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On the dynamical content of MAGs

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Abstract. Adopting a procedure borrowed from the effective field theory prescriptions, we study the dynamics of metric-affine theories of increasing order, that in the complete version include invariants built from curvature, nonmetricity and torsion. We show that even including terms obtained from nonmetricity and torsion to the second order density Lagrangian, the connection lacks dynamics and acts as an auxiliary field that can be algebraically eliminated, resulting in some extra interactions between metric and matter fields.

Introduction. The intriguing choice to treat alternative theories of gravity by means of the Palatini approach, namely elevating the affine connection to the role of independent variable, contains the seed of some interesting (usually under-explored) generalizations of General Relativity, the metric-affine theories of gravity. The peculiar aspect of these theories is to provide a natural way for matter fields to be coupled to the independent connection through the covariant derivative built from the connection itself.

The simplest prototypical version of a metric-affine theory of gravity is the Einstein-Cartan-Sciama-Kibble (ECSK) theory [1, 2]. In ECSK theory, it is still possible to show that the independent (and not necessarily symmetric) connection can be algebraically eliminated in favour of the metric and its derivatives, plus matter fields [3]. This means that the connection is not propagating in the spacetime, but it is indissolubly confined inside matter configurations.

Interestingly, once the independent connection is introduced as a new variable, nothing prevents us from adding further scalar invariants in the Einstein-Cartan action. The action describing a metric-affine theory is assumed to be a suitable limit, at a certain order, of a some fundamental theory; one can then follow the standard approach of effective field theory, and consider all the operators having the dimension of the specific order of the approximation. In the present case of ECSK, these operators are all the possible second order (namely, all the operators of the same dimension of the scalar curvature) invariants that can be formulated starting from the structures (torsion and nonmetricity) induced by the non-Riemannian nature of the actual spacetime.

Generalized metric-affine theories of gravity have been recently studied from different perspectives. Apart from the huge amount of work done in Einstein-Cartan gravity, progress was also made for what concerns the higher order versions of these theories [4, 5], also attracted by the tantalizing possibility of relaxing the theoretical puzzles associated with the recent discovery of accelerated expansion. The dynamical content of a metric-affine theory of gravity was already analyzed in [6] including the set of second order corrections built with torsion tensor. Here we are going to extend the previous result to the most general second order metric-affine theory of gravity, including quadratic operators in nonmetricity and torsion [7]. Note that, as it will be showed on the basis of a dimensional analysis, this will not include the quadratic Ricci terms, whose presence would introduce *brevi manu* new dynamical degrees of freedom.

The general action. We refer the reader to [7] for the notation and the conventions used also here. In a Lagrangian formulation of a metric-affine theory of gravity, the general action will be of the form

$$\mathcal{S} = \int d^4x\sqrt{-g}[\mathcal{L}_G(g_{\mu\nu}, \Gamma^\gamma{}_{\alpha\beta}) + \mathcal{L}_M(g_{\mu\nu}, \Gamma^\gamma{}_{\alpha\beta}, \psi)]\,, \tag{1}$$

where g is the determinant of the metric, $\mathcal{L}_G(g_{\mu\nu}, \Gamma^\gamma{}_{\alpha\beta})$ and $\mathcal{L}_M(g_{\mu\nu}, \Gamma^\gamma{}_{\alpha\beta}, \psi)$ are respectively the gravitational and the matter Lagrangian density (where we have made explicit the dependence of the matter from the connection), and ψ is a convenient way to refer collectively to the matter fields included in the theory under scrutiny.

We want to study the dynamics of the most general lowest order theory associated with the gravitational Lagrangian $\mathcal{L}_G(g_{\mu\nu}, \Gamma^\gamma{}_{\alpha\beta})$. Our prescription for constructing such general action is based on power counting of the dimensions of the gravitational terms. In natural units, the metric tensor is adimensional, the connection has the dimension of the inverse of a length and consequently the Ricci tensor has dimension [length]$^{-2}$. Since the total action must be dimensionless, the Lagrangian density must have dimension [length]$^{-4}$. We can think the Lagrangian density as the product of a geometrical scalar invariant times an opportune overall constant, in the form of a power of a length L_P, to adjust the total dimension. For the Einstein-Cartan action, for example, the correct Lagrangian density is $\mathcal{L}_G^{EC} = \mathcal{R}/(16\pi L_P^2)$.

Discarding for simplicity a cosmological constant term, it is not possible to build a Lagrangian density whose geometrical factor has dimension [length]$^{-1}$. This is trivially seen for the Ricci tensor case that is already of dimension [length]$^{-2}$. Nonmetricity and torsion tensor are also of dimension [length]$^{-1}$, but it is not possible to form a scalar invariant from just one rank-three tensor saturated with the metric (rank two); for such reason they will appear only as quadratic terms, hence only as terms of order [length]$^{-2}$, that is at the same order of the Ricci scalar $\mathcal{R}$.

The most general gravitational action with a Lagrangian density of dimension [length]$^{-2}$ is

$$\mathcal{S}_G = \frac{1}{16\pi L_P^2}\int d^4x\sqrt{-g}\left(\mathcal{R} + \sum_i a_i Q^2_{(i)} + \sum_i b_i Q_{(i)} * S_{(i)} + \sum_i c_i S^2_{(i)}\right)\,, \tag{2}$$

where the last three terms are a symbolic representation of all the possible independent contractions that can be obtained from nonmetricity Q and torsion tensor S (the symbol "*" in (2) indicates the tensorial product between torsion and nonmetricity). Using the symmetries of these two (Q is symmetric with respect to its last two indices, while S is antisymmetric with respect to its first two indices), eleven different combinations are found (see [7]).

It is interesting to note that the full Lagrangian density is free of terms obtained by the covariant derivative of nonmetricity and torsion. The reason is easily understood. The independent connection can be decomposed in the Levi-Civita connection of the metric plus a combination of terms in Q and S called the distortion tensor

$$K_{\alpha\beta}{}^\gamma = \frac{1}{2}\left(-Q_\alpha{}^\gamma{}_\beta + Q^\gamma{}_{\beta\alpha} - Q_{\beta\alpha}{}^\gamma\right) + S_{\alpha\beta}{}^\gamma - S_\beta{}^\gamma{}_\alpha + S^\gamma{}_{\alpha\beta}\,. \tag{3}$$

Using this last condition, one can always think to write the covariant derivative with respect to the independent Γ as a Riemannian part (namely a covariant derivative with respect to the metric) plus another one encoding the non-Riemannian structures of the spacetime; the Riemannian derivative leads to a total divergence resulting in a surface term, the second part is instead constituted of contractions of the distortion tensor with Q and S, i.e. (quadratic) combinations of torsion and nonmetricity, already included in the general action above.

The dynamical content of the independent connection. We are mostly interested in solving the Palatini field equation for the connection and to re-express it as a function of matter fields plus Christoffel symbols of the metric. Without loss of generality, the global structure of this field equation is

$$\frac{1}{\sqrt{-g}}[\overset{\Gamma}{\nabla}_\lambda\,(\sqrt{-g}g^{\mu\nu}) - \overset{\Gamma}{\nabla}_\sigma\,(\sqrt{-g}g^{\sigma\mu})\delta^\nu_\lambda] + \sum_j \alpha_j\, Q^{(j)} + \sum_j \beta_j\, S^{(j)} = (8\pi L_P^2)\Delta^{\mu\nu}{}_\lambda\,, \tag{4}$$

where

$$\begin{aligned}\sum_j \alpha_j\, Q^{(j)} &= \alpha_1\delta^\mu{}_\lambda Q^\nu{}_\alpha{}^\alpha + \alpha_2 g^{\mu\nu}Q_\alpha{}^\alpha{}_\lambda + \alpha_3\delta^\nu{}_\lambda Q^\alpha{}_\alpha{}^\mu + \alpha_4 Q^{\mu\nu}{}_\lambda + \alpha_5 Q_\lambda{}^{\nu\mu} + \alpha_6 Q^{\nu\mu}{}_\lambda + \\ &\quad + \alpha_7\delta^\mu{}_\lambda Q^\alpha{}_\alpha{}^\nu + \alpha_8 g^{\mu\nu}Q_{\lambda\alpha}{}^\alpha + \alpha_9\delta^\nu{}_\lambda Q^\mu{}_\alpha{}^\alpha\,, \end{aligned}\tag{5}$$

$$\sum_j \beta_j\, S^{(j)} = \beta_1 g^{\mu\nu}S_{\lambda\alpha}{}^\alpha + \beta_2\delta^\nu{}_\lambda S^\mu{}_\alpha{}^\alpha + \beta_3\delta^\mu{}_\lambda S^\nu{}_\alpha{}^\alpha + \beta_4 S^\nu{}_\lambda{}^\mu + \beta_5 S^{\nu\mu}{}_\lambda + \beta_6 S^\mu{}_\lambda{}^\nu\,, \tag{6}$$

and α_j and β_j are some linear combinations of the primitive coefficients a_i, b_i and c_i used in equation (2). The right hand side of (4) is given by the so-called hypermomentum tensor $\Delta^{\mu\nu}{}_\lambda \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M(g,\Gamma,\psi)}{\delta\Gamma^\lambda{}_{\mu\nu}}$, that is, the tensor describing the intrinsic properties of matter as spin angular momentum, shear and dilation current [8]. Also note that

$$\frac{1}{\sqrt{-g}}[\overset{\Gamma}{\nabla}_\lambda\,(\sqrt{-g}g^{\mu\nu}) - \overset{\Gamma}{\nabla}_\sigma\,(\sqrt{-g}g^{\sigma\mu})\delta^\nu_\lambda] = \delta^\nu{}_\lambda Q_\alpha{}^{\alpha\mu} + \frac{1}{2}g^{\mu\nu}Q_{\lambda\alpha}{}^\alpha - Q_\lambda{}^{\mu\nu} - \frac{1}{2}\delta^\nu{}_\lambda Q^\mu{}_\beta{}^\beta\,, \tag{7}$$

that is, we can reformulate the Palatini equation just in terms of an expression linear in the torsion tensor and in the nonmetricity tensor, that we can symbolically rewrite as

$$\sum_j \tilde{\alpha}_j\, Q^{(j)} + \sum_j \beta_j\, S^{(j)} = (8\pi L_P^2)\Delta^{\mu\nu}{}_\lambda\,, \tag{8}$$

with $\tilde{\alpha}_3 = \alpha_3 + 1$, $\tilde{\alpha}_5 = \alpha_5 - 1$, $\tilde{\alpha}_8 = \alpha_8 + \frac{1}{2}$, $\tilde{\alpha}_9 = \alpha_9 - \frac{1}{2}$ and $\tilde{\alpha}_{j\neq 3,5,8,9} = \alpha_j$. We can contract this equation in three different independent ways: with the metric $g_{\mu\nu}$, with δ^λ_μ and with δ^λ_ν, resulting in a simple linear system whose solution can be written as follows

$$\begin{aligned} Q^{\alpha\rho}{}_\alpha &= (8\pi L_P^2)(A_1\Delta_\alpha{}^{\alpha\rho} + B_1\Delta^{\alpha\rho}{}_\alpha + C_1\Delta^{\rho\alpha}{}_\alpha)\,, \\ Q^{\rho\alpha}{}_\alpha &= (8\pi L_P^2)(A_2\Delta_\alpha{}^{\alpha\rho} + B_2\Delta^{\alpha\rho}{}_\alpha + C_2\Delta^{\rho\alpha}{}_\alpha)\,, \\ S^{\rho\alpha}{}_\alpha &= (8\pi L_P^2)(A_3\Delta_\alpha{}^{\alpha\rho} + B_3\Delta^{\alpha\rho}{}_\alpha + C_3\Delta^{\rho\alpha}{}_\alpha)\,. \end{aligned}\tag{9}$$

where A_i, B_i and C_i are some elementary, but rather lenghty, expressions of the coefficients $\tilde{\alpha}_i$ and β_i. We can now use these three equations to substitute the corresponding terms in (8); their contribution is fully determined by the matter content of the theory, so we can move them on the right hand side, where they are collectively denoted as "$[f(\text{traces of }\Delta)]^{\mu\nu}{}_\lambda$". What remains is the equation

$$\tilde{\alpha}_4 Q^{\mu\nu}{}_\lambda + \tilde{\alpha}_5 Q_\lambda{}^{\nu\mu} + \tilde{\alpha}_6 Q^{\nu\mu}{}_\lambda + \beta_4 S^\nu{}_\lambda{}^\mu + \beta_5 S^{\nu\mu}{}_\lambda + \beta_6 S^\mu{}_\lambda{}^\nu = (8\pi L_P^2)\Delta^{\mu\nu}{}_\lambda + [f(\text{traces of }\Delta)]^{\mu\nu}{}_\lambda. \tag{10}$$

The antisymmetric part with respect to $\{\mu\nu\}$ pairs of indices of the previous equation gives an equation to express the torsion tensor in terms of the antisymmetric part of nonmetricity tensor plus terms in hypermomenta

$$\beta_5 S^{\nu\mu}{}_\lambda + (\beta_6 - \beta_4)S^{[\mu}{}_\lambda{}^{\nu]} = (\tilde{\alpha}_6 - \tilde{\alpha}_4)Q^{[\mu\nu]}{}_\lambda + (8\pi L_P^2)\Delta^{[\mu\nu]}_{\ \ \lambda} + [f(\text{traces of }\Delta)]^{\mu\nu}{}_\lambda \equiv \Theta^{\mu\nu}{}_\lambda\,, \tag{11}$$

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that can be solved considering a suitable combination of the three different permutations of the indices $(\mu\nu\lambda) \to (\lambda\mu\nu)$ and $(\mu\nu\lambda) \to (\lambda\nu\mu)$. At the end, we get:

$$\begin{aligned} S_{\mu\nu\lambda} &= \frac{2\beta_5\Theta_{\mu\nu\lambda} - \beta_4(\Theta_{\mu\nu\lambda} - \Theta_{\lambda\mu\nu} + \Theta_{\lambda\nu\mu}) + \beta_6(\Theta_{\mu\nu\lambda} - \Theta_{\lambda\mu\nu} + \Theta_{\lambda\nu\mu})}{(\beta_4 + 2\beta_5 - \beta_6)(\beta_4 - \beta_5 - \beta_6)} \\ &= [\widehat{f}(\Delta)]_{[\mu\nu]\lambda} - \frac{2(\tilde{\alpha}_6 - \tilde{\alpha}_4)}{\beta_4 + 2\beta_5 - \beta_6} Q_{[\mu\nu]\lambda} \,. \end{aligned} \tag{12}$$

Since we are not interested in the exact form of the contribution of matter to the torsion tensor, we have here defined another tensor $[\widehat{f}(\Delta)]_{\mu\nu\lambda}$ that includes all the contributions coming from the hypermomenta in $\Theta_{\mu\nu\lambda}$. Note that the tensor $\Theta_{\mu\nu\lambda}$, and hence the torsion tensor $S_{\mu\nu\lambda}$, is linear in nonmetricity $Q_{\mu\nu\lambda}$. Using this expression in equation (10) to eliminate the torsion, we can rewrite it in the form

$$\xi_1 Q_{\mu\nu\lambda} + \xi_2 Q_{\lambda\nu\mu} + \xi_3 Q_{\nu\mu\lambda} = [\overline{f}(\Delta)_{\mu\nu\lambda}] \,, \tag{13}$$

where $\xi_i \equiv \xi_i(\tilde{\alpha}_j, \beta_k)$ are some coefficients determined by the equations (10) and (12) and $[\overline{f}(\Delta)]_{\mu\nu\lambda}$ is defined, in analogy to f and $\hat{f}$, as the collective contribution from matter to the right hand side of the expression; equation (13) can be now solved with respect to $Q_{\mu\nu\lambda}$ adding and subtracting the further two equations obtained permuting the indices $(\mu\nu\lambda) \to (\lambda\mu\nu)$ and $(\mu\nu\lambda) \to (\nu\mu\lambda)$. Having expressed nonmetricity in terms of just matter fields, we can reuse it in the equation for torsion to have another expression using just the matter fields. At the end, the total connection, that can be written in terms of Christoffel symbols of the metric plus distortion (where we recall that the distortion tensor (3) is a combination of nonmetricity and torsion), is hence reduced to a (not trivial) expression of metric with its derivatives and of matter fields under the guise of the hypermomenta combination.

An important point to stress is the following. Given a matter Lagrangian that does contain at most linear terms in the covariant derivative, and hence linear terms in the connection, our demonstration shows the lack of dynamics, and the consequent reduction to an auxiliary field, of $\Gamma^{\lambda}{}_{\mu\nu}$. On the other hand, some specific and exotic forms of matter can evade this fulfillment. Anyway, equations of motion of matter fields are required to be at most of second order, which forces the matter Lagrangian to contain only linear derivatives. Therefore, even in the most convoluted case, the hypermomentum tensor will be algebraic in the connection and the latter can be still eliminated at the component level.

What is the physical consequence of the lack of dynamics of the connection? Once it has been shown that the connection can be algebraically written in terms of derivatives of the metric and matter fields, it is clear that we can substitute all the terms explicitly dependent on connection (or torsion, or nonmetricity) in the field equation obtained varying with respect to the metric. Due to the extreme length of the equation, we will omit to write it here completely. It is anyway clear that, varying the general action (2) with respect to the metric, we will basically obtain terms that are quadratic in the connection (torsion/nonmetricity), and hence that will carry extra contributions of the kind "(hypermomentum)^2" to the effective stress energy tensor. This is similar to what happens in Einstein-Cartan theory as shown in [3], with the main difference that now, because of the presence of nonmetricity terms, the field equations will contain new terms of different nature, coupling matter fields (in the form of hypermomenta) to Christoffel symbols.

Higher orders terms. It is an easy task to argue that scalar Lagrangian corrections of the order $[\text{length}]^{-2n}$, with integer n, are the only ones that can be written starting from our elementary geometrical objects. This is essentially due to the fact that the only quantities carrying dimension $[\text{length}]^{-1}$ are odd-rank objects (torsion, nonmetricity and covariant derivative), and they cannot be trivially saturated with the (even-rank) metric. For such reason, the next-order invariants are all the possible terms with dimension [length]$^{-4}$. It goes beyond the scope of this paper to enumerate all the possible invariants of the fourth order (just for comparison, it should be taken into account that in a spacetime with torsion and vanishing nonmetricity there are 151 independent scalar invariants [9]). It is anyway possible to show that these terms are inevitably introducing further degrees of freedom, even assuming the simplifying hypothesis of matter fields not coupled to the connection. It is what for example occurs for the generalized Palatini theories considered in [10], whose specific choice of a Lagrangian density of the form $\mathcal{R} + \mathcal{R}_{\mu\nu}\mathcal{R}_{\kappa\lambda}(ag^{\mu\kappa}g^{\nu\lambda} + bg^{\mu\lambda}g^{\nu\kappa})$ has been shown to be equivalent to Einstein gravity plus a (dynamical) Proca vector field.

A rather peculiar case is the metric-affine version of $f(\mathcal{R})$ theories of gravity [5, 11]. The Ricci scalar $\mathcal{R}$ is invariant under the projective transformation

$$\Gamma^{\rho}{}_{\mu\nu} \to \Gamma^{\rho}{}_{\mu\nu} + \delta^{\rho}{}_{\mu}\xi_{\nu}\,, \tag{14}$$

where ξ_ν is an arbitrary covariant vector field. Consequently also any function of $\mathcal{R}$ will respect the same symmetry. While this issue is not a problem when matter does not couple to the connection, for a metric-affine theory this feature can lead to inconsistent field equations. In general the matter Lagrangian is not projective invariant, neither it is reasonable to restrict the matter content to those fields that are fulfilling this property. To circumvent the problem, it is necessary to break the projective invariance of the gravitational sector, fixing the four degrees of freedom of the transformation (related to the four components of the vector field ξ_ν) by a Lagrange multiplier. Note that the projective invariance is automatically broken if distortion-squared terms and, *a fortiori*, higher order curvature invariants are added to the action; such theories have been shown already to carry further dynamics.

Since the number of degrees of freedom to be fixed is four, and since the projective transformation suggests that the goal of breaking projective invariance should be achieved by constraining the connection, it is reasonable to propose an additional term in the gravitational Lagrangian involving a contraction of either nonmetricity tensor or torsion tensor. As already shown in [12], the term of the form $A^\mu Q_\mu \equiv A^\mu g^{\alpha\beta}Q_{\mu\alpha\beta}$, previously proposed in [13] is not suitable for a generic $f(\mathcal{R})$ metric-affine theories, since it requires the function of the Ricci scalar to reduce to the Einstein-Cartan term and the theory results to have no solutions of the field equations whenever the $f(\mathcal{R})$ is non-linear. Interestingly, the result is still valid even if we try to fix the four degrees of freedom through the other independent contraction of the nonmetricity tensor, namely adding to the Lagrangian density the Lagrange multiplier the $B^\mu \widetilde{Q}_\mu \equiv B^\mu g^{\alpha\beta}Q_{\alpha\beta\mu}$, as it can be easily proved: in a torsion-less theory without matter fields, the independent Levi-Civita connection is written as [7]

$$\Gamma^{\gamma}{}_{\alpha\beta} = \{^{\gamma}_{\alpha\beta}\} + \frac{1}{2}\left(-Q_{\alpha}{}^{\gamma}{}_{\beta} + Q^{\gamma}{}_{\beta\alpha} - Q_{\beta\alpha}{}^{\gamma}\right)\,; \tag{15}$$

on the other hand, the field equation of the connection reduces to the usual $f(\mathcal{R})$-Palatini equation[1], that can be solved with respect to the connection to give

$$\Gamma^{\gamma}{}_{\alpha\beta} = \{^{\gamma}_{\alpha\beta}\} + \frac{1}{2f'(\mathcal{R})}(2\partial_{(\alpha}f'(\mathcal{R})\delta^{\gamma}_{\beta)} - g^{\gamma\sigma}g_{\alpha\beta}\partial_\sigma f'(\mathcal{R}))\,. \tag{16}$$

Equating (15) and (16) gives a condition expressing the contribution to nonmetricity coming from the gravitational sector of the theory

$$-Q_{\alpha}{}^{\gamma}{}_{\beta} + Q^{\gamma}{}_{\beta\alpha} - Q_{\beta\alpha}{}^{\gamma} = \frac{1}{f'(\mathcal{R})}(2\partial_{(\alpha}f'(\mathcal{R})\delta^{\gamma}_{\beta)} - g^{\gamma\sigma}g_{\alpha\beta}\partial_\sigma f'(\mathcal{R}))\,; \tag{17}$$

[1] The condition about the absence of matter fields implies also the vanishing of the Lagrange multipliers A_μ and B_μ that can be proven to be proportional to the trace of the hypermomentum tensor.

we can find two independent expressions by contracting respectively α and β indices or α and γ in the last equation; the resulting conditions being

$$\begin{aligned} Q_\gamma - 2\widetilde{Q}_\gamma &= -\frac{2}{f'(\mathcal{R})}\partial_\gamma f'(\mathcal{R})\,, \\ -Q_\beta &= \frac{4}{f'(\mathcal{R})}\partial_\beta f'(\mathcal{R})\,. \end{aligned} \tag{18}$$

It is now clear that both the constraint $Q_\mu = 0$ or $\widetilde{Q}_\mu = 0$ lead to the same result $Q_\mu = \widetilde{Q}_\mu = \partial_\mu f'(\mathcal{R}) = 0$, that makes the theory obviously inconsistent since it forces the function $f(\mathcal{R})$ to be at most linear.

A viable alternative [5, 6] is the theory obtained constraining the trace of the torsion tensor $S_{\mu\rho}{}^\rho$ through the term $C^\mu S_\mu \equiv C^\mu S_{\mu\rho}{}^\rho$

$$\mathcal{S} = \frac{1}{16\pi L_P^2}\int d^4x\sqrt{-g}(f(\mathcal{R}) + C^\mu S_\mu) + \mathcal{S}_M(g_{\mu\nu}, \Gamma^\gamma{}_{\alpha\beta}, \psi)\,, \tag{19}$$

whose field equations are written as

$$\begin{aligned} f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} &= (8\pi L_P^2)T_{\mu\nu}\,, \\ -\overset{\Gamma}{\nabla}_\lambda\left(\sqrt{-g}f'g^{\mu\nu}\right) + \overset{\Gamma}{\nabla}_\sigma\left(\sqrt{-g}f'g^{\sigma\mu}\right)\delta^\nu{}_\lambda + 2\sqrt{-g}f'S^\mu{}_\lambda{}^\nu &= (8\pi L_P^2)\sqrt{-g}\left(\Delta^{\mu\nu}{}_\lambda - \frac{2}{3}\Delta^{\sigma[\nu}{}_\sigma\delta^{\mu]}{}_\lambda\right), \\ S_{\alpha\mu}{}^\alpha &= 0\,. \end{aligned} \tag{20}$$

Note that this choice is fully consistent: if we require vacuum solutions, the second of equations (20) simply reduces to the two conditions $S_{\mu\nu\rho} = 0$ and $\overset{\Gamma}{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) = 0$, that are the usual field equations found in Palatini-$f(\mathcal{R})$ theories of gravity. On the other side, this version of $f(\mathcal{R})$-metric-affine theories has the further feature to avoid propagation of torsion waves in vacuum. In fact (modulo the dependence of the matter Lagrangian on the covariant derivative, that must be at most linear), taking the antisymmetric part of second equation in (20) with respect to μ and ν, and adding suitable permutations of the obtained expression, we can show that torsion tensor is

$$S_{\mu\nu}{}^\lambda = \frac{8\pi L_P^2}{f'(\mathcal{R})}g^{\rho\lambda}(\Delta_{[\rho\mu]\nu} + \Delta_{[\nu\rho]\mu} - \Delta_{[\mu\nu]\rho})\,, \tag{21}$$

namely, the torsion tensor is algebraically defined by the antisymmetric part of the hypermomentum tensor, $\Delta^{[\mu\nu]}{}_\lambda$.

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