

Astrophysical constraints and insights on extended relativistic gravity

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Abstract. It is discussed how observations of gravitational lensing at scales of individual, groups and clusters of galaxies can be understood in terms of non-Newtonian gravitational interactions with a relativistic structure compatible with the Einstein Equivalence Principle. This result is derived on very general grounds without knowing the underlying structure of the gravitational field equations. As such, any developed gravitational theory built to deal with these astrophysical scales (relevant to MOND phenomenology) needs to reproduce those results.

1. Introduction

In the second decade of the 17th century, the observed residuals of Neptune's orbit by were postulated to be due to [1]: (i) the effect of the Sun's gravity, at such a great distance might differ from Newton's description; or (ii) the discrepancies might simply be observational error; or (iii) perhaps Uranus was being pulled, or perturbed, by an as-yet undiscovered planet. Future work by Le Verrier and Adams [2] lead to the discovery of Neptune (the baryonic dark matter perturber of the motion of Uranus). When anomalies on the advance of the perihelium of Mercury were later observed, Le Verrier postulated the existence of an hypothetical unknown planet called Vulcan, closer to the sun than Mercury [cf. 3]. Vulcan was never observed and the relativistic extension made by Einstein when building up his general theory of relativity showed that almost all of the perihelium's advance of Mercury came from relativistic corrections of Newton's theory of gravity [cf. 4]. The important lessons to learn from these stories are that: (i) There could be unseen objects that perturb gravitational orbits and (ii) there may be cases in which gravity needs to be extended. Note however that, in the previous cases, the anomalies were tiny as compared to the observed dynamics. As opposed to this, the current dark matter/energy hypothesis require huge energy corrections in order to leave Einstein's or Newton's field equations untouched. As such, one may be more tempted to search for non-relativistic and relativistic extensions of gravity.

The flattening of rotation curves in spiral galaxies is usually taken as the basis for either modifying gravity or incorporating unknown dark non-baryonic matter into dynamics. It has always been a complicated issue to distinguish between the path to follow, although it is quite clear that finding dynamical systems with accelerations greater than Milgrom's acceleration constant $a_0 \approx 10^{-10} \text{m/s}^2$ which require dark matter, will contradict all types of MODified Newtonian Dynamics (MOND) proposals. The absence of high accelerated systems requiring dark matter and the non-experimental detection of the mysterious dark matter particle is a good



starting point in order to question the validity of gravity on mass and length scales such that the accelerations induced by the gravitational system on test particles are smaller than a_0 .

The flattening of rotation curves in spiral galaxies is best described by the Tully-Fisher relation:

$$v \propto M^{1/4}, \quad (1)$$

where v represents the velocity of a test star about the centre of a galaxy and M is the mass of that galaxy. This relation is only valid in the outer regions of spiral galaxies and it is well known that the inner parts are well described by Kepler's third law, which for circular motions take the form:

$$v \propto \sqrt{M/r}. \quad (2)$$

Globular clusters are traditionally considered perfect Newtonian systems since the velocity dispersion as a function of radius is well modelled with Newtonian gravity. However, recent observations show flattening profiles of these "rotation curves" in the outer parts. As shown by [5, 6], this flattening occurs when the equivalent Newtonian acceleration reaches a value $\approx a_0$. Furthermore, at these scales, the systems obeys Tully-Fisher scaling (1) for a dispersion velocity v .

The simplest gravitational system available in astrophysics is a binary star system. Studies of wide open binaries as a test of Newtonian gravity [7] show that gravity deviates from its Newtonian behaviour when the accelerations $\lesssim a_0$, i.e. for systems with $\sim 1M_\odot$, separated by $\sim 1\text{pc}$.

Many astrophysical and cosmological data can either be explained by unknown dark matter/energy entities or by a modification to the theory of gravity [e.g. 8–11]. The fact that the outer parts of globular clusters follow a Tully-Fisher law and that wide open binaries do not follow Kepler's third law are unfeasible to explain with the introduction of dark matter.

As such as an starting point one can think that Kepler's third law is valid at sufficiently small scales, where the gravitational acceleration a of test particles is $a \gtrsim a_0$ and that it gets substituted by the Tully-Fisher law at sufficiently large scales -where $a \lesssim a_0$, making gravity non-scale invariant.

2. The law of gravity

As explained by [12, 13], consider a test particle in circular orbit about a central mass source M . Using equation (2) and centrifugal balance means that the acceleration $a = v^2/r = -GM/r^2$, where G_N is Newton's gravitational constant. The minus sign is introduced in order to account for the attractive force of gravity. This calculation was performed by Newton in his *Philosophiæ Naturalis Principia Mathematica* book [14]. If we now assume that there is a scale for which Kepler's third law is substituted by the Tully-Fisher law, then using equation (1) and centrifugal balance it follows that $a = v^2/r = -G_M M^{1/2}/r$, where a Modified gravitational constant G_M has been introduced. This relation can be transformed into the basic traditional MOND with the definition $G_M^2 := a_0 G$ to obtain:

$$a = -\frac{\sqrt{a_0 G M}}{r}. \quad (3)$$

Note that G_M and/or a_0 are as fundamental as Newton's gravitational constant since they are applied to different sets of observations at different scales. The purpose of any extended theory of gravity not dealing with dark matter is to extend equation (3) into a proper theory at both relativistic and non-relativistic regimes with a transition from the Newtonian to this modified regime occurring at accelerations $\sim a_0$.

3. Relativistic constraints

As noted by [12], a constraint on the PPN γ parameter can be built following the same procedure as it is done in general relativity [15]. To do so it is only necessary to acknowledge a curved space-time, where the Einstein Equivalence Principle is valid and where particles follow geodesic trajectories. In Schwarzschild spherical coordinates, a point mass source M generating a gravitational field is described by the following line element:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00} c^2 dt^2 + g_{11} dr^2 - r^2 d\Omega^2, \quad (4)$$

for a metric tensor $g_{\mu\nu}$ and space-time coordinates $x^\alpha = (ct, r, \theta, \varphi)$, where t represents time, r the radial coordinate and the polar and azimuthal angles are given by θ and φ respectively, with an angular displacement $d\Omega^2 := d\theta^2 + \sin^2 \theta d\varphi^2$ and velocity of light c . The symmetry of the problem means that the unknown metric components g_{00} and g_{11} are functions that depend on the radial coordinate r only. For non-relativistic particles, we must consider the limit where the speed of light $c \rightarrow \infty$. In this case, the radial component of the geodesic equation is given by [see e.g. 16]:

$$\frac{1}{c^2} \frac{d^2 r}{dt^2} = \frac{1}{2} g_{11} \frac{\partial g_{00}}{\partial r}. \quad (5)$$

In this weak-field slow-motion approximation, a particle bounded to a circular orbit about the mass M has a centrifugal radial acceleration given by:

$$\frac{d^2 r}{dt^2} = -\frac{v^2}{r}, \quad (6)$$

for a circular tangential velocity v , and so equating this with equation (5) it follows that:

$$\frac{v^2}{c^2 r} = -\frac{1}{2} \frac{\partial g_{00}}{\partial r}. \quad (7)$$

Substitution of equation (3) in the previous relation yields:

$$^{(2)}g_{00} = \frac{2\phi}{c^2} = -2 \left(\frac{v}{c} \right)^2 \ln \left(\frac{r}{r_\star} \right) = -\frac{2G_M M^{1/2}}{c^2} \ln \left(\frac{r}{r_\star} \right), \quad (8)$$

where r_\star is an arbitrary length and ϕ is the Newtonian gravitational potential.

Having obtained the $^{(2)}g_{00}$ component, which determines the non-relativistic motion of massive particles, we now proceed to obtain the $^{(2)}g_{11}$ component. In the literature it is customary to define a new scalar potential ψ as:

$$^{(2)}g_{11} = -\frac{2\psi}{c^2}, \quad (9)$$

in complete analogy with the definition of the Newtonian gravitational potential ϕ . The introduction of this potential can be justified considering a more general scenario. Without requiring spherical symmetry, the spatial part of the metric can be written as $g_{ik} dx^i dx^k$, with $^{(0)}g_{kl} = \delta_{kl}$ being the Minkowskian part. The second order perturbation corrections of g_{kl} could in principle involve other potentials (and not only ϕ or ψ). By a suitable choice of coordinates, one can get rid of the anisotropic contributions at the same perturbation order, which turns g_{kl} into a diagonal form. Given the isotropy of space, there is no preferred direction and so $^{(2)}g_{ik} \propto \delta_{ik}$. It is natural to expect that the leading order $\mathcal{O}(2)$ correction must be of the same order of magnitude as the gravitational potential ϕ . Accordingly $g_{kl} = (1 + 2\gamma\phi/c^2) \delta_{kl}$, where γ is a proportionality constant, and so

$$ds^2 = g_{00}dt - (1 + 2\gamma\phi/c^2) \delta_{kl}dx^k dx^l. \quad (10)$$

Since spherical Schwarzschild coordinates are widely used in astrophysical literature, let us calculate the metric component g_{11} in such coordinates.

The conversion to spherical Schwarzschild coordinates is straightforward since:

$$g_{11}dr^2 + r^2d\Omega^2 = (1 + 2\gamma\phi/c^2) (d\tilde{r}^2 + \tilde{r}^2d\Omega^2), \quad (11)$$

for spherical isotropic coordinates $(ct, \tilde{r}, \theta, \varphi)$. Substitution of equation (8) in the previous relation yields:

$$r = \tilde{r} \left[1 - \gamma \left(\frac{G_M M^{1/2}}{c^2} \right) \ln \left(\frac{r}{r_\star} \right) \right], \quad (12)$$

and so,

$$dr = d\tilde{r} \left[1 - \frac{G_M M^{1/2}}{c^2} \ln \left(\frac{\tilde{r}}{r_\star} \right) - \frac{G_M M^{1/2}}{c^2} \right], \quad (13)$$

at perturbation order $\mathcal{O}(2)$. This means that:

$$\psi = -\gamma G_M M^{1/2}, \quad \text{and so} \quad {}^{(2)}g_{11} = -\frac{2\gamma G_M M^{1/2}}{c^2}. \quad (14)$$

4. Weak gravitational lensing at large scales

As pointed out by [17], recent observations have shown that gravitational lensing on individual, groups and clusters of galaxies can be modelled with the standard Schwarzschild solution of general relativity, assuming the existence of a total dark plus baryonic isothermal halo, where the Tully-Fisher law holds for the baryonic matter. As such, the bending angle of light can be calculated using the standard lensing equation. The result is that this bending angle does not depend on the impact parameter and scales with the square root of the total baryonic mass. Under a modified theory of gravity scheme with no dark matter component, the bending angle has the same value and so, since the time metric component ${}^{(2)}g_{00}$ is known from the Tully-Fisher law as shown in equation (8), then a reconstruction of the spatial metric component ${}^{(2)}g_{11}$ can be made [17].

To be more specific, take a static spherically symmetric total matter distribution M_T , assuming the validity of Einstein's general relativity so that Schwarzschild's metric holds. Therefore $g_{00S} = -1/g_{11S}$ and so:

$$g_{00S} = 1 - \frac{2r_g}{r} = 1 - \frac{2GM_T(r)}{c^2 r} = 1 - 2 \left(\frac{v}{c} \right)^2. \quad (15)$$

The subscript S identifies the coefficients of the Schwarzschild metric, and $M_T(r) = v^2 r / G$ refers to the hypothetical isothermal total matter distribution [cf. 18] needed to explain the observed lensing, when assuming general relativity. From this it follows that the dark matter hypothesis provides a self-consistent interpretation of observed phenomenology: the same dark matter halos, which are required to explain the observed rotation curves, have been solved for by analysing extensive lensing observations.

From equation (15) it follows that for isothermal total matter halos under Einstein's general relativity, the metric coefficient g_{00} does not depend on the radial coordinate. We can see this by using the empirical Tully-Fisher relation (1) between the velocity and the total baryonic mass in the last identity above. With the aid of the lens equation and making approximations to $\mathcal{O}(2)$ it follows that [17]:

$$g_{11}(r) = -1 - 2\left(\frac{v}{c}\right)^2 = -1 - \frac{2(GM_b a_0)^{1/2}}{c^2} \quad (16)$$

Thus, any metric theory of gravity where g_{11} matches the above expression in the regime where gravitational lenses are observed will accurately reproduce all the observed lensing phenomenology, with M_b the total baryonic mass of the object in question (galaxies or group of galaxies), and no hypothetical dark matter assumed to exist. Equations (8) and (16) give empirical mathematical relations for the metric coefficients at perturbation order $\mathcal{O}(2)$ which reproduce all observed rotation velocity and gravitational lensing phenomenology, without the inclusion of any dark matter component.

In terms of the non-relativistic scalar potentials ϕ and ψ defined the above results yield:

$$\phi = -G_M M^{1/2} \ln\left(\frac{r}{r_\star}\right), \quad \psi = -G_M M^{1/2}, \quad \text{and so} \quad \gamma = 1. \quad (17)$$

As shown by [12], this is an extremely important result meaning that the relativistic structure of the underlying theory of gravity at individual, groups and cluster of galaxies scales is compatible with that found in the solar system, where the PPN parameter γ has also a value of one. The difference however, lies on the fact that the gravitational potential ϕ appearing in equation (10) is not the one associated with Kepler's third law, but the one inferred from the Tully-Fisher law.

5. Discussion

The fact that the PPN parameter $\gamma = 1$ serves as a basis for the construction of any extended theory of gravity based on the Tully-Fisher relation and/or MOND. Theories of gravity built to account for MOND phenomenology with $\gamma \neq 1$ are not compatible with weak lensing observations of individual groups and clusters of galaxies and should be disregarded.

The metric extension of gravity by [19], built with the idea of extending MOND to a relativistic regime, is an example of a theory of gravity with $\gamma = 1$. As explained by (author?) [13, 20] this proposal can be only applied to systems with a high degree of symmetry. A much general approach to be published elsewhere is being developed in which this requirement is not necessary.

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References

- [1] Bouvard A 1821 *Tables Astronomiques Publiées Par Le Bureau Des Longitudes De France, Contenant Les Tables De Jupiter, De Saturne Et D'Uranus, Construites D'Après La Théorie De La Mécanique Céleste* (Bachelier et Huzard)
- [2] Kollerstrom N Neptune's discovery: The british case for co-prediction <http://www.ucl.ac.uk/sts/nk/neptune/chron.htm> accessed: 2014-01-27
- [3] Hsu J and Fine D 2005 *100 Years of Gravity and Accelerated Frames: The Deepest Insights of Einstein and Yang-Mills* Advanced series on theoretical physical science (World Scientific) ISBN 9789812563354

- [4] Clemence G M 1947 *Rev. Mod. Phys.* **19**(4) 361–364
- [5] Hernandez X and Jiménez M A 2012 *ApJ* **750** 9 (*Preprint* 1108.4021)
- [6] Hernandez X, Jiménez M A and Allen C 2013 *MNRAS* **428** 3196–3205 (*Preprint* 1206.5024)
- [7] Hernandez X, Jiménez M A and Allen C 2012 *European Physical Journal C* **72** 1884 (*Preprint* 1105.1873)
- [8] Famaey B and McGaugh S S 2012 *Living Reviews in Relativity* **15** 10 (*Preprint* 1112.3960)
- [9] Sotiriou T P and Faraoni V 2010 *Reviews of Modern Physics* **82** 451–497 (*Preprint* 0805.1726)
- [10] Capozziello S and de Laurentis M 2011 *Physics Reports* **509** 167–321 (*Preprint* 1108.6266)
- [11] Nojiri S and Odintsov S D 2011 *Physics Reports* **505** 59–144 (*Preprint* 1011.0544)
- [12] Mendoza S and Olmo G J 2014 *ArXiv e-prints* (*Preprint* 1401.5104)
- [13] Mendoza S 2015 *Canadian Journal of Physics* **93**(2) 217–231
- [14] Newton I 1729 *The Mathematical Principles of Natural Philosophy* 3rd ed The Mathematical Principles of Natural Philosophy (B. Motte)
- [15] Will C 1993 *Theory and Experiment in Gravitational Physics* (Cambridge University Press) ISBN 9780521439732
- [16] Landau L and Lifshitz E 1975 *The classical theory of fields* Course of theoretical physics (Butterworth Heinemann) ISBN 9780750627689
- [17] Mendoza S, Bernal T, Hernandez X, Hidalgo J C and Torres L A 2013 *MNRAS* **433** 1802–1812 (*Preprint* 1208.6241)
- [18] Binney J and Tremaine S 2008 *Galactic Dynamics: Second Edition* (Princeton University Press)
- [19] Bernal T, Capozziello S, Hidalgo J C and Mendoza S 2011 *Eur. Phys. J. C* **71** 1794 (*Preprint* 1108.5588)
- [20] Mendoza S 2012 *Open Questions in Cosmology* ed Olmo D G J (InTech) ISBN 978-953-51-0880-1 available from: <http://www.intechopen.com/books/open-questions-in-cosmology/extending-cosmology-the-metric-approach>