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Inertial forces in General Relativity

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Abstract. An exact equation describing the inertial forces for arbitrary orthonormal frames on arbitrary spacetimes is presented. It manifests that the so-called gravitomagnetic field consists of a combination of two effects of independent origin: the vorticity of the observer congruence, plus the angular velocity of rotation, relative to Fermi-Walker transport, of the triad of spatial axes that each observer "carries" with it. Such formulation encompasses different gravitomagnetic fields in the literature. The formalism is applied to notable reference frames in Kerr spacetime, and their inertial forces shown to have familiar Newtonian analogues.

1. Introduction

Inertial forces are fictitious forces that arise in the description of the motion of test particles when the reference frame is not inertial. That is always the case in a curved spacetime, since no globally inertial frames exist therein. The inertial forces have been defined in different ways in the literature (see [1] and Refs. therein), from the best known linearized theory approaches, to exact formulations (e.g. [2, 3, 1]). Herein we will follow the exact approach in [1], to which we refer for more details and notation/conventions; it embodies many of the approaches in the literature as special cases. The problem may the stated as follows. Consider a congruence of observers of 4-velocity u^α, and a test particle of worldline $z^\alpha(\tau)$ and 4-velocity $dz^\alpha/d\tau = U^\alpha$. Let $h^\alpha{}_\beta \equiv u^\alpha u_\beta + \delta^\alpha{}_\beta$ be the projector orthogonal to u^α (the "space" projector). The spatial projection of the particle's 4-velocity, $U^{\langle\alpha\rangle} \equiv h^\alpha{}_\beta U^\beta$, gives a notion[1] of velocity of the particle *relative to the observers.* For simplicity, we consider only particles in geodesic motion. It is the variation of $U^{\langle\alpha\rangle}$ along $z^\alpha(\tau)$, that one casts as *inertial forces.* To determine it one must specify the reference frame. The observers' 4-velocity $\mathbf{u}$ defines the time axis; but one still needs to specify the spatial axes, in particular their transport law along the observer congruence (see Fig. 1 of [1]), which is where the different approaches mainly differ. If we take an orthonormal frame (a "tetrad") $\mathbf{e}_{\hat\alpha}$, with $\mathbf{e}_{\hat 0} = \mathbf{u}$, the general form for the transport law reads

$$\nabla_{\mathbf{u}}\mathbf{e}_{\hat\beta} = \Omega^{\hat\alpha}{}_{\hat\beta}\mathbf{e}_{\hat\alpha}; \quad \Omega^{\alpha\beta} = 2u^{[\alpha}a^{\beta]} + \epsilon^{\alpha\beta}{}_{\nu\mu}\Omega^\mu u^\nu \ , \tag{1}$$

where $a^\alpha \equiv \nabla_{\mathbf{u}} u^\alpha$, and Ω^α the angular velocity of rotation of the spatial triad $\mathbf{e}_{\hat i}$ relative to Fermi-Walker transport. We will leave Ω^α *arbitrary* for now.

In order to measure the rate of change of $U^{\langle\alpha\rangle}$ with respect to such frame, we need a connection $\tilde\nabla$ (i.e., a *covariant derivative*) for spatial vectors with the following properties: i) with respect

[1] More precisely, one can decompose $U^\alpha = \gamma(u^\alpha + v^\alpha)$, where $\gamma = -U^\alpha u_\alpha = \sqrt{1 - v^\alpha v_\alpha}$ is the Lorentz factor, and $v^\alpha = U^{\langle\alpha\rangle}/\gamma$. v^α is the usual 3-velocity of the particle relative to the observers; in a coordinate system $\{t, x^i\}$ where the observers are at rest ($u^i = 0$), one has $v^i = dx^i/dt$.

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to it, the spatial triad vectors $\mathbf{e}_{\hat{i}}$ are constant along the observer congruence: $\tilde{\nabla}_{\mathbf{u}}\mathbf{e}_{\hat{i}} = 0$. That is, $\tilde{\nabla}_{\mathbf{u}}$ should become an ordinary time derivative ($\partial_{\mathbf{u}}$) *in the prescribed frame*[2]. ii) Along the directions orthogonal to u^α, $\tilde{\nabla}$ should equal the projected Levi-Civita spacetime connection, $\tilde{\nabla}_{\mathbf{X}} Z^\alpha = h^\alpha{}_\beta \nabla_{\mathbf{X}} Z^\beta$, for all X^α and Z^α orthogonal to u^α. This is to correct for the variation of the axes $\mathbf{e}_{\hat{i}}$ in the directions orthogonal to u^α, which are not related to inertial forces (e.g. the trivial variation from point to point of the $\mathbf{e}_{\hat{i}}$ of a non-rectangular coordinate system in flat spacetime). This is achieved by the connection $\tilde{\nabla}$ whose action on a spatial vector X^α reads

$$\tilde{\nabla}_\alpha X^\beta = (h^u)^\beta{}_\gamma \nabla_\alpha X^\gamma + u_\alpha \epsilon^\beta{}_{\delta\gamma\lambda} u^\gamma X^\delta \Omega^\lambda \ . \tag{2}$$

The first term $(h^u)^\beta{}_\gamma \nabla_\alpha X^\gamma \equiv \nabla^\perp_\alpha X^\beta$ is called [2] the "Fermi-Walker connection", since $\nabla^\perp_{\mathbf{u}} X^\alpha$ is the Fermi-Walker derivative of X^α along the congruence. The second term adjusts the connection to the transport law chosen for the $\mathbf{e}_{\hat{i}}$. It is easily seen that $\tilde{\nabla}$ preserves the spatial metric $h_{\alpha\beta}$.

Differentiating $U^{\langle\alpha\rangle}$, with respect to $\tilde{\nabla}$, along the particle's worldline (of tangent U^α) yields

$$\tilde{\nabla}_{\mathbf{U}} U^{\langle\alpha\rangle} = -\gamma \nabla_{\mathbf{U}} u^\beta + \epsilon^\alpha{}_{\beta\gamma\delta} u^\delta U^\beta \Omega^\gamma \equiv F^\alpha_{\rm GEM} \tag{3}$$

which is the *inertial* (or "gravitoelectromagnetic") "force". It consists of two terms of distinct origin: a first term that depends only on the variation of the observers' 4-velocity u^α along the particle's worldline, and a second *independent* term, that arises from the transport law for the spatial triads $\mathbf{e}_{\hat{i}}$. Decomposing $\nabla_\beta u_\alpha$ in the congruence's kinematics

$$\nabla_\beta u_\alpha = -a_\alpha u_\beta - \epsilon_{\alpha\beta\gamma\delta}\omega^\gamma u^\delta + K_{(\alpha\beta)}; \qquad K_{(\alpha\beta)} = h_\alpha^{\ \mu} h_\beta^{\ \nu} \nabla_\mu u_\nu = \sigma_{\alpha\beta} + \theta h_{\alpha\beta} \ , \tag{4}$$

where $\omega^\alpha \equiv \epsilon^{\alpha\beta\gamma\delta}\nabla_\beta u_\gamma u_\delta/2$ is the vorticity of the observer congruence, and $K_{(\alpha\beta)}$ the shear/expansion tensor ($\theta \equiv$ expansion, $\sigma_{\alpha\beta} \equiv$ traceless shear), we get

$$F^\alpha_{\rm GEM} = \gamma \left[\gamma G^\alpha + \epsilon^\alpha{}_{\beta\gamma\delta} u^\delta U^\beta H^\gamma - K^{(\alpha\beta)} U_\beta \right], \tag{5}$$

where

$$G^\alpha = -\nabla_{\mathbf{u}} u^\alpha \ ; \qquad H^\alpha = \omega^\alpha + \Omega^\alpha \tag{6}$$

are, respectively, the "gravitoelectric" and "gravitomagnetic" fields, that play analogous roles to the electric and magnetic fields in the Lorentz force. H^α thus consists of two parts of independent origin: the vorticity ω^α of the observers, plus the angular velocity Ω^α of rotation of the spatial triads relative to Fermi-Walker transport. Ω^α has a physical interpretation in terms of the "precession" of gyroscopes. According to the Mathisson-Papapetrou equations, no torque is exerted on a gyroscope (taken as a *pole-dipole* particle) in a gravitational field; its spin vector (under the Mathisson-Pirani spin condition [4]) undergoes Fermi-Walker transport, $DS^\alpha/d\tau = S_\nu a^\nu U^\alpha$. In a comoving tetrad $\mathbf{e}_{\hat{\alpha}}$ (where $U^{\hat{\alpha}} = \delta^{\hat{\alpha}}_{\hat{0}}$, $S^{\hat{0}} = 0$), this is

$$\frac{DS^{\hat{i}}}{d\tau} = 0 \ \Leftrightarrow \ \frac{dS^{\hat{i}}}{d\tau} = -\Gamma^{\hat{i}}_{\hat{0}\hat{k}} S^{\hat{k}} = \left(\vec{S} \times \vec{\Omega}\right)^{\hat{i}} . \tag{7}$$

The inverted commas are because the gyroscopes, in rigor, do not precess (it is the frame that rotates); their rotation axes actually determine the local *compass of inertia*; and Fermi-Walker transport mathematically the non-rotating frame carried by an accelerated observer [1, 4, 7].

Ω^α is up until now arbitrary; different choices are made in the literature. The natural choice would be to choose spatial axes pointing to fixed neighboring observers; in general, however, that

[2] Just like $\nabla_{\mathbf{u}} = \partial_{\mathbf{u}}$ in an inertial frame, or $\nabla^\perp_{\mathbf{u}} = \partial_{\mathbf{u}}$ in a Fermi-Walker transported frame.

is not compatible with orthonormal axes. Let η^α be a connecting vector between the observers' worldlines, $\mathcal{L}_{\mathbf{u}}\eta^\alpha = 0$, and $Y^\alpha = h^\alpha{}_\beta \eta^\beta$ its space projection. Y^α evolves in the tetrad as [1]

$$\dot{Y}_{\hat{i}} = \left[\sigma_{\hat{i}\hat{j}} + \frac{1}{3}\theta\delta_{\hat{i}\hat{j}} + \epsilon_{\hat{i}\hat{k}\hat{j}}(\omega^{\hat{k}} - \Omega^{\hat{k}})\right] Y^{\hat{j}} \ . \tag{8}$$

If the congruence is rigid ($\sigma_{\hat{i}\hat{j}} = \theta = 0$), and one chooses spatial triads $\mathbf{e}_{\hat{i}}$ *co-rotating* with the observers, which amounts to choosing $\Omega^\alpha = \omega^\alpha$, then Y^α is constant in the tetrad, $\dot{Y}_{\hat{i}} = 0$. If the congruence is not rigid but only expands ($\theta \neq 0$, $\sigma_{\hat{i}\hat{j}} = 0$), then $\dot{Y}_{\hat{i}} = \theta Y_{\hat{i}}/3$, i.e., Y^α, albeit not constant, has a fixed direction on the tetrad, so the $\mathbf{e}_{\hat{i}}$ still point to fixed neighboring observers. If $\sigma_{\hat{i}\hat{j}} \neq 0$, then that is not possible, as the connecting vectors cannot remain orthogonal; the choice $\Omega^\alpha = \omega^\alpha$ means in this case that the triads $\mathbf{e}_{\hat{i}}$ co-rotate with the observers, but without undergoing their shearing effects. This is as much as an orthonormal frame can adapt to the observer congruence; for this reason we dub it *the congruence adapted frame*. It is argued in [5] to be the closest generalization of the Newtonian concept of reference frame.

Other transport laws for the spatial frame (with $\Omega^\alpha \neq \omega^\alpha$) are used in the literature, as they can be suitable for some applications; an example are the so-called "locally non rotating frames" (e.g. [6, 7, 3]), defined in stationary axisymmetric spacetimes, and where the observer congruence are the zero angular momentum observers (whose vorticity is zero, $\omega^\alpha = 0$), and the triads $\mathbf{e}_{\hat{i}}$ are tied to the background symmetries (implying in general that $\Omega^\alpha \neq 0$). Another choice [2] is a Fermi-Walker transported spatial frame ($\Omega^\alpha = 0$). These reference frames might seem strange at first; in the next section we will discuss them in the Kerr spacetime, and show that each of them has actually familiar Newtonian analogues.

2. Rigid observer congruences

2.1. Rigidly rotating frames

Rigidly rotating frames are reference frames *adapted* to a rigidly rotating congruence of observers. The simplest case is the rigidly rotating frame in flat spacetime. Let α = constant be their angular velocity. The metric is obtained from the Minkowski metric in cylindrical coordinates $\{t, r, \varphi, z\}$ by the transformation $\varphi = \phi + \alpha t$. A spatial orthonormal triad $\mathbf{e}_{\hat{i}}$ adapted to the observers is obtained by taking (using $h_{\alpha\beta}$, see Sec. 1) the normalized projection, orthogonal to $\mathbf{u}$, of the coordinate basis vectors $\{\partial_r, \partial_\phi, \partial_z\}$. The $\mathbf{e}_{\hat{i}}$ point to fixed observers (since the ∂_i are connecting vectors), and (in agreement with Eq. (8)) rotate relative to Fermi-Walker transport with an angular velocity that matches the congruence's vorticity, $\vec{\Omega} = \vec{\omega} = \alpha/(1 - \alpha^2 r^2)\partial_z$. The inertial forces in this frame are obtained by setting $K_{(\alpha\beta)} = 0$, $H^\alpha = 2\omega^\alpha$, in Eq. (5),

$$\vec{F}_{\rm GEM} = \gamma \left[\gamma \vec{G} + 2\vec{U} \times \vec{\omega}\right] \tag{9}$$

with $\vec{G} = \alpha^2 r/(1 - \alpha^2 r^2)\partial_r$. These are the inertial forces felt in a merry go round; $\gamma^2 \vec{G}$ and $2\gamma\vec{U} \times \vec{\omega}$ are relativistic versions of the well known centrifugal and Coriolis forces.

2.1.1. Frame adapted to the "static" observers in Kerr spacetime Take the Kerr metric in Boyer-Lindquist coordinates. The so-called "static" observers are the observers at rest in that coordinate system, whose worldlines are tangent to the time Killing vector field, $\mathbf{u} \propto \partial/\partial t$. This is a *rigid* observer congruence, and a spatial orthonormal triad $\mathbf{e}_{\hat{i}}$ adapted to it is obtained by projecting the coordinate basis vectors $\{\partial_r, \partial_\theta, \partial_\phi\}$ in the hyperplanes orthogonal to $\mathbf{u}$, and normalizing. The triad $\mathbf{e}_{\hat{i}}$ points to fixed observers (as the ∂_i are connecting vectors), and rotate with respect to Fermi-Walker transport with angular velocity $\vec{\Omega} = \vec{\omega}$, in agreement with Eq. (8). Since the congruence is rigid, and the triads are $\vec{e}_{\hat{i}}$ locked to neighboring observers, this is a *rigid frame*. The vorticity $\vec{\omega}$ decreases as r increases, vanishing at infinity where the $\mathbf{e}_{\hat{i}}$ become

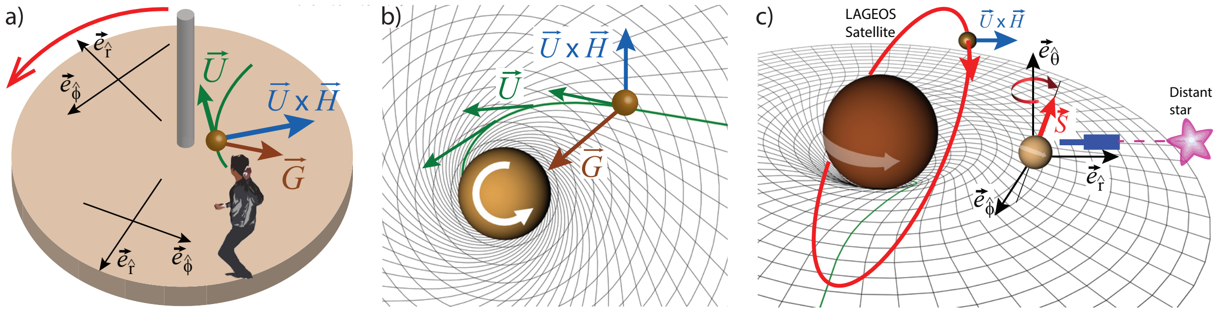

Figure 1. Two examples of rigidly rotating frames: a) a merry go round and b)-c) the frame adapted to the "static" observers in the Kerr spacetime. The inertial forces have the same form (9). $\vec{G}$ has opposite directions (centrifugal in a)); in Kerr $\vec{\omega} \neq 0$ for finite r, whilst $\lim_{r\to\infty} \vec{\omega} = 0$ (manifestation of frame-dragging). The axes at infinity are inertial ("star fixed"); since the frame is rigid, measuring the precession of gyroscopes (e.g. Gravity-Probe B) or deflection of test particles (e.g. the LAGEOS satellite) at any point relative to this frame, amounts to measuring them with respect to the distant stars.

inertial axes. This manifests the frame-dragging effect (absent in the merry go round of Fig. 1a)): the frame is inertial at infinity, and rotating close to the black hole (relative to the local compass of inertia, see Sec. 1) whilst, at the same time, being rigid.

A gyroscope whose center of mass is at rest in this frame is seen to precess as (cf. Eq. (7)) $d\vec{S}/d\tau = \vec{S} \times \vec{H}/2$. Since the frame is rigid, the triads $\mathbf{e}_{\hat{i}}$ are locked to the inertial axes at infinity; experimentally this amounts to anchor the axes to telescopes pointing at the distant stars (Fig. 1c)). This means that, at any point, $-\vec{H}/2$ is in fact the precession with respect to an inertial frame at infinity. It is known as the *Lense-Thirring* gyroscope precession, and has been measured, in the gravitational field of the Earth, by the Gravity Probe B mission [9].

The expression for the inertial forces is formally similar to the one in a merry go round, Eq. (9), only now $\vec{G} = -\vec{a}$ and $\vec{\omega} = \vec{\Omega}$ are minus the acceleration and the vorticity of the static observers in Kerr spacetime (for explicit expressions for $\vec{a}$ and $\vec{\Omega}$, see Eqs. (107)-(108) and (110) of [7]). The gravitomagnetic force, second term of Eq. (9), leads to a precession of the orbits of particles in the field of a rotating source, which has been detected by Laser Ranging to the LAGEOS satellites [10], and is under further scrutiny by the ongoing LARES mission [11].

2.2. Non-adapted frame — the "Fermi-Walker" gravitomagnetic field
Consider now a reference frame composed of the same congruence of static observers in Kerr spacetime, but now carrying spatial triads $\mathbf{e}_{\hat{i}}$ that undergo Fermi-Walker transport along the congruence; i.e., triads attached to the spinning axis of local guiding gyroscopes, which is given by the condition $\Omega^\alpha = 0$. These triads are not *adapted* to the congruence: the congruence has vorticity ($\omega^\alpha \neq 0$), thus we have observers that are rotating, but carrying with them spatial axes that do not rotate. The gravitomagnetic field reduces to the observer's vorticity, $\vec{H} = \vec{\omega} + \vec{\Omega} = \vec{\omega}$ and is known as the "Fermi-Walker gravitomagnetic field" [2]; it is one half of the $\vec{H}$ of the start-fixed rigid frame of Sec. 2.1.1. The inertial forces in such frame are thus

$$\vec{F}_{\rm GEM} = \gamma \left[\gamma \vec{G} + \vec{U} \times \vec{\omega} \right] . \qquad (10)$$

These are the inertial forces involved in the "hidden momentum" of a spinning particle [4]. Such reference frame might seem strange, but it actually has a familiar Newtonian analogue: a family of rigidly rotating observers in flat spacetime, carrying a system of axes that remain fixed to the basis vectors of the Cartesian inertial frame (see Fig. 2b) of [1]), which is the situation in a Ferris wheel, for observers sitting in the cabins.

3. Zero angular momentum observers (ZAMOs)

The quantity $u_\phi = \mathbf{u} \cdot \partial_\phi$ is identified with the observer's "orbital" angular momentum. Zero angular momentum observers (ZAMOs) are stationary observers for which $u_\phi = 0$. They have angular velocity $\dot{\phi}_{\rm ZAMO} = -g_{0\phi}/g_{\phi\phi}$, and 4-velocity $u^\alpha_{\rm ZAMO} = u^0_{\rm ZAMO}(1, 0, \dot{\phi}_{\rm ZAMO}, 0)$, the component $u^0_{\rm ZAMO}$ being determined by the normalization condition $\mathbf{u}^2_{\rm ZAMO} = -1$.

These observers, also called "fiducial" or "Eulerian" [8], have special properties. They have *no vorticity* ($\omega^\alpha = 0$), being orthogonal to the hypersurfaces $t = constant$. This means that they do not rotate relative to their neighbors (are locally irrotational). They also measure zero Sagnac effect around the black hole. That is, consider a circular loop, made of optical fiber, at fixed (r, θ) around the black hole, and let a flash of light be emitted at some point, so that light rays travel along the fiber both in the positive and negative ϕ directions (i..e, both co-rotating and counter-rotating with the black hole); it turns out that the ZAMOs will receive back both light signals *at the same time*. Hence the ZAMOs are not rotating in any sense relative to the spacetime geometry — locally and "globally". However, relative to the fixed stars (i.e., to the static observers), they are seen to be rotating, with angular velocity $\dot{\phi}_{\rm ZAMO} \neq 0$, which manifests frame-dragging. The congruence has zero expansion, $\theta = 0$, but shears, $\sigma_{\alpha\beta} \neq 0$.

Analogy with a free vortex.— The ZAMOs congruence exhibits many formal similarities with the irrotational flow of a free vortex (Fig. 2) well known from fluid dynamics. In a free vortex (in flat spacetime) the fluid has angular velocity $\dot{\phi} = K/r^2$. In spite of being rotating relative to the (globally inertial) star-fixed frame, the vorticity of the flow is zero ($\omega^\alpha = 0$), i.e., it is *locally irrotational*, just like the ZAMOs in Kerr spacetime. Also, similarly to the ZAMOs, the congruence does not expand ($\theta = 0$) but shears ($\sigma_{\alpha\beta} \neq 0$). The qualitative difference is that the observers comoving with the flow have non-zero angular momentum, and thus measure non-zero Sagnac effect along a circular optical fiber centered at $r = 0$. This is because the frame-dragging effect of Kerr spacetime has no parallel in this setup: herein, rotating with respect to the distant stars implies having non-zero angular momentum (unlike the ZAMOs in Kerr, which, for finite r, rotate with respect to the distant stars, whilst having zero angular momentum).

3.1. Congruence adapted frame

Since the ZAMOs have no vorticity, the congruence adapted frame amounts to setting $\Omega^\alpha = \omega^\alpha = 0$, i.e., choosing spatial triads Fermi-Walker transported along the congruence. This means that there is no gravitomagnetic field in this frame, $H^\alpha = \omega^\alpha + \Omega^\alpha = 0$, the inertial forces reduce to the gravitoelectric field plus the shear force, see Fig. 2a),

$$F^\alpha_{\rm GEM} = \gamma \left[\gamma G^\alpha - \sigma^\alpha{}_\beta U^\beta \right] \equiv F^\alpha_{\rm G} + F^\alpha_{\rm shear} \ . \tag{11}$$

This reference frame has a close analogue in fluid dynamics. As is well known, a boat circling a free vortex keeps its tip pointing always in (nearly) the same direction (Fig. 2b)); it is so because the boat behaves approximately as a vorticity meter. In a reference frame *adapted* to observers comoving with the flow of a free vortex (where the spatial axes are tied to vorticity meters), the inertial forces take likewise the form (11).

3.2. Locally "non-rotating" frames (LNR)

The so-called "locally non-rotating frames" [6] (also known as "proper reference frames of the fiducial observers" [8, b]) are frames associated with the ZAMOs, where the spatial triads $\mathbf{e}_{\hat{i}}$ are tied to the background symmetries. These frames are regarded as important for black hole physics, because they are defined everywhere (unlike the star fixed static observers of Sec. 2, that do not exist past the ergosphere). For the Kerr spacetime the $\mathbf{e}_{\hat{i}}$ are obtained by simply normalizing the Boyer-Lindquist coordinate basis $\{\partial_r, \partial_\theta, \partial_\phi\}$ (as it is orthogonal to $\mathbf{u} = \mathbf{u}_{\rm ZAMO}$). These triads clearly are *not* Fermi-Walker transported along the congruence (i.e., $\Omega^\alpha \neq 0$), see

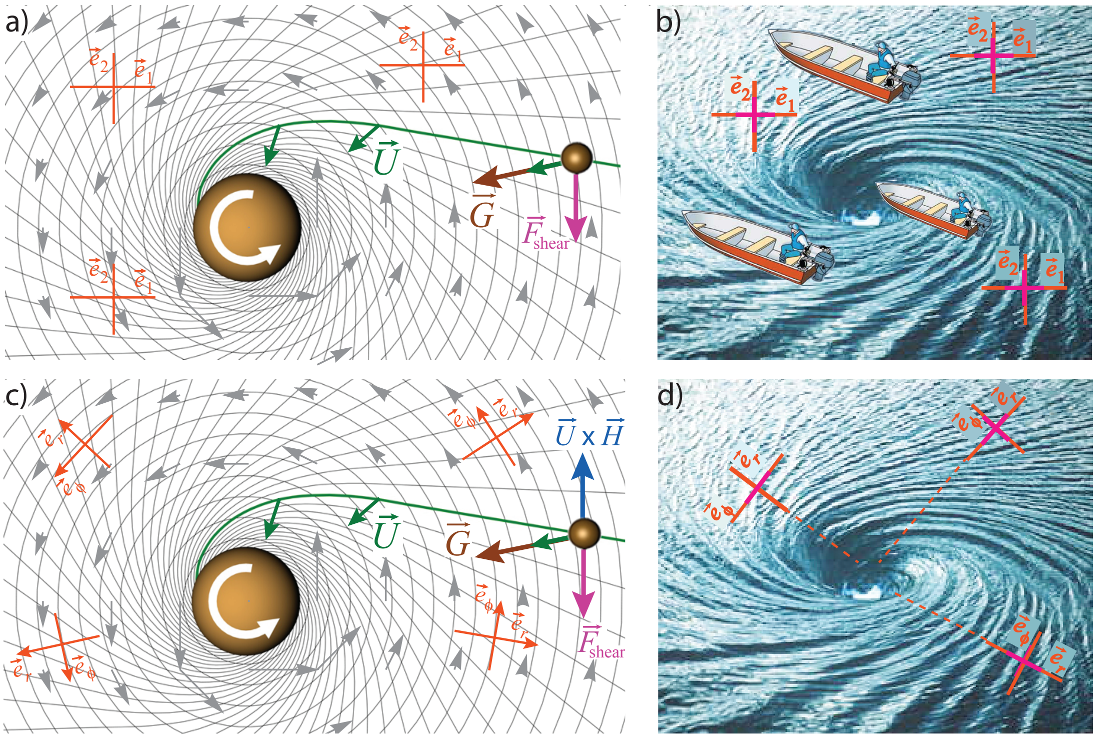

Figure 2. a) Frame adapted to the ZAMOs in Kerr spacetime; gray arrows represent the ZAMOs velocity (relative to the star fixed frame), $\vec{U}$ are the space components of the test particle's velocity *relative* to the ZAMOs, $U^{\langle\alpha\rangle} = h^{\alpha}{}_{\beta}U^{\beta}$. b) Frame adapted to observers comoving with the flow of a free vortex; the spatial frame is set up by tying the axes to vorticity meters (e.g. a boat). c) Locally "non-rotating" frames in Kerr spacetime. d) Frame composed of observers comoving with the flow of a free vortex, carrying spatial triads tied to a cylindrical coordinate system. In a) and b), $H^{\alpha} = \theta = 0$, and the inertial forces reduce to Eq. (11). In c) and d), $H^{\alpha} = \Omega^{\alpha}$, and the inertial forces are given by (12).

Fig. 2c); the explicit expression for $\Omega^{\alpha} \equiv \Omega^{\alpha}_{\rm LNR}$ is given in e.g. Eqs. (73)-(74) of [7]. Hence, the denomination locally "non-rotating" frames is misleading (cf. [7]); indeed the spatial frame rotates relative to the local compass of inertia. This is manifest in the inertial forces; the gravitomagnetic field in this frame is (since $\omega^{\alpha} = 0$) $H^{\alpha}_{\rm LNR} = \Omega^{\alpha}_{\rm LNR}$, and thus

$$\vec{F}_{\rm GEM} = \gamma\left[\gamma\vec{G} + \vec{U}\times\vec{\Omega}_{\rm LNR} - \sigma^{\hat{i}}{}_{\hat{j}}U^{\hat{j}}\vec{e}_{\hat{i}}\right] \equiv \vec{F}_{\rm G} + \vec{F}_{\rm GM} + \vec{F}_{\rm shear} \ , \tag{12}$$

i.e., consist of the gravitoelectric, gravitomagnetic, and shear forces. This reference frame again has an analogue in fluid dynamics: the frame associated to the observers comoving with the flow of a free vortex, and carrying spatial triads tied to the basis vectors of a cylindrical coordinate system, see Fig. 2d); therein, likewise, $H^{\alpha} = \Omega^{\alpha}$ and the inertial forces take the form (12).

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