

A study of the electrical properties of complex resistor network based on NW model

Yunfeng Chang, Yunting Li, Liu Yang, Lu Guo and Gaochao Liu

College of Science, China Three Gorges University, Yichang, Hubei Province, China

E-mail: changyf_ctgu@163.com

Abstract. The power and resistance of two-port complex resistor network based on NW small world network model are studied in this paper. Mainly, we study the dependence of the network power and resistance on the degree of port vertices, the connection probability and the shortest distance. Qualitative analysis and a simplified formula for network resistance are given out. Finally, we define a branching parameter and give out its physical meaning in the analysis of complex resistor network.

1. Introduction

Complex networks have recently emerged as an invaluable tool for describing and quantifying complex systems in many branches of science[1, 2, 3, 4]. Suppose the edges are resistors and the vertices are junctions between resistors, and suppose we apply a voltage or current between two vertices s and t such that a current I flows from s to t through the network. What then is the property of complex network composed by resistors? In 1993, Klein proposed the concept of resistance distance of graphs[5]. In 2010, Newman gave a brief introduction on how to solve vertex voltage in active two-port complex resistor network[4]. Along with the in-depth study of the physical meaning and the topological feature of complex network, many complex networks are found to display community structure[6, 7, 8], dividing naturally into communities or modules with dense connections within communities but sparse connections among them. Communities have proven to be of interest both in their own right as functional building blocks within networks and for the insights they offer into the dynamics or modes of the formation of networks. A large volume of research has been devoted to the development of algorithmic tools for discovering communities[9, 10, 11, 12, 13]. In 2004, Wu and Huberman presented a fast spectrum segmentation method based on resistor network voltage spectrum, combining with the Kirchhoff's current law, the segmentation of community structure was solved in a better way[14].

For the explicit computation of two-port resistance, much work has been done on arbitrary finite and infinite lattices in terms of the eigenvalues and eigenvectors of the Laplacian matrix[15, 16, 17, 18]. That is, most studies of resistor network focus on the calculation of equivalent resistance of regular networks[19] while the study of resistor network based on complex network model is relatively less. In this paper, two-port resistor network based on NW small world complex network model[20] are studied. Mainly, we study the dependence of the network power and resistance on the degree of port vertices, the connection probability and the shortest distance. Qualitative analysis and a simplified formula for network resistance are given out.



Finally, we define a branching parameter and give out its physical meaning in the analysis of complex resistor network.

2. Electrical properties of resistor network

Suppose we apply a voltage or current between two vertices s and t such that there is a current I flows from s to t through the network, where s is the input vertex and t is the output vertex ($s \neq t$). By setting the reference voltage of the output vertex t as $U_t = 0$, the relative voltage of the input vertex s as U_s , the electrical parameters of complex resistor network are given below.

2.1. The resistance of complex resistor network

In this paper, network resistance is divided into two types: network resistance of vertex-pair $R(s, t)$ and network resistance of vertex-degree $R(k_s, k_t)$. $R(s, t)$ is the equivalent total network resistance from the input vertex s to the output vertex t , such that:

$$R(s, t) = \frac{U_s}{I}. \quad (1)$$

As for fixed input-output vertex-pair (s, t) , $R(s, t)$ is a fixed value independent of the applied voltage or current, we get $R(s, t) = R(t, s)$. Network resistance of vertex-degree $R(k_s, k_t)$ is defined as the average value of network resistance of vertex-pair with the same input-output degree (k_s, k_t) :

$$R(k_s, k_t) = \frac{1}{N(k_s, k_t)} \sum_{s \in V_s, t \in V_t} R(s, t). \quad (2)$$

Where $N(k_s, k_t)$ is the number of vertex-pair whose degrees are (k_s, k_t) , V_s is the collection of vertices whose degree is k_s , V_t is the collection of vertices whose degree is k_t . $R(k_s, k_t)$ is also independent of the applied voltage and current $R(k_t, k_s) = R(k_s, k_t)$.

2.2. The power of complex resistor network

Network power is also divided into two types: network power of vertex-pair $P(s, t)$ and network power of vertex-degree $P(k_s, k_t)$. $P(s, t)$ is the total power consumption of the complex resistor network when a current I is inputted at vertex s and flow out from vertex t :

$$P(s, t) = IU_s. \quad (3)$$

On account of that different vertex-pairs in the network may have the same degree as (k_s, k_t) , the average value of the total power of these vertex-pairs is called the network power of vertex-degree, denoted by $P(k_s, k_t)$:

$$P(k_s, k_t) = \frac{1}{N(k_s, k_t)} \sum_{s \in V_s, t \in V_t} P(s, t). \quad (4)$$

Here $P(s, t) = P(t, s)$, $P(k_s, k_t) = P(k_t, k_s)$

3. The relationship between the electrical properties and the topological structure of complex resistor network

In general, small resistor network can be solved by its properties of series connection and parallel connection: the more the branches of the network have, the more parallel resistors it have and the smaller the total resistance will be, thus the network current streaming capability will be strong. For large resistor network based on a complex network, network voltage division

capability and current streaming capability can be used to describe the affection of complex network topological structure on network electrical properties. In this paper, we select network power and resistance to describe network's voltage division capability and current streaming capability. For the same complex resistor network, when the input current is constant, the smaller the network resistance of vertex-pair $R(s, t)$, the smaller the power of vertex-pair $P(s, t)$ and the stronger the current streaming capability, otherwise the current streaming capability will be weak correspondingly; when the input voltage remains constant, the bigger the network resistance of vertex-pair $R(s, t)$, the smaller the network power of vertex-pair $P(s, t)$ and the stronger the voltage division capability, or it will be weak. As for different complex resistor network, we can evaluate the current streaming capability of the network by comparing the network resistance of vertex-degree $R(k_s, k_t)$ and the network power of vertex-degree $P(k_s, k_t)$.

Evidently, the electrical properties of complex resistor network have great relationship with its underlying topological structure. Even for the same network model, when the input and output vertices are different, the corresponding electrical parameters will be different. However, the statistical features such as degree distribution, shortest distance distribution, etc. will remain unchanged. Then, what's the relationship between these electrical parameters and the statistical features of the complex networks? We study this problem on the basis of NW small-world complex network model.

3.1. The Relationship between network power and input-output vertex degree

The resistor network here is a small world complex network with $N = 1000$ vertices based on NW small world complex network model. The unit of resistance and current was omitted for convenience. By setting all the edge resistance $R_0 = 1$, the input current $I = 10$, the connection probability $p = 0.002$ (average degree $\langle k \rangle = 5.84$) and $p = 0.04$ (average degree $\langle k \rangle = 80.53$), and the shortest distance from the input vertex to the output vertex $d = 1$, Figure 1-6 give out the relationship between network power $P(s, t)$, $P(k_s, k_t)$ and the input-output degree (k_s, k_t) . Figure 1 and Figure 2 are three dimensional evolution of the network power of vertex-pair $P(s, t)$ along with the input and output vertex degree k_s and k_t with different connection probabilities. Figure 3 and Figure 4 are the change of network power of vertex-pair $P(s, t)$ with the change of output vertex degree k_t while the input vertex degree k_s remains unchanged. Figure 5 and Figure 6 are the change of network power of vertex-degree $P(k_s, k_t)$ with the change of output vertex degree k_t while the input vertex degree k_s remains unchanged.

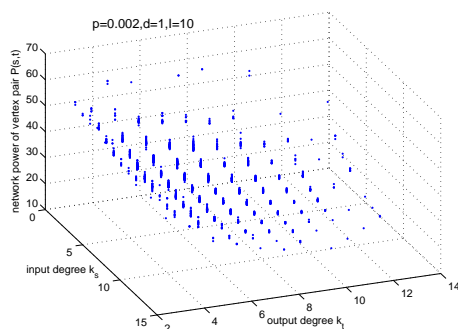


Figure 1. Three dimensional evolution of network power of vertex-pair $P(s, t)$ along with the input and output vertex degree k_s and k_t when the connection probability $p = 0.002$ ($\langle k \rangle = 5.84$).

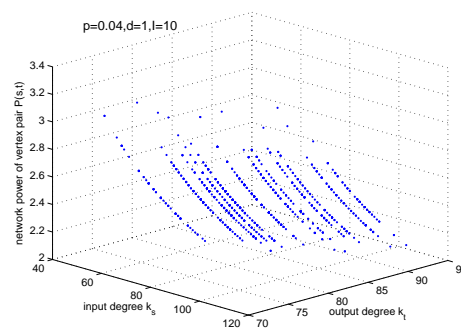


Figure 2. Three dimensional evolution of network power of vertex-pair $P(s, t)$ along with the input and output vertex degree k_s and k_t when the connection probability $p = 0.04$ ($\langle k \rangle = 80.53$).

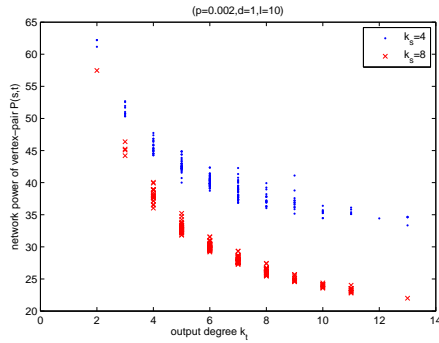


Figure 3. The change of network power of vertex-pair $P(s,t)$ with the change of output vertex degree while the input vertex degree remains unchanged when $p = 0.002(\langle k \rangle = 5.84)$.

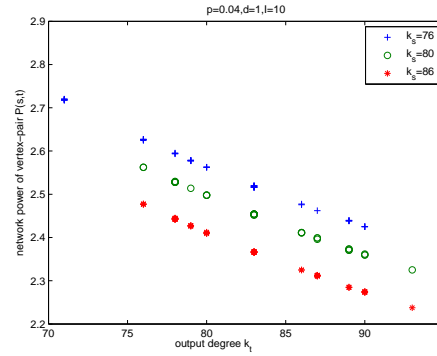


Figure 4. The change of network power of vertex-pair $P(s,t)$ with the change of output vertex degree while the input vertex degree remains unchanged when $p = 0.04(\langle k \rangle = 80.53)$.

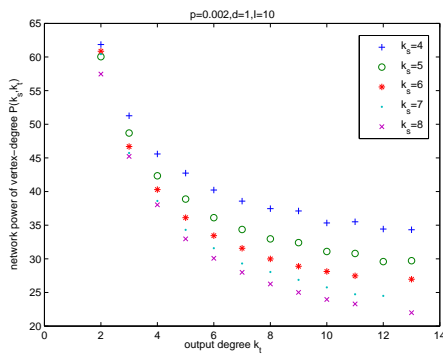


Figure 5. The change of network power of vertex-degree $P(k_s, k_t)$ with the change of output vertex degree while the input vertex degree remains unchanged when $p = 0.002(\langle k \rangle = 5.84)$.

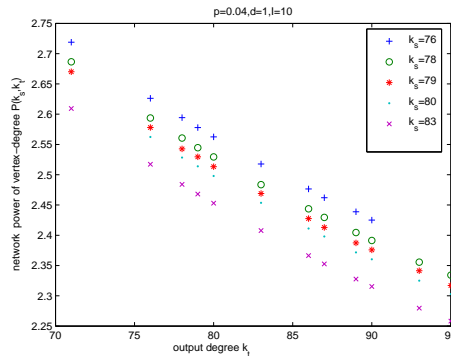


Figure 6. The change of network power of vertex-degree $P(k_s, k_t)$ with the change of output vertex degree while the input vertex degree remains unchanged when $p = 0.04(\langle k \rangle = 80.53)$.

We can find from Figure 1-4 that when the shortest distance between input and output vertices is constant, network power of vertex-pair $P(s,t)$ shows a trend of decreasing along with the increase of the input/output vertex degree k_s/k_t when the output/input vertex degree k_s/k_t remains unchanged. These results can also be observed from Figure 5 and Figure 6, where the network power of vertex-degree $P(k_s, k_t)$ shows a trend of decreasing along with the increasing of the input/output degree k_s/k_t . The reason for the above results is that the number of parallel branches between the input and output vertices is determined by the input and output vertex degree. The greater the k_s and k_t , the more the parallel branches, the smaller $R(s,t)$, the smaller $P(s,t)$ and $P(k_s, k_t)$, such that the current streaming capability of the network will be stronger, whereas weaker.

3.2. The relationship between network power and edge connection probability

In Figure 1 and Figure 3, when the input and output vertex degree k_s, k_t keeps the same, the network power of vertex-pair $P(s,t)$ corresponded to them does not completely identical, but changes in an interval. However, the diversity in Figure 2 and Figure 4 is not obvious,

where the network power of vertex-pair $P(s, t)$ is approximately equal. What's more, Figure 5 and Figure 6 show that the change of the network power of vertex-degree $P(k_s, k_t)$ when the connection probability $p = 0.002$ is a curve while the change of the network power of vertex-degree $P(k_s, k_t)$ when the connection probability $p = 0.04$ is almost a straight line. And the network power when the connection probability $p = 0.002$ is obviously larger than the network power when the connection probability $p = 0.04$. These results can be illustrated by network resistance. Assuming that:

- The number of branches between vertex s and vertex t is n . In general, the larger the degree k_s and k_t , the bigger the n .
- The resistance of each branch from vertex s to vertex t is simplified into one same-value R_s (this is true when the size of the complex resistor network is large), then the total resistance R can be formulated as follows:

$$\frac{1}{R} = \underbrace{\frac{1}{R_s} + \frac{1}{R_s} + \cdots + \frac{1}{R_s}}_n \quad (5)$$

Thus,

$$R = \frac{R_s}{n} \quad (6)$$

Here R can be $R(s, t)$ or $R(k_s, k_t)$. The smaller n , the more R changes. Such as when $n = 1, 2$, $R = R_s, 0.5R_s$, the $R - n$ figures are curves of inversely proportional function. But the larger n and the less R changes, such as when $n = 700, 800$, $R = 0.0014R_s, 0.00125R_s$, $R - n$ curves are approximately straight lines. When $p = 0$, the network resistance is the largest and the current streaming capability is the weakest. And when the connection probability p is small, the branch number n from s to t with the same degree (k_s, k_t) fluctuates strongly, so does $P(s, t)$, as shown in Figure 1 and Figure 3: the same input-output degree (k_s, k_t) has different network power $P(s, t)$. As the edge connection probability p keeps on increasing, the average degree of the network $\langle k \rangle$ gradually increases, then n also increases, so $R(k_s, k_t)$ and $P(k_s, k_t)$ reduce. With the increase of n , the fluctuation of n tends to be gentle, thus the change of the network resistance is less and less obvious, and the difference of $P(s, t)$ is smaller and smaller, as shown in Figure 2 and Figure 4: different $P(s, t)$ corresponded to the same (k_s, k_t) gradually concentrate into points. When $p = 1$, all the vertex degrees reach the maximum same value so does n , and $P(s, t)$ and $P(k_s, k_t)$ reach the minimum, $P(k_s, k_t) = P(s, t)$, the current streaming capability is the strongest.

3.3. The relationship between network power and the shortest distance

In this part, we choose proper connection probability p , so that the generated resistor network model has wide range of shortest distance. And the vertices whose degree are close to the average degree $\langle k \rangle$ are selected as input-output vertices in order to get powerful results. Detailedly, we choose the NW complex small world resistor network with connection probabilities $p = 0.001$, $p = 0.002$ and $p = 0.005$, the corresponding average degrees are $\langle k \rangle = 4.02$, $\langle k \rangle = 5.84$ and $\langle k \rangle = 12.07$, the average shortest distances are $\langle d \rangle = 5.3023$, $\langle d \rangle = 4.1497$ and $\langle d \rangle = 3.0327$, and the shortest distances range among vertices are $d = [1, 10]$, $d = [1, 7]$ and $d = [1, 5]$. Figure 7 to Figure 10 give out the simulation results of network power vs. shortest distance with $k_s = k_t = 4$ when $p = 0.001$, $k_s = k_t = 6$ when $p = 0.002$ and $k_s = k_t = 12$ when $p = 0.005$.

From the three groups of scatters in Figure 7, Figure 8 and Figure 9, it can be seen that the maximum and minimum value of the network power of vertex-pair $P(s, t)$ have no obvious trends

vs. the shortest distance when $d > 1$. Figure 10 shows that when the connection probability p is small, the network power of vertex-degree $P(k_s, k_t)$ increases with the increase of the shortest distance d . However, with the increase of the connection probability p , the difference of the network power of vertex-degree $P(k_s, k_t)$ corresponded to different d are getting smaller, the curves of $P(k_s, k_t)$ vs. d will be an approximately straight line. What's more, $P(k_s, k_t)$ exhibits bigger change when the difference between the value of d and $\langle d \rangle$ is large than the case when d is closes to $\langle d \rangle$. $P(s, t)$ and $P(k_s, k_t)$ with $d = 1$ are both smaller than other cases of d .

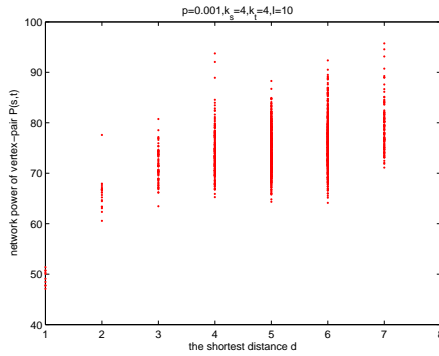


Figure 7. Network power of vertex-pair vs. shortest distance with $k_s = k_t = 4$ when $p = 0.001$.

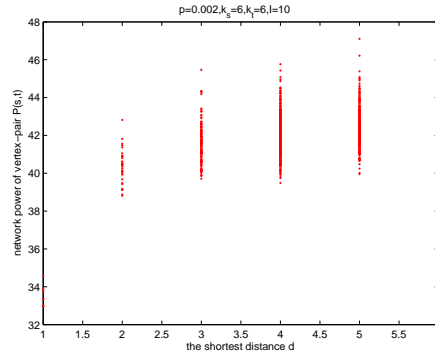


Figure 8. Network power of vertex-pair vs. shortest distance with $k_s = k_t = 6$ when $p = 0.002$.

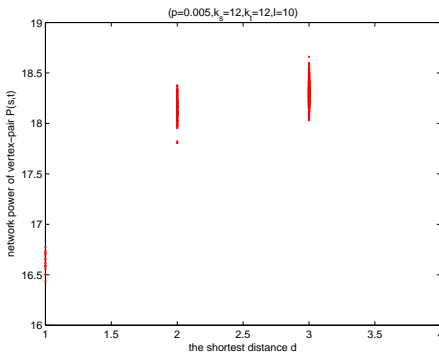


Figure 9. Network power of vertex-pair vs. shortest distance d with $k_s = k_t = 12$ when $p = 0.005$.

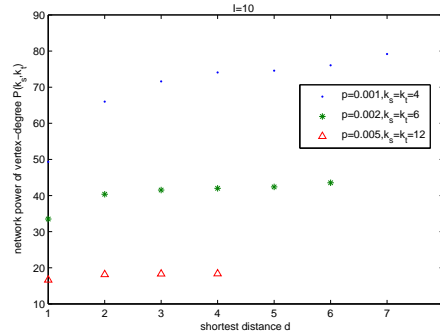


Figure 10. Network power of vertex-degree vs. shortest distance d with different connection probabilities.

For a small resistor network, due to limited number of branches, the total resistance is mainly determined by the branch with the smallest resistance. In a complex network of resistors, however, because of the large number of branches, the total network resistance will not only related to the branch with the smallest resistance but also the number of these branches. Assuming that the shortest distance from vertex s to vertex t is d , the maximum length of loop-free path from s to t is d_{max} , and the simplified resistance between adjacent nodes are all R_0 , the property of complex resistor network can be studied by the following simplification:

- Regarding each loop-free path from vertex s to vertex t as a branch.
- The number of the loop-free paths from s to t with length d_i is m_i . The corresponding branch resistance is $d_i R_0$, $d_i \in (d, d_{max})$.

Assuming that $d_i = d + i - 1$, then $i \in (1, d_{max} - d + 1)$. When k_s and k_t are fixed, $m_i = m_i(s, t, d)$. The network resistance of vertex-pair with the same shortest distance $R(s, t, d)$ can be expressed as:

$$R(s, t, d) = \frac{1}{\sum_{i=1}^{d_{max}-d+1} \frac{m_i(s, t, d)}{d_i R_0}} = \frac{R_0}{\sum_{i=1}^{d_{max}-d+1} \frac{m_i(s, t, d)}{d+i-1}} \quad (7)$$

Here we introduce the branching factor:

$$f(s, t, d) = \sum_{i=1}^{d_{max}-d+1} \frac{m_i(s, t, d)}{d+i-1} \quad (8)$$

The network resistance of vertex-degree with the same shortest distance $R(k_s, k_t, d)$ can be written as:

$$R(k_s, k_t, d) = \frac{1}{N(d)} \sum_{(s,t) \in V(k_s, k_t, d)} R(s, t, d) \quad (9)$$

or

$$R(k_s, k_t, d) = \frac{1}{N(d)} \sum_{(s,t) \in V(k_s, k_t, d)} \frac{R_0}{f(s, t, d)} \quad (10)$$

$N(d)$ is number of vertex-pair with the same shortest distance d and vertex degree (k_s, k_t) , $V(k_s, k_t, d)$ is the collection of these vertices. For different vertex-pairs with the same degree (k_s, k_t) and shortest distance d , $m_i(s, t, d)$ and d_{max} are usually different, this results in the fluctuation of $P(s, t)$ in Figure 7, Figure 8 and Figure 9. For most of the vertex-pairs in complex resistor network, when d_i is close to the average shortest distance $\langle d \rangle$ of the network, $m_i(s, t, d)$ is large, and when the difference between d_i and $\langle d \rangle$ is large, $m_i(s, t, d)$ is small: $m_i(s, t, d)$ firstly increases and then reduces with the increase of d_i . Thus, for different vertex-pairs with the same vertex degree, when $d = 1$, the value range of $(d_{max} - d + 1)$ for d_i is the maximum, the number of parallel branches is the maximum, f is maximum (Equation 8), $R(k_s, k_t, d)$ of vertex-degree is minimum (Equation 10), the network power of vertex-degree $P(k_s, k_t)$ is minimum, the current streaming capability is the strongest. With the increase of d , the value range of $(d_{max} - d + 1)$ gradually reduces, the number of parallel branches gradually reduces, f reduces, so that the network resistance of vertex-degree $R(k_s, k_t)$ and the network power of vertex-degree $P(k_s, k_t)$ both increase. When d is close to the average shortest distance $\langle d \rangle$, the number of parallel branches decreases, $f(d)$ changes gently, so that $R(k_s, k_t)$ and $P(k_s, k_t)$ change gently, as shown in Figure 10. When $d = d_{max}$, the range of $(d_{max} - d + 1)$ is the smallest, the number of parallel branches reaches the minimum, f is minimum, $R(k_s, k_t)$ and $P(k_s, k_t)$ reach the maximum, the streaming capability is the weakest. Generally speaking, it is the reduction of the number of branches when the shortest distance from the input vertex to the output vertex increases, which leads to the the increase of network resistance and network power.

4. Summary and Outlook

In this paper, we established an undirected complex resistor network based on NW small world network model. The relationship between network current streaming capability and network structure is studied. By simulation and analytical analysis, we got the following results:

a) Complex network resistance and power have a decreasing trend with the increase of the input or output vertex degree.

b) With the increase of edge connection probability, network resistance and power decrease, and the current streaming capability is enhanced. At the same time, the differences between the network resistance and power of vertex-pair and vertex-degree with different connection probabilities are getting smaller.

c) With the increase of the shortest distance, the network resistance and power both increase, and the current streaming capability is weakened.

The underlying reason for the above results is also studied. And in the analysis of network resistance and power with different shortest distance, we gave a simplified formula for network resistance, and introduced a branching factor which could be a new statistical parameters in the analysis of complex resistor network.

More research on the local performance of different complex resistance network will be done in our future work, such as the study of the edge-current, edge-power, vertex-current, vertex-power, the specific form of $m_i = m_i(s, t, d)$ in branch factor $f = f(s, t, d)$. These local study of complex resistor network can provide useful estimation of the importance of different vertex and edge.

Acknowledgement

This work is supported by the National Science Foundation of China under grant Nos. 11147155, 61174216, 61074091 and 11303020, and Education Commission of Hubei Province of China under grant No. Q20121311, Natural Science Foundation of CTGU under grant No. KJ2008B033.

References

- [1] Wasserman S, Faust K 1994 *Social Network Analysis* (Cambridge University Press, Cambridge)
- [2] Albert R, Barabasi A L 2002 *Rev. Mod. Phys.* 74 47
- [3] Newman M E J 2003 *SIAM Review* 45 167
- [4] Newman M E J 2010 *Network: An Introduction* (Oxford University Press, New York)
- [5] Klein D J, Randicc M 1993 Resistance Distance *Journal of Mathematical Chemistry* 12(1): 81-95
- [6] Flake G W, Lawrence S, Giles C L, et al. 2002 Self-organization and identification of web communities *IEEE computer* 35(3):66-70
- [7] Wilkinson D M and Huberman B A 2004 A method for finding communities of related genes *PNAS* 101 5241
- [8] Newman M E J and Girvan M 2004 Finding and evaluating community structure in networks *Phys. Rev. E* 69 026113
- [9] Danon L, Duch J, Diaz-Guilera A and Arenas A 2005 Comparing Community Structure Identification *J. Stat. Mech.* P09008
- [10] Newman M E J 2013 Community detection and graph partitioning *Europhysics Letter* 103 28003
- [11] Newman M E J 2013 Spectral methods for community detection and graph partitioning *Phys. Rev. E* 88 042822
- [12] Schuetz P and Cafilisch A 2008 Efficient modularity optimization by multistep greedy algorithm and vertex mover refinement *Phys. Rev. E* 77 046112
- [13] Chang Y F and Ping Y 2011 Emerging cluster analysis of SCI journals and its efficiency *Int. J. Mod. Phys. C* 22(10):1247-1256
- [14] Wu F, Huberman B A 2004 Finding communities in linear time: A physics approach *Eur. Phys. J. B.* 38 331-338
- [15] Wu F Y 2004 Theory of resistor networks: the two-point resistance *J. Phy. A:Math. Gen* 37 6653
- [16] Izmailian N Sh and Ming-Chang Huang 2010 Asymptotic expansion for the resistance between two maximally separated nodes on an M by N resistor network *Phys. Rev. E* 82 011125
- [17] Izmailian N Sh, Kenna R and Wu F Y 2014 The two-point resistance of a resistor network: a new formulation and application to the cobweb network *J. Phy. A:Math. Gen* 47 035003
- [18] Zhi-zhong Tan, Ling Zhou and Jian-hua Yang 2013 The equivalent resistance of a $3 \times n$ cobweb network and its conjecture of an m by n cobweb network *J. Phy. A:Math. Gen* 46 195202
- [19] Tan Z Z, Essam J W and Wu F Y 2014 Two-point resistance of a resistor network embedded on a globe *Phys. Rev. E* 90 012130
- [20] Newman M E J and Watts D J 1999 Renormalization group analysis of the small-world network model *Phys. Lett. A* 263: 341-346