiple scales, as was proposed in [18].

### 3. The scale-dependence of the BLM generalizations at the NNLO and beyond

Consider first the  $\{\beta\}$ -expanded form of the  $O(a_s^4)$  approximation for the Bjorken polarized sum rule defined in the  $\overline{\text{MS}}$ -scheme by Eqs.(2)-(5) and transform it to the multiple-scale case using the solution of the RG-equation presented in [18] and written down in the form  $R_\delta$  relation, defined in [18] as

$$a_s(Q^2) = a_s(Q_\delta^2) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n a_s(Q^2)}{d \ln Q^2} \Big|_{Q^2=Q_\delta^2} (-\delta)^n \quad (9)$$

where  $\ln Q^2/Q_\delta^2 = -\delta$ . This transformation relation leads to the  $O(a_s^4)$  multiscale approximation of  $C_{NS}^{\text{Bjp}}(a_s)$ , which can be obtained from the straightforward RG-transformations of the power-series for  $C_{NS}^{\text{Bjp}}(a_s)$  ( see e.g. [18]), namely,

$$C_{NS}^{\text{Bjp}}(a_s) = 1 + c_1[0]a_s(Q_1^2) + \left[ \beta_0 c_2[1] + c_2[0] + \beta_0 c_1[0]\delta_1 \right] a_s^2(Q_2^2) \quad (10)$$

$$+ \left[ \left( \beta_0^2 c_3[2] + \beta_1 c_3[0, 1] + \underline{\beta_0 c_3[1]} + c_3[0] \right) + \left( \beta_0^2 \delta_1^2 + \beta_1 \delta_1 \right) c_1[0] \right. \\ \left. + 2\beta_0 \left( \beta_0 c_2[1] + c_2[0] \right) \delta_2 \right] a_s^3(Q_3^2) \quad (11)$$

$$+ \left[ \left( \beta_0^3 c_4[3] + \beta_1 \beta_0 c_4[1, 1] + \beta_2 c_4[0, 0, 1] + \beta_0^2 c_4[2] + \underline{\beta_1 c_4[0, 1]} + \underline{\beta_0 c_4[1]} + c_4[0] \right) \right. \\ \left. + \left( \beta_0^3 \delta_1^3 + \frac{5}{2} \beta_1 \beta_0 \delta_1^2 + \beta_2 \delta_1 \right) c_1[0] + \left( 3\beta_0^2 \delta_2^2 + 2\beta_1 \delta_2 \right) \left( \beta_0 c_2[1] + c_2[0] \right) \right. \\ \left. + 3\beta_0 \left( \beta_0^2 c_3[2] + \beta_1 c_3[0, 1] + \underline{\beta_0 c_3[1]} + c_3[0] \right) \delta_3 \right] a_s^4(Q_4^2) + O(a_s^5) \quad (12)$$

Here the scales are defined as  $Q_K^2 = Q^2 \exp(\delta_k)$ . Note that the  $\{\beta\}$ -dependent structures of the  $\delta_k$  expressions of the terms in Eqs.(10), (11) and Eq.(12) **coincide** with the introduced in [26]  $\{\beta\}$ -expanded structure of Eqs.(4)-(6) introduced in [26]. This feature was already observed in [18] and [27]. This fact is not **accidental** at all, but follows from the general principles of the RG method. Moreover, contrary to the claims of refs.[18], it is impossible to neglect in this expansion

the terms proportional to  $\beta_0 a_s^3(Q^2)$  in Eq.(5) (namely, to put **to zero** the  $c_3[1]$  coefficient in Eq.(5) of the  $\{\beta\}$ -expanded expression for  $C_{\text{NS}}^{\text{Bjp}}(a_s)$ ). Indeed, this will automatically lead to disappearance of the  $3\beta_0^2 r_{4,2}$  in Eq.(6) of [18], which corresponds to the  $3\beta_0^2 c_3[1]\delta_3 a_s^4(Q_4^2)$ -term, underlined in part in Eq.(12).

To conclude, within the multiple-scale considerations of [18],[27], [28] the neglected and underlined in Eqs.(11),(12) terms will affect the results for the scales  $Q_3^2$  and  $Q_4^2$ , obtained and discussed in [18],[27], [28]. They should be corrected by taking into account the  $\{\beta_i\}$ -dependent terms, underlined in Eq.(11),(12).

#### 4. The prediction of the $\{\beta\}$ -dependent NNLO expression for the Bjorken polarized sum rule with $n_{gl}$ multiplet of gluinos

One of the ways to confirm the  $\{\beta\}$ -dependent structure of the NNLO expression for  $C_{\text{NS}}^{\text{Bjp}}(a_s)$  from Eqs.(3)-(5), which is in agreement with the analytical result of the direct NNLO QCD calculation of [38], is to evaluate analytically at the  $a_s^3$ -level additional contributions from the  $n_{gl}$  multiplets of SUSY QCD gluinos, as was done [33] in the case of the  $e^+e^-$  characteristic  $C_D^{NS}(Q^2)$ . After this, it will be extremely interesting to use the ideas of [26] and combine this possible new result with the analytical expressions for first two coefficients of the RG  $\beta$ -function  $\beta_0(n_f, n_{gl})$  and  $\beta_1(n_f, n_{gl})$ . We believe, this possible study will coincide with the prediction made in [25] namely, with the following  $O(a_s^3)$  approximation for  $C_{\text{NS}}^{\text{Bjp}}(a_s)$ :

$$\begin{aligned} C_{\text{NS}}^{\text{Bjp}}(a_s) = & 1 - \frac{3}{4}C_F a_s + \left( -\frac{3}{2}C_F\beta_0(n_f, n_{gl}) + \frac{21}{32}C_F^2 - \frac{1}{16}C_F C_A \right) a_s^2 \\ & + \left[ -\beta_0^2(n_f, n_{gl})\frac{115}{24}C_F - \beta_1(n_f, n_{gl})\left(\frac{59}{16} + 3\zeta_3\right)C_F \right. \\ & + \beta_0(n_f, n_{gl})\left(\left(\frac{83}{24} - \zeta_3\right)C_F^2 + \left(\frac{215}{192} - 6\zeta_3 + \frac{5}{2}\zeta_5\right)C_F C_A\right) \\ & \left. + -\frac{3}{128}C_F^3 - \frac{65}{64}C_F^2 C_A - \left(\frac{523}{768} - \frac{27}{8}\zeta(3)\right)C_F C_A^2 \right] a_s^3 + O(a_s^4) \end{aligned} \quad (13)$$

where

$$\beta_0(n_f, n_{gl}) = \frac{11}{12}C_A - \frac{1}{3}\left(T_F N_F + \frac{1}{2}n_{gl}C_A\right) \quad (14)$$

$$\beta_1(n_f, n_{gl}) = \frac{17}{24}C_A^2 - \frac{5}{12}C_A\left(T_F N_F + \frac{1}{2}n_{gl}C_A\right) - \frac{1}{4}\left(T_F N_F C_F + \frac{1}{2}n_{gl}C_A^2\right) \quad (15)$$

are the coefficients of the corresponding  $\beta$ -function, defined by Eq.(7). The prediction of Eq.(7) was obtained in [25] using a similar expression for  $C_D^{NS}(Q^2)$ , which result from the calculations of [33], detailed considerations of [26] and following from the conformal symmetry original Crewther relation [30] between the  $C_D^{NS}(Q^2)$  and  $C_{\text{NS}}^{\text{Bjp}}(Q^2)$  coefficient functions. It was also checked in [25] that the same expression can be obtained from the  $O(a_s^3)$ -generalization of the Crewther relation, discovered in [31]. In this relation the defined by the conformal symmetry term is modified by the to the conformal symmetry breaking term, namely by the factorized  $\beta$ -function of QCD with  $n_{gl}$  multiplets of massless gluino. We are sure that a direct check of the prediction of Eq.(13) may give an additional argument in favour of the used in our works form of  $\{\beta\}$ -expansion at the level of NNLO corrections, considered in this report. Moreover, possible N<sup>3</sup>LO evaluation of  $C^{NS}(a_s)$  in QCD with  $n_{gl}$  gluino multiplet should clarify how to extract still unknown analytical coefficients of the  $\{\beta\}$ -dependence pattern of the  $O(a_s^4)$ -correction to  $C^{NS}(a_s)$ .

## 5. Theoretical advantages of one-scale seBLM/PMC and its phenomenological troubles

Let us transform the general multiple-scale NNLO approximation for  $C_{\text{NS}}^{\text{Bjp}}(a_s)$ , defined in Eq.(10),(11), to the single-scale approximation, studied in [25]. This can be simply done by fixing  $\delta_1=\delta_2=\delta$ . Absorbing now the  $\beta_0$ - dependent term into the scale  $\delta_1$  at the  $O(a_s^2)$  level we obtain the standard BLM expression for  $C_{\text{NS}}^{\text{Bjp}}$ , namely

$$C_{\text{NS}}^{\text{Bjp}}(a_s) = 1 + c_1[0]a_s(Q_{\text{BLM}}^2) + c_2[0]a_s^2(Q_{\text{BLM}}^2) + O(a_s^3) \quad (16)$$

Taking into account that  $c_1[0] = -\frac{3}{4}C_F = -1$  and  $c_2[1] = -\frac{3}{2}C_F = -2$  we get the value of the standard BLM scale  $Q_{\text{BLM}}^2 = Q^2 \exp(-c_2[1]/c_1[0]) = Q^2 \exp[-2] = Q^2 0.135$ . The  $c_2[0]$  term was obtained in [25]. Its expression is  $c_2[0] = -\frac{21}{32}C_F^2 + \frac{1}{16}C_F C_A = -11/12 = -0.91(6)$ .

In the work of Ref.[25] we absorb into the BLM scale of the NNLO  $O(a_s^3(Q_{\text{BLM}}^2))$  single-scale coefficient from Eq.(11) the terms proportional to  $\beta_0^2 c_3[2]$ ,  $\beta_0 c_3[0, 1]$ ,  $\beta_0 c_3[1]$  as well. To avoid lengthy discussions, given in the original work [25] we will consider the case of  $N_F = 3$  number of massless flavours and present only part of analytical and numerical expressions of the corresponding coefficients of the  $\{\beta\}$ -expansion procedure, which define the BLM scale of the NNLO approximation for  $C_{\text{NS}}^{\text{Bjp}}(a_s)$ . They are taken from the results of [25] and read  $c_3[2] = -\frac{151}{24}C_F = -115/18 = -6.38(8)$ ,  $c_3[0, 1] = (-\frac{59}{16} + 3\zeta_3)C_F = (-\frac{59}{12} + 4\zeta_3) = -0.108$  and  $c_3[1] = (\frac{83}{24} - \zeta_3)C_F^2 + (\frac{215}{192} - 6\zeta_3 + \frac{5}{2}\zeta_5)C_F C_A = \frac{4591}{432} - \frac{232}{9}\zeta_3 + 10\zeta_5 = -9.989$ . Using these numbers and the expressions for  $c_1[0]$ ,  $c_2[0]$  and  $c_2[1]$ , as given above, we obtain the expression for the NNLO BLM scale for  $C_{\text{NS}}^{\text{Bjp}}(a_s)$ , given in [25] in a bit different normalization

$$Q_{\text{NNLO}}^2 = Q_{\text{BLM}}^2 \exp[-7.32\beta_0 a_s(Q_{\text{BLM}}^2)] = Q^2 \exp[-2 - 7.32\beta_0 a_s(Q_{\text{BLM}}^2)] \quad (17)$$

. The expressions for the  $O(a_s^3(Q_{\text{NNLO}}^2))$  coefficient to  $C_{\text{NS}}^{\text{Bjp}}(a_s)$  read [29]:

$$c_3[0] = -\frac{3}{128}C_F^3 - \frac{65}{64}C_F^2 C_A - \left(\frac{523}{768} - \frac{27}{8}\zeta_3\right)C_F C_A^2 \approx 35.034 \quad (18)$$

where the numerical value is given in the case of  $SU(N_c = 3)$  colour gauge group. Thus, the final result we are interested in reads

$$C_{\text{NS}}^{\text{Bjp}}(a_s) = 1 - a_s(Q_{\text{NNLO}}^2) - 0.91(6)a_s^2(Q_{\text{NNLO}}^2) + 35.034a_s^3(Q_{\text{NNLO}}^2) + O(a_s^4) \quad (19)$$

Considering the one-scale PMC-type expression we conclude that

- its analytical coefficients extracted from the  $\{\beta\}$ -expansion procedure of [26] and the  $\overline{\text{MS}}$ -scheme generalizations of the Crewther relation of [31], [29] do not depend on the number of flavours and on the terms proportional to the QCD  $\{\beta\}$ -function. Combined with similar expressions for the coefficients  $d_i[0]$  of the  $\{\beta\}$ - expansion representation of the  $\overline{\text{MS}}$ -scheme expression for the  $e^+e^-$  characteristic  $C_D^{\text{NS}}(a_s)$  these coefficients satisfy the scheme-independent relations, which follow from the original Crewther relation of [30], which is based on the conformal symmetry.
- Unfortunately, in view of not small  $O(a_s^3)$ -term application of the single-scale realisation of the PMC approach of Eq.(19), which is very similar to the seBLM method, spoils the satisfactory convergence of the  $O(a_s^3)$   $\overline{\text{MS}}$ -scheme approximation of the polarized Bjorken sum rule perturbative QCD expression, while the NNLO generalization of the BLM-scale of Eq.(17) moves the applicability of the resulting perturbative expressions to the region of higher energies.

## 6. Conclusions

At the current stage of our studies, presented in detail in [25], we discover several phenomenological disadvantages of the applicability of the BLM approach, generalized to  $O(a_s^3)$ -level even following the application of the  $\{\beta\}$ -expansion formalism, proposed in [26] and studied in [29]. We raise the question of the inappropriate use of this formalism within the Principle of Maximal Conformality considered in [18, 27, 28] and propose the calculating test, which may give extra argument in favour of the self-consistency of the structure of the  $\{\beta\}$ -expansion approach, as used by us at the  $O(a_s^3)$ -level. We would like also to mention that it is possible to invent the way how to make the generalized BLM approach more suitable for high-energy studies. However, we are still unable to give any straightforward theoretically motivated prescription for its formulation. At present, this work, which was started in [25], is based on some empirical observations and necessitates more careful analysis of  $O(a_s^4)$ -contributions to the RG-invariant quantities, still unknown within the used version of the theoretically consistent  $\{\beta\}$ -expansion approach. Note, that our studies are also of importance in view of the necessity of better understanding of the applications of the NNLO generalizations of the BLM approach to other important observables, like the event-shape distributions [39], which are measured precisely at  $e^+e^-$ -colliders. .

## 7. Acknowledgements

I would like to thank Sergey Mikhailov for our productive collaboration and for useful discussions of the material of this talk. This work, as well as the participation at the ACAT-20014 Workshop was supported in part by RSCF, Grant N 14-22-00161. I am also grateful to the organizers of this Workshop and in particular to Milos Lokajicek for the warm hospitality in Prague and partial financial support.

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