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Symmetry operators and conserved currents

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Abstract. In these proceedings we present recent progress concerning existence of symmetry operators for the conformal scalar wave equation, the Dirac-Weyl equation and the source-free Maxwell equation. Furthermore, we consider conserved currents for the source-free Maxwell equations.

1. Introduction

In these proceedings we consider *symmetry operators*, i.e. linear differential operators which take solutions of a differential equation to solutions. These are versatile tools for the analysis of fields on spacetimes with special structure. A well-known example of a symmetry operator for the scalar wave equation is provided by the Lie derivative along a Killing vector field. However, in the Kerr spacetime, there is also another symmetry operator that arises from a Killing spinor rather than a Killing vector. This was an essential tool in a proof of decay of scalar waves on the Kerr background by Andersson and Blue [4].

As the symmetry operators are so useful, it is natural to ask when they exist. Here we will consider what conditions on a spacetime are necessary and sufficient for existence of symmetry operators for the conformal wave equation, the Dirac-Weyl equation, and the Maxwell equation, i.e. for massless test fields of spins 0, 1/2 and 1. We will investigate how the conditions for the symmetry operators for the different field equations are related, and how they are related to existence of conserved currents. This exposition is mainly based on the paper [3] about symmetry operators, but also ideas from [2] about conserved currents are presented.

2. Preliminaries

2.1. Spinors

Throughout, we will work on an orientable and time orientable globally hyperbolic spacetime with metric g_{ab} satisfying Einstein's field equations. We will use the $(+---)$ sign convention and extensively use spinor notation from [6].

2.2. Irreducible decompositions

Any spinor can be decomposed so it is written entirely in terms of the spin metric and symmetric spinors formed from the original spinor by taking traces and symmetrizing the remaining indices. For instance, a general valence $(3,0)$ spinor is decomposed as

$$T_{ABC} = T_{(ABC)} - \tfrac{1}{6}T^{D}{}_{CD}\epsilon_{AB} - \tfrac{1}{6}T^{D}{}_{DC}\epsilon_{AB} - \tfrac{1}{6}T^{D}{}_{BD}\epsilon_{AC} - \tfrac{1}{6}T^{D}{}_{DB}\epsilon_{AC} - \tfrac{1}{2}T_{A}{}^{D}{}_{D}\epsilon_{BC}.$$

We call this an *irreducible decomposition*. This is an important simplification because now we *only need to work with symmetric spinors*. From an algebraic point of view this is of great importance when one wants to simplify complicated spinorial expressions and tries to write them in a canonical form. Most calculations for the research presented here were done using the *xAct* suite for *Mathematica*. In fact I have written the *SymManipulator* package which is a part of *xAct*, and includes an efficient algorithm to do irreducible decompositions of arbitrary spinor expressions.

As we have seen, one can work with only symmetric spinors, but a covariant derivative of a symmetric spinor is not necessarily symmetric, so we need to do an irreducible decomposition of that too. This gives us four natural differential operators that maps symmetric spinors to symmetric spinors. For instance, consider the irreducible decomposition of the covariant derivative of a symmetric valence $(3,2)$ spinor.

$$\begin{aligned}\nabla_{A}{}^{A'}T_{BCD}{}^{B'C'} ={}& (\mathscr{T}_{3,2}T)_{ABCD}{}^{A'B'C'} - \tfrac{2}{3}\bar{\epsilon}^{A'(B'}(\mathscr{C}_{3,2}T)_{ABCD}{}^{C')} \\ &- \tfrac{3}{4}\epsilon_{A(B}(\mathscr{C}^{\dagger}_{3,2}T)_{CD)}{}^{A'B'C'} + \tfrac{1}{2}\epsilon_{A(B}\bar{\epsilon}^{A'(B'}(\mathscr{D}_{3,2}T)_{CD)}{}^{C')},\end{aligned}$$

where

$$\begin{aligned}(\mathscr{D}_{3,2}T)_{AB}{}^{A'} &= \nabla^{CB'}T_{ABC}{}^{A'}{}_{B'},\\ (\mathscr{C}_{3,2}T)_{ABCD}{}^{A'} &= \nabla_{(A}{}^{B'}T_{BCD)}{}^{A'}{}_{B'},\\ (\mathscr{C}^{\dagger}_{3,2}T)_{AB}{}^{A'B'C'} &= \nabla^{C(A'}T_{ABC}{}^{B'C')},\\ (\mathscr{T}_{3,2}T)_{ABCD}{}^{A'B'C'} &= \nabla_{(A}{}^{(A'}T_{BCD)}{}^{B'C')}.\end{aligned}$$

These operators generalizes to arbitrary valence as long as enough indices exist to form the index contractions, so we get $\mathscr{D}_{k,l}$, $\mathscr{C}_{k,l}$, $\mathscr{C}^{\dagger}_{k,l}$ and $\mathscr{T}_{k,l}$ for symmetric valence (k,l) spinors. More details about these operators including commutator relations can be found in [3].

Definition 2.1. *A symmetric valence* (k,l) *spinor* $L_{A\ldots F}{}^{A'\ldots F'}$ *is called a valence* (k,l) Killing spinor *if* $(\mathscr{T}_{k,l}L)_{A\ldots G}{}^{A'\ldots G'}=0$.

3. Symmetry operators

With our notation the field equations we would like to study now takes the forms:

- The conformal wave equation (Spin-0): $(\square + 4\Lambda)\phi = 0$.
- The Dirac-Weyl equation (Spin-1/2): $(\mathscr{C}^{\dagger}_{1,0}\phi)_{A'}=0$.
- The Maxwell equation (Spin-1): $(\mathscr{C}^{\dagger}_{2,0}\phi)_{AA'}=0$.

As mentioned in the introduction, what we mean by a symmetry operator is a linear differential operator that maps a solution of a differential equation to a solution of the differential equation. However, we can slightly extend this notion so that an operator that map to solutions of the complex conjugate version of the differential equation instead is also considered to be a symmetry operator. As the complex conjugate of the Dirac-Weyl equation is $(\mathscr{C}_{0,1}\bar{\phi})_{A}=0$ and similarly for the Maxwell equation, we can formulate the two different kinds of symmetry operators for spin-1/2 and spin-1 as:

- First kind: $\ker\mathscr{C}^{\dagger}\to\ker\mathscr{C}^{\dagger}$. (Preserves parity)
- Second kind: $\ker\mathscr{C}^{\dagger}\to\ker\mathscr{C}$. (Changes parity)

The conformal wave equation is real so we do not have a notion of second kind symmetry operators for that case. In the paper [3], we study both the first and second kind symmetry operators up to second order, but here we will restrict ourselves to the first kind of operators.

To give clear statements for existence of symmetry operators, we introduce the following operators that depends on the curvature.

Definition 3.1. *For a valence $(2,2)$ Killing spinor $L_{AB}{}^{A'B'}$, we define*

$$\begin{aligned}(\mathcal{O}^{(0)}_{2,2}L)_{A}{}^{A'} \equiv{}& \tfrac{1}{3}\Psi_{ABCD}(\mathscr{C}_{2,2}L)^{BCDA'} + \tfrac{1}{3}\bar{\Psi}^{A'}{}_{B'C'D'}(\mathscr{C}^{\dagger}_{2,2}L)_{A}{}^{B'C'D'}\\ &+ L^{BCA'B'}(\mathscr{C}_{2,2}\Phi)_{ABCB'} + L_{A}{}^{BB'C'}(\mathscr{C}^{\dagger}_{2,2}\Phi)_{B}{}^{A'}{}_{B'C'},\\ (\mathcal{O}^{(1)}_{2,2}L)_{AB}{}^{A'B'} \equiv{}& L^{CDA'B'}\Psi_{ABCD} - L_{AB}{}^{C'D'}\bar{\Psi}^{A'B'}{}_{C'D'}.\end{aligned}$$

It turns out that the conditions for existence of symmetry operators involves existence of valence $(2,2)$ Killing spinors related to the curvature through the operators in definition 3.1.

Definition 3.2. *Let $L_{AB}{}^{A'B'}$ be a Killing spinor of valence $(2,2)$. Define the conditions*

(A0) There is a function Q such that $(\mathcal{O}^{(0)}_{2,2}L)_{A}{}^{A'} = (\mathscr{T}_{0,0}Q)_{A}{}^{A'}$.

(A1) There is a vector field $P_{A}{}^{A'}$ such that $(\mathcal{O}^{(1)}_{2,2}L)_{AB}{}^{A'B'} = (\mathscr{T}_{1,1}P)_{AB}{}^{A'B'}$.

The conditions for existence of symmetry operators up to second order for the conformal wave equation was already settled by Michel, Radoux & Šilhan [5], but here we present it in our notation. In the paper [3], we derived conditions for existence of symmetry operators of the first kind up to second order for the Dirac-Weyl and Maxwell equations.

Theorem 3.3 ([5], [3])**.** *(i) The conformal wave equation has a symmetry operator $\phi \to \chi$, with order less or equal to two, if and only if there is a valence $(2,2)$ Killing spinor satisfying condition A0.*

(ii) There exists a symmetry operator of the first kind for the Dirac-Weyl equation $\phi_A \to \chi_A$, with order less or equal to two, if and only if there is a valence $(2,2)$ Killing spinor satisfying condition A0 and condition A1.

(iii) There exists a symmetry operator of the first kind $\phi_{AB} \to \chi_{AB}$ for the Maxwell equation, with order less or equal to two, if and only if there is a valence $(2,2)$ Killing spinor satisfying condition A1.

To see how one can construct valence $(2,2)$ Killing spinors that satisfies these conditions, we observe that if they are built from conformal Killing vectors or valence $(2,0)$ Killing spinors, they naturally satisfy both conditions A0 and A1.

Proposition 3.4 ([3])**.** *Let $\xi^{AA'}$ and $\zeta^{AA'}$ be (not necessarily distinct) conformal Killing vectors and let κ_{AB} be a Killing spinor of valence $(2,0)$.*

(i) The symmetric spinor $\xi_{(A}{}^{(A'}\zeta_{B)}{}^{B')}$ is a Killing spinor of valence $(2,2)$, which admits solutions to the auxiliary conditions A0 and A1.

(ii) The symmetric spinor $\kappa_{AB}\bar{\kappa}_{A'B'}$ is also a Killing spinor of valence $(2,2)$, which admits solutions to the auxiliary conditions A0 and A1.

It is worth to note that the conditions A0 and A1 are not equivalent to each other. In fact in [3] we give an example of a spacetime and a valence $(2,2)$ Killing spinor that satisfies the A1 condition, but not the A0 condition.

3.1. Maxwell equation

Although we have found expressions for all the symmetry operators in the paper [3], we will here restrict ourselves to symmetry operators of the first kind for the Maxwell equation.

Theorem 3.5 ([3])**.** *There exists a symmetry operator of the first kind up to order 2 for the Maxwell equation iff*

$$(\mathscr{T}_{2,2}L)_{ABC}{}^{A'B'C'} = 0, \qquad (\mathscr{T}_{1,1}P)_{AB}{}^{A'B'} = -\tfrac{2}{3}(\mathcal{O}^{(1)}_{2,2}L)_{AB}{}^{A'B'}, \qquad (\mathscr{T}_{0,0}Q)_{BA'} = 0.$$

The symmetry operator takes the form

$$\begin{aligned}&\phi_{AB} \to \chi_{AB} = Q\phi_{AB} + (\mathscr{C}_{1,1}A)_{AB},\\ &A_{AA'} = -L^{BC}{}_{A'}{}^{B'}(\mathscr{T}_{2,0}\phi)_{ABCB'} + \tfrac{1}{3}\phi^{BC}(\mathscr{C}_{2,2}L)_{ABCA'} - \tfrac{4}{9}\phi_{AB}(\mathscr{D}_{2,2}L)^{B}{}_{A'} - P^{B}{}_{A'}\phi_{AB}. \quad (3.1)\end{aligned}$$

We also note that the potential satisfies $(\mathscr{C}^{\dagger}_{1,1}A)_{A'B'} = 0$. Furthermore, note that (3.1) is the general form of a first order operator $\ker\mathscr{C}^{\dagger}_{2,0} \to \ker\mathscr{C}^{\dagger}_{1,1}$ up to addition of a gradient.

4. Conserved currents

In this section we will consider conserved currents for the Maxwell field, and see how these are connected with the symmetry operators. Here, any divergence-free vector field formed as a concomitant of the Maxwell field will be called a *conserved current*. Typically, we will study the flux of a conserved currents through a timelike hypersurface, which is then a *conserved energy*. A special class of conserved currents are the *trivial currents*, which should not be confused with the zero current.

Definition 4.1. *A* trivial current $\tilde{J}^{AA'}$ *is a field that can be written as*

$$\tilde{J}_{AA'} = (\mathscr{C}^{\dagger}_{2,0}S)_{AA'} + (\mathscr{C}_{0,2}T)_{AA'}$$

for some symmetric spinor fields S_{AB} and $T_{A'B'}$.

The flux through a hypersurface of a trivial current $\tilde{J}^{AA'}$ is given by a pure boundary term. We will study *equivalence classes* of currents up to trivial currents. For most applications it is natural to study currents which are bilinear in ϕ_{AB} and $\bar{\phi}_{A'B'}$. We can write this class of currents in a simple form.

Lemma 4.2 ([2])**.** *Assume that $J^{AA'} \in \ker\mathscr{D}_{1,1}$ is a conserved current that is a differential operator that is bilinear in ϕ_{AB} and $\bar{\phi}_{A'B'}$, where $(\mathscr{C}^{\dagger}_{2,0}\phi)_{AA'} = 0$, then it can be written as*

$$J^{AA'} = A^{A}{}_{B'}\bar{\phi}^{A'B'} + \tilde{J}^{AA'},$$

where $A_{AA'}$ satisfies $(\mathscr{C}^{\dagger}_{1,1}A)_{A'B'} = 0$ and $\tilde{J}^{AA'}$ is a trivial current.

Remark 4.3. *Observe that $\phi_{AB} \to (\mathscr{C}_{1,1}A)_{AB}$ is a symmetry operator of the first kind for the Maxwell equation.*

Remark 4.4. *The general first order $A_{AA'}$ is given by* (3.1)*. Hence, the conditions for existence of a first order conserved current of this form are the same as for a second order symmetry operator of the first kind.*

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