

A new approach for development of rheological relations for saturated porous media

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Abstract. Equations of hydro-geo-mechanics in filtrating porous media with porous variable-mass skeleton are examined. Variation of skeleton mass occurs due to heterogeneous chemical reactions. Regularities of mass transfer and deformations in such kinds of media are analyzed. Peculiarities of obtaining of rheological relations are investigated. A new approach to obtain rheological relations is proposed. Numerical simulations in order to verify obtained model are performed.

1. Introduction

Hydrogeomechanical models are important for solution of various problems in hydrogeology, hydrogeocology, oil production and geophysics. Main modern concepts of the hydrogeomechanics are presented, for example, in [1, 2]. Oil depletion on some of Russian oil fields necessitates formulating models that take into account the variation of the stress-strain state of the rock mass caused by chemical interactions between components of underground fluid and the material of the porous skeleton to perform effective enhanced oil recovery. Those models are also essential in problems of hydrogeology, such as filtration of solutions in clay layers, suffosion processes and karst processes.

Above-mentioned chemical interactions usually cause the variation of the mass of the porous matrix [3]. That is why it is important to perform an additional research on the influence of this variation on rheological relations, which are required to obtain a closed model of deformations of filtrating porous media.

It is also necessary to perform systematic development of main equations of underground mass-transfer in this case. Whereas those questions did not receive exhaustive explanation in specialized literature, it makes sense to obtain required equations and examine most important applications.

2. Mathematical model

2.1. Mass balance equations of porous skeleton and percolating liquid

First, it is essential to develop set of equations of filtration in a deformable porous medium with porous variable-mass skeleton. From the definition of the volume strain of the porous medium θ :

$$\theta = (V - V_0)/V_0 \quad (1)$$

Assuming values of θ are small, we can obtain the following ratio:



$$V = V_0 \exp \theta \quad (2)$$

V is the representative volume of porous medium, lower index "0" stands for initial values in the zero time. Therefore, for the mass of the porous medium we have:

$$M_s = (1 - m) \rho_s V_0 \exp \theta \quad (3)$$

Where: ρ_s is the density of the solid phase; m is the porosity of the rock. Last equation was differentiated with respect to time.

$$\frac{\partial m}{\partial t} = \frac{(1 - m)}{\rho_s} \frac{\partial \rho_s}{\partial t} + (1 - m) \frac{\partial \theta}{\partial t} - \frac{(1 - m)}{M_s} \frac{\partial M_s}{\partial t} \quad (4)$$

Mass balance of the solid material of the porous skeleton is described by the equation

$$\partial[(1 - m)\rho_s] / \partial t + \text{div}[(1 - m)\rho_s \mathbf{W}] = j. \quad (5)$$

Here \mathbf{W} is the velocity of the solid phase; j denotes the source/drain of the mass of the porous skeleton caused by interface interaction. The porous skeleton is assumed to loose its mass during interface interaction, so hereinafter j will represent the drain of the mass. The mass of the material of the porous skeleton in the representative volume there can be defined as

$$\rho_s V_s = \rho_s (1 - m) V = M_s \quad (6)$$

Here V_s is the volume of the solid phase in the representative volume. Drain in the equation (5) is the loss of the mass of the porous skeleton caused by processes like dissolution, leaching or suffosion. It can be written as

$$j = \frac{1}{V} \frac{\partial M_s}{\partial t} \quad (7)$$

From the equation (5), after the expanding of derivatives we have

$$-\frac{\partial m}{\partial t} \rho_s + (1 - m) \frac{\partial \rho_s}{\partial t} + (1 - m) \rho_s \text{div} \mathbf{W} + \mathbf{W} \text{grad}[(1 - m)\rho_s] = j \quad (8)$$

Taking into account the equations (4) and (7) and assuming the last member of the (8) to be the second order infinitesimal:

$$\partial \theta / \partial t = \text{div} \mathbf{W} \quad (9)$$

The last assumption is explained in the traditional way of the poromechanics [4]. From the Terzaghi's Principle $\mathbf{P} = \sigma^f + p$, where \mathbf{P} is the external load, σ^f is the effective stress, and p is the water pressure in pores, it follows that $\text{grad} \sigma^f = -\text{grad} p$ if $\text{grad} \mathbf{P} = 0$. Since the porosity of the rock m and the density of the solid phase ρ_s are functions of σ^f and p and taking into account the last equation it follows that $\mathbf{W} \text{grad}[(1 - m)\rho_s]$ from the (7) is in proportion to $\mathbf{W} \text{grad} p$. According to the Darcy's law, this member is also in proportion to the product of the velocity \mathbf{W} and the filtration velocity \mathbf{Q} . Values of velocities in examined processes are relatively low, so the second degree of the velocity can be ignored as infinitesimal value [4].

Mass balance of the dissolving fluid in saturated rock is described by the equation:

$$\partial(m\rho) / \partial t + \text{div}(m\rho \mathbf{V}) = 0. \quad (10)$$

Here ρ is the density of the fluid, \mathbf{V} is the velocity of the fluid. Relative velocity of the fluid in rock (filtration velocity) is $\mathbf{q} = m\mathbf{s}(\mathbf{V} - \mathbf{W})$. Then, from the equations (9) and (10), we have:

$$m\partial\rho/\partial t + \rho\partial m/\partial t + \text{div}(\rho\mathbf{q}) + \text{div}(\rho m\mathbf{W}) = 0. \quad (11)$$

After transformation of the last equation, while $\mathbf{q}\text{grad}\rho$, $\mathbf{W}\text{grad}(m\rho)$ are considered infinitesimal values, it leads to the following equation:

$$m\rho^{-1}\partial\rho/\partial t + \partial m/\partial t + \text{div}\mathbf{q} + m\partial\theta/\partial t = 0. \quad (12)$$

Combined with (4) the equation (12) gives the following expression:

$$[m\rho^{-1}\frac{\partial\rho}{\partial t} + (1-m)\rho_s^{-1}\frac{\partial\rho_s}{\partial t}] + \frac{\partial\theta}{\partial t} + \text{div}\mathbf{q} = \frac{(1-m)}{M_s}\frac{\partial M_s}{\partial t}. \quad (13)$$

It is obvious from $\rho_s = M_s/V_s$ that:

$$(1-m)\rho_s^{-1}\frac{\partial\rho_s}{\partial t} = \frac{(1-m)V_s}{M_s}\frac{\partial(M_s/V_s)}{\partial t}. \quad (14)$$

Expanding the derivative in the right side and taking (13) into the account:

$$(1-m)\rho^{-1}\frac{\partial\rho}{\partial t} + \frac{\partial\theta}{\partial t} + \text{div}\mathbf{q} = \frac{(1-m)}{V_s}\frac{\partial V_s}{\partial t}. \quad (15)$$

The right side of (15) can be expressed as:

$$\frac{(1-m)}{V_s}\frac{\partial V_s}{\partial t} = (1-m)\frac{\partial\varepsilon}{\partial t} \quad (16)$$

Here ε is the volume strain of the skeleton material (sum of the diagonal elements of the strain tensor). Then (15) can be written as:

$$(1-m)\rho^{-1}\frac{\partial\rho}{\partial t} + \frac{\partial\theta}{\partial t} + \text{div}\mathbf{q} = (1-m)\frac{\partial\varepsilon}{\partial t}. \quad (17)$$

Water is a slightly compressible fluid, so first member in the left side of (17) can be neglected:

$$\frac{\partial\theta}{\partial t} + \text{div}\mathbf{q} = (1-m)\frac{\partial\varepsilon}{\partial t}. \quad (18)$$

Time integration of the last equation gives following result:

$$\theta + \int_0^t \text{div}\mathbf{q} \cdot d\tau = \int_0^t (1-m)\frac{\partial\varepsilon}{\partial t} \cdot d\tau. \quad (19)$$

Partial integration of the right side leads to the equation:

$$\theta + \int_0^t \text{div}\mathbf{q} \cdot d\tau = (1-m)\varepsilon + \int_0^t \varepsilon \frac{\partial m}{\partial t} \cdot d\tau. \quad (20)$$

Using the equation (4) for the time derivative of the porosity and (14) for the member $(1-m)\rho_s^{-1}\partial\rho_s/\partial t$ we can transform (20) to the following equation:

$$\theta + \int_0^t \operatorname{div} \mathbf{q} \cdot d\tau = (1-m)\varepsilon + \int_0^t \varepsilon \left[(1-m) \frac{\partial \theta}{\partial t} - (1-m) \frac{\partial \varepsilon}{\partial t} \right] \cdot d\tau. \quad (21)$$

It is obvious that $\varepsilon \leq \theta$ so for the last member of the right side of (21) it is true that:

$$\int_0^t \varepsilon \left[(1-m) \frac{\partial \theta}{\partial t} - (1-m) \frac{\partial \varepsilon}{\partial t} \right] \cdot d\tau \leq \int_0^t \left[(1-m)\theta \frac{\partial \theta}{\partial t} - (1-m)\varepsilon \frac{\partial \varepsilon}{\partial t} \right] \cdot d\tau \quad (22)$$

Last integral contains time derivatives of the squares of ε and θ so it can be neglected. Therefore, finally we receive:

$$\theta + \int_0^t \operatorname{div} \mathbf{q} \cdot d\tau = (1-m)\varepsilon. \quad (23)$$

The obtained equation has clear physical sense: total strain of the porous rock is made up of two terms. The first term is the water that was forced out during the filtration. The second term is deformation of the porous skeleton.

2.2. Rheological relations for the filtrating porous media with mass-variable porous skeleton

Usually rheological relations are obtained from the expression for the free energy of the porous medium [1]. Dissipation caused by chemical reactions is not included in this approach, and that is why it can hardly be applied in our case. In order to obtain rheological relations we consider that solid skeleton is elastic, then from the [5, 6] we have:

$$\sigma_{ij}^{(s)} = -\left(K - \frac{2}{3}G\right)\varepsilon\delta_{ij} - 2G\varepsilon_{ij}, \quad \varepsilon = \sum_i \varepsilon_{ii} \quad (24)$$

ε_{ij} is the strain tensor of the skeleton, G is the shear modulus. In one-dimensional case, vertical compressive stress σ_{zz} can be expressed as:

$$\sigma_{zz}^{(s)} = -\left(K + \frac{4}{3}G\right)\varepsilon \quad (25)$$

Denoting $\alpha = \left(K + \frac{4}{3}G\right)$, we can write (23) as:

$$\alpha \left[\theta + \int_0^t \operatorname{div} \mathbf{q} \cdot d\tau \right] = -(1-m)\sigma_{zz}^{(s)}. \quad (26)$$

From the definition of actual stresses [5], the right side of (26) can be written as:

$$-(1-m)\sigma_{zz}^{(s)} = P - mp \quad (27)$$

From the Terzaghi's Principle and (27), it follows that:

$$\sigma^f = \alpha\theta, \quad \alpha \int_0^t \operatorname{div} \mathbf{q} \cdot d\tau = (1-m)p. \quad (28)$$

The first equation is the rheological relation for the volume strains of the filtrating medium. Shear deformations obviously coincide with deformations of the porous medium, so rheological relations can be written as:

$$\sigma_{ij}^f = -(K - \frac{2}{3}G)\theta\delta_{ij} - 2G\varepsilon_{ij} \quad (29)$$

Introducing the porosity coefficient $e = m / (1 - m)$ and differentiating last equation of the system (28) with respect to time, we receive the equation for the pressure in the filtrating porous medium with mass-variable porous skeleton:

$$(1 - m)\partial p / \partial t - p\partial m / \partial t = -\alpha \cdot \text{div} \mathbf{q} \quad (30)$$

If the porosity is constant, this equation turns into the equation of the filtrational consolidation of the porous medium with the porous skeleton of constant mass. This equation is widely used in civil engineering [7].

3. Results and Discussion

3.1. Numerical simulation of filtrational dissolution

When the density of the solid phase is constant during the chemical interaction, the following set of equation can be written:

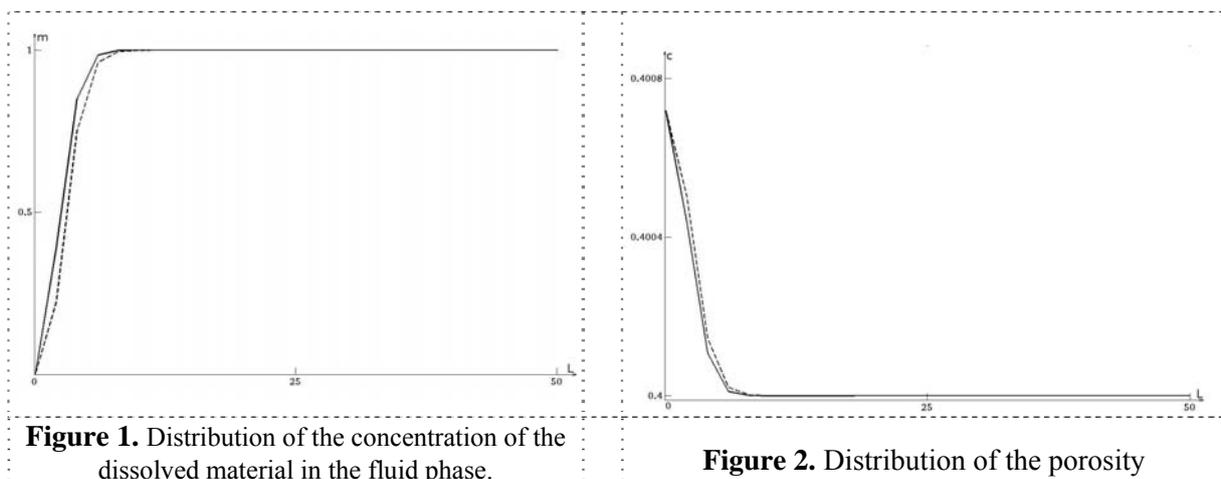
$$\begin{aligned} \partial m / \partial t &= (1 - m)\partial \theta / \partial t - [V_0(1 + \theta)]^{-1}\partial V_s / \partial t \\ \partial m / \partial t + m\partial \theta / \partial t + \text{div} \vec{v} &= 0, \quad \vec{v} = -k\mu^{-1}\nabla(p + \rho gz) \\ \partial (mc) / \partial t + [V_0(1 + \theta)]^{-1}\partial V_s / \partial t + mc\partial \theta / \partial t &= \text{div} (D\nabla c - \vec{v}c), \\ [V_0(1 + \theta)]^{-1}\partial V_s / \partial t &= \beta(c - c_0)^k; \beta, k, c_0 = \text{const} \end{aligned} \quad (31)$$

This set also includes the equation for the concentration of active component in the solution.

Here c is the concentration of the skeleton material; k is the permeability coefficient; μ is the viscosity of the filtrating fluid. The viscosity of the solution is constant; the permeability is dependent on the porosity in compliance with the Archie's law [1].

First we performed a comparison with one-dimensional analytical solution [10] to verify results of the numerical simulation. The analytical solution was calculated with the same parameters as the numerical solution above.

The results of the comparison are presented below. Graphics represent the dependence of the correspondent value on distance to the borehole (in meters). The dash line stands for numerical solution.



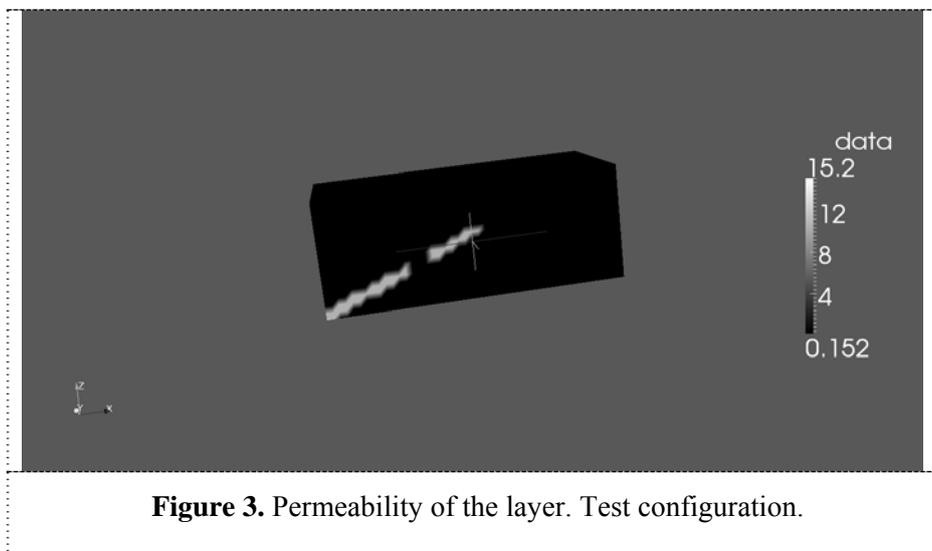
3.1. Simulation of injection of dissolving fluid

We performed three-dimensional numerical simulation of an injection of dissolving agent into homogeneous porous layer with vertical borehole as an example of the real physical problem. The carbonate rock was considered as the layer, water was considered as the dissolving agent. Data for speed of dissolution, solubility and other parameters were taken from [8].

Implicit finite difference scheme with regular computational grid was used to perform discretization of the (31). Simulated area had dimensions 100x100x40 meters, dimensional step was 2 meters, time step 10 seconds. Due to the large size of the area, PETSc solver was used to perform efficient parallel computations on those big systems [9]. GMRES method in combination with LU preconditioner showed the best performance for our case. This method showed good convergence in studied case. The residual was set to 10^{-8} .

Visualization of obtained results was performed in ParaView open-source software. (Contributors: National Center for Supercomputing Applications (NCSA) at the University of Illinois at Urbana-Champaign (UIUC), Lawrence Livermore National Laboratory (LLNL), Sandia National Laboratories (SNL), Los Alamos National Laboratory (LANL), Jean-Loup Gailly and Mark Adler (gzip library)).

We performed calculations with different configurations and properties of the clot. Permeability in the layer for one of the tests is presented on Fig. 1. Dark area stands for low permeability region, bright area designates extremely high permeable channel, which has a low permeable gap.



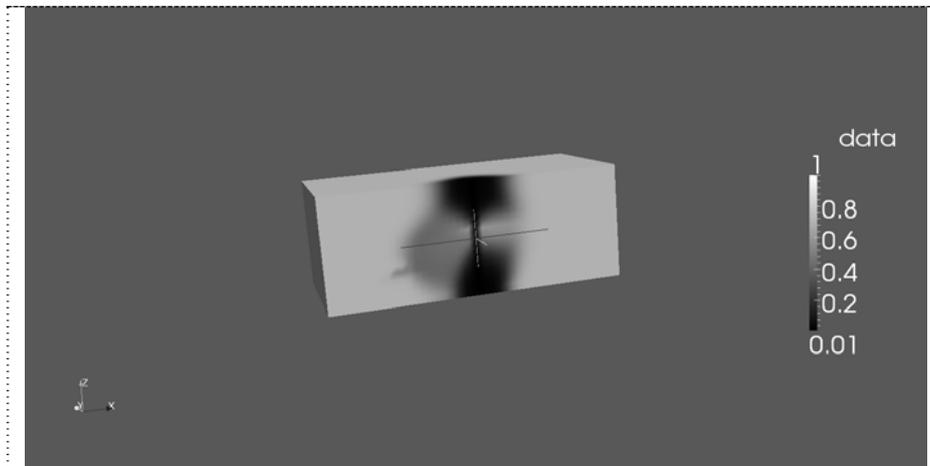


Figure 4. Distribution of concentration. Dark area designates the expansion of the injected fluid

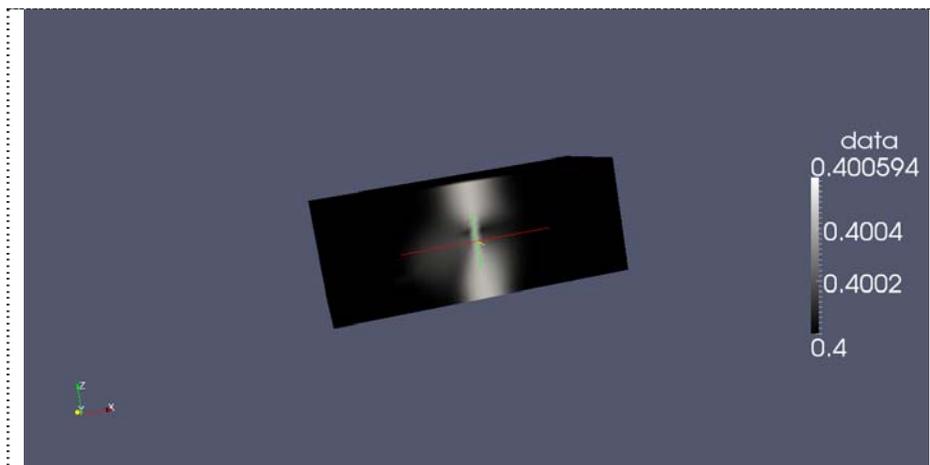


Figure 5. Distribution of porosity. Bright area is the region with larger porosity

4. Conclusion

New approach to the derivation of equations of the underground mass transfer for filtrating porous medium with the variable-mass skeleton is proposed. Obtained equations can be used to solve important practical problems, which require taking into account variations of the stress-strain state of the rock mass. Those variations are caused by chemical interactions between components of the underground fluid and the material of the porous skeleton. The applicability domain of the model includes problems of enhanced oil recovery, simulation of filtration of brines through clay layers, simulation of suffosion and karst processes.

Rheological relations for saturated porous media were obtained in the same way as equations of mechanics of continuum for solid body with dislocations [11,12]. Tensor of dislocational polarization

is introduced in that method. Its analog in our case is the integral $\int_0^t \text{div} \mathbf{q} \cdot d\tau$ in (19) and other

equations. Fluid in pores is isotropic, so the tensor is reduced to the contraction. Our approach gives an opportunity to obtain rheological relations for saturated porous media in the new way and to avoid some difficulties of the traditional method.

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