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Type III and II universal spacetimes

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Abstract. We briefly summarize our recent results on universal spacetimes. We show that universal spacetimes are necessarily CSI, i.e. for these spacetimes, all curvature invariants constructed from the Riemann tensor and its derivatives are constant. Then, we focus on type III universal spacetimes and discuss a proof of universality for a class of type III Kundt spacetimes. We also mention explicit examples of type III and II universal spacetimes.

1. Introduction

In the contribution [1] in this volume, we have introduced universal spacetimes obeying the following definition [2]

Definition 1.1. A metric is called *universal* if all conserved symmetric rank-2 tensors constructed from the metric, the Riemann tensor and its covariant derivatives of arbitrary order are multiples of the metric.

We have argued that universal metrics solve the vacuum equations of *all* theories of gravity with the Lagrangian of the form

$$L = L(g_{ab}, R_{abcd}, \nabla_{a_1} R_{bcde}, \ldots, \nabla_{a_1 \ldots a_p} R_{bcde}). \tag{1}$$

In general, it seems too difficult to study universal spacetimes in arbitrary spacetime dimension in full generality. However, employing the algebraic classification of the Weyl tensor [3] (see also [4] for a recent review) leads, in particular for types N and III, to considerable simplification of the problem and allows us to prove universality for various classes of metrics.

In [1], we summarized the main results of [5] for type N universal spacetimes. Here, let us briefly present further results on universal spacetimes (for details see [5]).

First, in section 2, we describe the main points of the proof of the following theorem.

Theorem 1.2. *A universal spacetime is necessarily a CSI spacetime.*

CSI (constant curvature invariant) spacetimes are spacetimes that have all curvature invariants constructed from the metric, the Riemann tensor and its derivatives of arbitrary order constant, see e.g. [6].

Then, we discuss type III spacetimes. Although we expect that type III universal spacetimes are necessarily Kundt, in contrast to the type N case, we cannot use Theorem 1.2 to prove this statement in full generality and in [5], we provide a proof only in the "generic" case. Thus, in

section 3, we focus on the sufficient conditions for universality. We identify a universal subclass of type III Kundt metrics[1]

Theorem 1.3. *Type III, $\tau_i = 0$ Einstein Kundt spacetimes obeying*

$$C_{acde}C_b{}^{cde} = 0 \tag{2}$$

are universal.

Finally, in section 4, we present explicit examples of type III and type II universal metrics.

2. Universal spacetimes are CSI

Let us here briefly summarize the main points of the proof of the theorem 1.2.

Let us consider the Lagrangian (1) containing covariant derivatives of the Riemann tensor up to a fixed p. In [7], it has been shown that by varying the action, one arrives at the field equations

$$\begin{aligned} -T^{ab} &= \frac{\partial L}{\partial g_{ab}} + E^a{}_{cde}R^{bcde} + 2\nabla_c\nabla_d E^{acdb} + \frac{1}{2}g^{ab}L, \\ E^{bcde} &= \frac{\partial L}{\partial R_{bcde}} - \nabla_{a_1}\frac{\partial L}{\partial \nabla_{a_1} R_{bcde}} + \cdots + (-1)^p \nabla_{(a_1}\cdots\nabla_{a_p)}\frac{\partial L}{\partial \nabla_{(a_1}\cdots\nabla_{a_p)}R_{bcde}}, \end{aligned} \tag{3}$$

where T^{ab} is the associated conserved tensor.

Now, let us consider any polynomial invariant I and let us assume that it contains derivatives of the Riemann tensor of orders at most p. By [7], we can assume it is of the form

$$I = I[g_{ab}, R_{abcd}, \nabla_{a_1}R_{bcde}, ..., \nabla_{(a_1...a_p)}R_{bcde}].$$

Let us consider the (infinite) series of Lagrangians $L = I^n$, $n = 1, 2, 3, ...$ By variation, we get a conserved tensor $T[n]^a{}_b$ for each n. For universal spacetimes, traces of $T[n]^a{}_b$ are constant. By studying the explicit forms of the corresponding expressions [5], one arrives at a conclusion that I has to be a constant and since I is an arbitrary curvature invariant it follows that universal spacetimes are CSI.

3. Type III universal spacetimes

Type III spacetimes by definition [3, 4] admit a multiple Weyl aligned null direction, mWAND, $\boldsymbol{\ell}$. Let us complete a frame with another null vector $\boldsymbol{n}$ and $n-2$ spacelike orthonormal vectors $\boldsymbol{m^{(i)}}$ with the only non-vanishing products being $\ell^a n_a = 1$, $m^{(i)a}m^{(j)}_a = \delta_{ij}$ (coordinate indices $a, b, \ldots = 0 \ldots n-1$, frame indices $i, j, \ldots = 2 \ldots n-1$).

In an appropriately chosen frame, the Weyl tensor for type III spacetime reads [3, 4]

$$C_{abcd} = 8\Psi'_i\,\ell_{\{a}n_b\ell_c m^{(i)}_{d\}} + 4\Psi'_{ijk}\,\ell_{\{a}m^{(i)}_b m^{(j)}_c m^{(k)}_{d\}} + 4\Omega'_{ij}\,\ell_{\{a}m^{(i)}_b\ell_c m^{(j)}_{d\}}, \tag{4}$$

where the frame components satisfy $\Psi'_i = \Psi'_{jij}$, $\Omega'_{ij} = \Omega'_{ji}$, and $\Omega'_{ii} = 0$ and for an arbitrary tensor T_{abcd}

$$T_{\{abcd\}} \equiv \frac{1}{2}(T_{[ab][cd]} + T_{[cd][ab]}). \tag{5}$$

As discussed in [5], for type III Einstein spacetimes, the symmetric rank-2 tensor

$$S_{ab} \equiv C_{acde}C_b{}^{cde} \tag{6}$$

is conserved. While for type N, S_{ab} vanishes identically, for type III Einstein spacetimes, it is in general a boost weight -2 tensor proportional to $\ell_a\ell_b$. Therefore, for type III universal spacetimes, we have an additional necessary condition (2).

[1] However, note that other type III Einstein Kundt spacetimes may also exist.

3.1. Main points of the proof of the theorem 1.3
The key intermediate result proven in [5] is that

Proposition 3.1. *For type III Einstein Kundt spacetimes, the boost order of $\nabla^{(k)}C$ (a covariant derivative of an arbitrary order of the Weyl tensor) with respect to the multiple WAND is at most -1.*

Proof of this result relies on the precise form of various Bianchi and Ricci identities [8, 9].
A direct consequence of proposition 3.1 is that

Lemma 3.2. *For type III Ricci-flat Kundt spacetimes, a non-vanishing rank-2 tensor constructed from the metric, the Weyl tensor and its covariant derivatives of arbitrary order is at most quadratic in the Weyl tensor and its covariant derivatives.*

We find that for type III Ricci-flat Kundt spacetimes, for which the FKWC basis [10] of rank-2, order-6 Weyl polynomials consists of

$$F_1 \equiv C^{pqrs}C_{pqrs;ab}, \quad F_2 \equiv C^{pqrs}{}_{;a}C_{pqrs;b}, \quad F_3 \equiv C^{pqr}{}_{a;s}C_{pqrb}{}^{;s}, \tag{7}$$

F_1 and F_2 are in general non-vanishing. In our case, F_1 and F_2 are conserved and thus, in general, type III Ricci-flat Kundt spacetimes are not universal. However, both F_1 and F_2 vanish for $\tau_i = 0$. In this case, a rather technical proof [5] allows us to arrive at the theorem 1.3.

4. Kundt spacetimes

Since all universal spacetimes we have discussed belong to the Kundt class, let us briefly discuss Kundt metrics. Kundt spacetimes are spacetimes admitting a null geodetic vector field $\boldsymbol{\ell}$ with vanishing shear, expansion and twist. In appropriately chosen coordinates,

$$\ell_{a;b} = L_{11}\ell_a\ell_b + \tau_i(\ell_a m_b^{(i)} + m_a^{(i)}\ell_b) \tag{8}$$

and the metric reads [6, 11]

$$\mathrm{d}s^2 = 2\mathrm{d}u\,[\mathrm{d}r + H(u,r,x^\gamma)\mathrm{d}u + W_\alpha(u,r,x^\gamma)\mathrm{d}x^\alpha] + g_{\alpha\beta}(u,x^\gamma)\mathrm{d}x^\alpha\mathrm{d}x^\beta, \tag{9}$$

with $\alpha,\beta,\gamma = 2\ldots n-1$.

Since universality implies CSI (theorem 1.2) we restrict ourselves to the *Kundt CSI metrics*, where [6, 12]

$$\begin{aligned} W_\alpha(u,r,x^\gamma) &= rW_\alpha^{(1)}(u,x^\gamma) + W_\alpha^{(0)}(u,x^\gamma), \\ H(u,r,x^\gamma) &= \frac{r^2}{8}\left(a + W_\alpha^{(1)}W^{(1)\alpha}\right) + rH^{(1)}(u,x^\gamma) + H^{(0)}(u,x^\gamma), \end{aligned} \tag{10}$$

$g_{\alpha\beta}(x^\gamma)$ (note that $g_{\alpha\beta,u} = 0$) is the metric of a locally homogeneous space and a is a constant. Note that (10) are necessary but not sufficient conditions for Kundt CSI.

4.1. type III
Note also that for type III and $\tau_i = 0$, the Bianchi identities imply $\Lambda = 0$ and thus type III, $\tau_i = 0$, Kundt universal spacetimes are in fact Ricci-flat and VSI (vanishing scalar invariants) with a metric of the form [13, 14]

$$\mathrm{d}s^2 = 2\mathrm{d}u\,[\mathrm{d}r + H(u,r,x^\gamma)\mathrm{d}u + W_\alpha(u,r,x^\gamma)\mathrm{d}x^\alpha] + \delta_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta, \tag{11}$$

with

$$W_2 = 0, \tag{12}$$

$$W_M(u, r, x^\gamma) = W_M^{(0)}(u, x^\gamma), \tag{13}$$

$$H(u, r, x^\gamma) = rH^{(1)}(u, x^\gamma) + H^{(0)}(u, x^\gamma). \tag{14}$$

Since further constraints on H and W_M follow from the Einstein equations we conclude with an explicit example of a four-dimensional type III, $\tau_i = 0$ Ricci-flat Kundt universal metric (expressed in other coordinates) [15]

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - x(v + \mathrm{e}^x)\mathrm{d}u^2 + \mathrm{e}^x(\mathrm{d}x^2 + \mathrm{e}^{-2u}\mathrm{d}y^2). \tag{15}$$

4.2. type II
An example of a four-dimensional type II universal Kundt spacetime

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v + \left(-v^2\lambda + H(u, x, y)\right)\mathrm{d}u^2 + \frac{1}{\lambda}(\mathrm{d}x^2 + \sinh^2 x\mathrm{d}y^2), \quad \Box H = 0 \tag{16}$$

has been given already in [2]. As we will discuss elsewhere, for some type II classes of Kundt spacetimes, universality depends on dimension of the spacetime.

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