Extended Kerr–Schild spacetimes of any dimension

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Abstract. We study geometric and algebraic properties of extended Kerr–Schild spacetimes (xKS), i.e. an extension of the Kerr–Schild (KS) ansatz where, in addition to the null KS vector, a spacelike vector field appears in the metric. In contrast to the KS case, it turns out that xKS spacetimes with a geodetic KS vector are not necessarily algebraically special and we obtain, in general, only a necessary condition under which the KS vector is geodetic. However, it is shown that this condition becomes sufficient if we appropriately restrict the geometry of the null and spacelike vector fields. Examples of xKS spacetimes belonging to the Kundt class and also expanding xKS spacetimes, namely the CCLP black hole, are provided and briefly discussed.

1. Kerr–Schild spacetimes

The Einstein field equations are a very complex system of partial differential equations of the 2nd order and finding exact solutions is a considerably non-trivial task especially in dimension $n > 4$. A possible approach to this problem is to assume an appropriate form of the unknown metric and an important example of such a technique is the Kerr–Schild (KS) ansatz [1]

$$g_{ab} = \eta_{ab} - 2\mathcal{H} k_a k_b, \qquad k_a k^a = 0, \tag{1}$$

where η_{ab} is a flat background metric and the KS vector $\boldsymbol{k}$ is null with respect to both g_{ab} and η_{ab}. This choice ensures a simple form of the inverse metric

$$g^{ab} = \eta^{ab} + 2\mathcal{H} k_a k_b \tag{2}$$

and therefore the full metric corresponds exactly to its linear approximation around the flat background. The cosmological constant Λ can be included by taking the corresponding maximally symmetric spacetime as the background metric $\bar{g}_{ab}$

$$g_{ab} = \bar{g}_{ab} - 2\mathcal{H} k_a k_b. \tag{3}$$

The analysis of metrics (3) in [2, 3] shows that if $\boldsymbol{k}$ is geodetic, these metrics are algebraically special[1] with $\boldsymbol{k}$ being the multiple Weyl aligned null direction. It also turns out that non-expanding Einstein KS spacetimes are of the Weyl type N and belong to the Kundt class, on

[1] In the context of the algebraic classification of the Weyl tensor in arbitrary dimension based on the existence of preferred null directions and their multiplicity, see e.g. the recent review [4].

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the other hand, expanding Einstein KS spacetimes are of genuine types II or D and the optical matrix[2] $\rho_{ij} \equiv \ell_{a;b} m^a_{(i)} m^b_{(j)}$ satisfies the optical constraint [2, 7]

$$\rho_{ik}\rho_{jk} = \frac{\rho_{lk}\rho_{lk}}{(n-2)\theta}\rho_{(ij)} \tag{4}$$

implying

$$\rho_{ij} = \mathrm{diag}\left(\mathcal{M}_1, \dots, \mathcal{M}_p, \frac{1}{r}, \dots, \frac{1}{r}, 0, \dots, 0\right), \tag{5}$$

where

$$\mathcal{M}_\mu = \frac{1}{r^2 + a_\mu^2}\begin{bmatrix} r & a_\mu \\ -a_\mu & r \end{bmatrix}. \tag{6}$$

Despite the simplicity of the KS ansatz (3), the class of KS metrics contains many physically interesting exact solutions of four-dimensional general relativity and also some of their higher dimensional analogues such as, for instance, the Schwarzschild black hole, the Vaidya radiating star, the Kinnersley photon rocket, the Kerr–(A)dS rotating black hole and type N *pp*-waves, see e.g. [8]. In fact, the KS ansatz has led to the discovery of the rotating black holes in higher dimensional general relativity with a (non-)vanishing cosmological constant [9, 10], respectively, and has been successfully applied also in the context of higher order gravities such as the Gauss–Bonnet theory [11] or quadratic gravity [12, 13].

We are motivated to generalize the KS ansatz for several reasons. More general ansatz could lead to exact solutions of more general Weyl types, e.g. black rings which are of type I_i. Another reason is that although the Kerr–Newman black hole can be cast to the KS form

$$\eta_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{7}$$

$$k_a\,\mathrm{d}x^a = \mathrm{d}t + \frac{rx + ay}{r^2 + a^2}\,\mathrm{d}x + \frac{ry - ax}{r^2 + a^2}\,\mathrm{d}y + \frac{z}{r}\,\mathrm{d}z \tag{8}$$

$$\mathcal{H} = -\frac{r^2}{r^4 + a^2 z^2}\left(Mr - \frac{Q^2}{2}\right), \qquad A = \frac{Qr^3}{r^4 + a^2 z^2}\boldsymbol{k}, \tag{9}$$

an exact charged rotating black hole solution of higher dimensional Einstein–Maxwell theory is unknown. It is also known that a straightforward generalization of five-dimensional rotating black hole solutions of general relativity in the KS form to the Gauss–Bonnet theory [11] do not represent rotating black holes [14]. Moreover, as will be mentioned later, some already known exact solutions can be cast to an extended KS form.

In the following, we briefly present our main results published in [15].

2. Extended Kerr–Schild spacetimes

Let us consider extended Kerr–Schild (xKS) metrics as an extension of the KS ansatz in the form

$$g_{ab} = \bar{g}_{ab} - 2\mathcal{H}k_a k_b - 2\mathcal{K}k_{(a}m_{b)} \tag{10}$$

involving an additional unit spacelike vector $\boldsymbol{m}$

$$k^a k_a = 0, \qquad k^a m_a = 0, \qquad m^a m_a = 1, \tag{11}$$

where $\bar{g}_{ab}$ is a maximally symmetric background metric

$$\bar{g}_{ab} = \Omega\eta_{ab}, \qquad \eta_{ab}\,\mathrm{d}x^a\mathrm{d}x^b = -\mathrm{d}t^2 + \mathrm{d}x_1^2 + \ldots + \mathrm{d}x_{n-1}^2 \tag{12}$$

[2] Throughout the paper, we employ the higher dimensional generalization of the Newman–Penrose formalism [5, 6].

Journal of Physics: Conference Series **600** (2015) 012040 doi:10.1088/1742-6596/600/1/012040

with a corresponding conformal factor

$$\Omega_{\mathrm{M}} = 1, \quad \Omega_{\mathrm{dS}} = -\frac{(n-2)(n-1)}{2\Lambda {x_1}^2}, \quad \Omega_{\mathrm{AdS}} = \frac{(n-2)(n-1)}{2\Lambda t^2}. \tag{13}$$

The inverse metric can be expressed as

$$g^{ab} = \bar{g}^{ab} + \left(2\mathcal{H} - \mathcal{K}^2\right) k^a k^b + 2\mathcal{K} k^{(a} m^{b)}. \tag{14}$$

It is appropriate to identify the vectors $\boldsymbol{k}$, $\boldsymbol{m}$ with the null and spacelike frame vectors

$$\boldsymbol{k} \equiv \boldsymbol{\ell}, \qquad \boldsymbol{m} \equiv \boldsymbol{m}^{(2)} \tag{15}$$

and define indices $\tilde{\imath}, \tilde{\jmath} = 3, \ldots, n-1$ so that $\boldsymbol{m}$ is excluded in the notation $\boldsymbol{m}^{(\tilde{\imath})}$.

2.1. Geodeticity of the KS vector $\boldsymbol{k}$

In the case of KS spacetimes, the null KS vector $\boldsymbol{k}$ is geodetic if and only if the boost weight 2 component of the Ricci tensor $R_{00} = R_{ab} k^a k^b$ vanishes. For the xKS metric (10), R_{00} reads

$$R_{00} = 2\mathcal{H} L_{i0} L_{i0} - \frac{1}{2}\mathcal{K}^2 L_{\tilde{\imath}0} L_{\tilde{\imath}0} + \mathcal{K}(2L_{i(i}L_{2)0} + L_{\tilde{\imath}0} M_{\tilde{\imath}0} + \mathrm{D}L_{20}) + 2\mathrm{D}\mathcal{K} L_{20} \tag{16}$$

and therefore R_{00} vanishes if $\boldsymbol{k}$ is geodetic, but the converse implication does not hold.

If we appropriately restrict the arbitrariness in the choice of the vectors $\boldsymbol{k}$ and $\boldsymbol{m}$

$$k_{[a;b]} m^b = 0, \qquad (\zeta m_{[a})_{;b]} k^b = 0, \tag{17}$$

which can be expressed in terms of the Lie derivative as

$$\mathcal{L}_{\boldsymbol{m}} k_a = 0, \quad \mathcal{L}_{\boldsymbol{k}} (\zeta m_a) = 0, \tag{18}$$

the boost weight 2 component of the Ricci tensor reduces to

$$R_{00} = \left(2\mathcal{H} - \frac{1}{2}\mathcal{K}^2\right) L_{\tilde{\imath}0} L_{\tilde{\imath}0}. \tag{19}$$

Therefore, assuming $\mathcal{K}^2 \neq 4\mathcal{H}$, the KS vector $\boldsymbol{k}$ is geodetic if and only if R_{00} vanishes.

Note that, in the context of the Einstein field equations the vanishing of the boost weight 2 component of the Ricci tensor is related to the vanishing of the boost weight 2 component of the energy-momentum tensor $R_{00} = \kappa T_{00}$ and the case $R_{00} = 0$ not only includes the vacuum case, i.e. Einstein spacetimes

$$R_{ab} = \frac{2\Lambda}{n-2} g_{ab}, \tag{20}$$

but also an aligned Maxwell field

$$R_{ab} = \frac{\kappa}{4}\left(F^c{}_a F_{cb} - \frac{1}{2(n-2)} F^2 g_{ab}\right), \qquad F_{ab} k^b \propto k_a, \tag{21}$$

or aligned pure radiation

$$R_{ab} = \Phi k_a k_b. \tag{22}$$

Note also that the relation (18) is compatible with the optical constraint (4) if $\boldsymbol{m}$ does not correspond to any 2×2 block $\mathcal{M}_\mu$ in the optical matrix (5) implying $\boldsymbol{k}$ and $\boldsymbol{m}$ are surface forming.

2.2. Algebraic types of xKS spacetimes

As already mentioned above, KS spacetimes (3) of any dimension with a geodetic $\boldsymbol{k}$ are algebraically special. On the other hand, xKS spacetimes (10) with a geodetic $\boldsymbol{k}$ are in general of Weyl type I with $\boldsymbol{k}$ being the Weyl aligned null direction. However, in case one assumes the relation (18) between the vectors $\boldsymbol{k}$ and $\boldsymbol{m}$ holds, xKS spacetimes are algebraically special with $R_{0i} = 0$ if and only if the optical matrix ρ_{ij} and the function $\mathcal{K}$ take one of the following forms

$$\rho^{(1)}_{ij} = 0, \quad \mathcal{K}^{(1)} = c_1 r + c_2, \tag{23}$$

$$\rho^{(2)}_{ij} = \mathrm{diag}\left(\frac{1}{r}, 0, \ldots, 0\right), \quad \mathcal{K}^{(2)} = c_1 r + \frac{c_2}{r}, \tag{24}$$

$$\rho^{(3)}_{ij} = \frac{1}{1 + c_1^2 r^2}\,\mathrm{diag}\left(\begin{bmatrix} \frac{1}{r} & c_1 \\ c_1 & c_1^2 r \end{bmatrix}, 0, \ldots, 0\right), \quad \mathcal{K}^{(3)} = \frac{\sqrt{1 + c_1^2 r^2}}{c_2 r}, \quad c_1 \neq 0, \tag{25}$$

$$\rho^{(4)}_{ij} = \mathrm{diag}\left(0, \frac{1}{r}, \frac{1}{r + c_2}, \ldots, \frac{1}{r + c_p}, 0, \ldots, 0\right), \quad \mathrm{rank}\,\rho^{(4)}_{ij} \geq 1, \quad \mathcal{K}^{(4)} = c_1, \tag{26}$$

$$\rho^{(5)}_{ij} = \mathrm{diag}\left(\frac{1}{r}, \frac{1}{r + c_2}, \ldots, \frac{1}{r + c_p}, 0, \ldots, 0\right), \quad \mathrm{rank}\,\rho^{(5)}_{ij} \geq 2, \quad \mathcal{K}^{(5)} = c_1 r, \tag{27}$$

$$\rho^{(6)}_{ij} = \mathrm{diag}\left(\frac{1}{r}, \mathcal{M}, \ldots, \mathcal{M}\right), \quad \mathcal{K}^{(6)} = c_1 r + \frac{c_2}{r}, \quad (c_1 \neq 0) \wedge (c_2 \neq 0), \tag{28}$$

$$\mathcal{M} = \begin{bmatrix} s & A \\ -A & s \end{bmatrix}, \quad s = \frac{r}{r^2 + \frac{c_2}{c_1}}, \quad A = \sqrt{\frac{c_2}{c_1}}\frac{1}{r^2 + \frac{c_2}{c_1}}, \tag{29}$$

where r is an affine parameter along the null geodesics $\boldsymbol{k}$ and c_i are arbitrary scalar functions independent of r.

3. Kundt xKS spacetimes

It turns out that for Kundt xKS metrics the following statements are equivalent:

(i) the spacetime is algebraically special,

(ii) the boost weight 1 components of the Ricci tensor $R_{0i} \equiv R_{ab} k^a m^b_{(i)} = 0$ vanish,

(iii) the function $\mathcal{K}$ and the Ricci rotation coefficients $M_{\tilde{\imath}0}$ take one of the forms

$$\mathcal{K} = d\sqrt{(r + b)^2 + \mu_{\tilde{\jmath}}\mu_{\tilde{\jmath}}}, \quad M_{\tilde{\imath}0} = \frac{\mu_{\tilde{\imath}}}{(r + b)^2 + \mu_{\tilde{\jmath}}\mu_{\tilde{\jmath}}}, \tag{30}$$

or

$$\mathcal{K} = fr + e, \quad M_{\tilde{\imath}0} = 0, \tag{31}$$

where r is an affine parameter along the null non-expanding, non-shearing, and non-twisting geodesics $\boldsymbol{k}$ and b, d, e, f, $\mu_{\tilde{\imath}}$ are arbitrary scalar functions independent of r.

3.1. Examples of Kundt xKS spacetimes

The most straightforward examples of Kundt xKS spacetimes are metrics with vanishing scalar invariants (VSI) [16, 17]

$$\mathrm{d}s^2 = 2\mathrm{d}u\,\mathrm{d}r + \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + 2H(u, r, x^k)\,\mathrm{d}u^2 + 2W_i(u, r, x^k)\,\mathrm{d}u\,\mathrm{d}x^i \tag{32}$$

which are of the Weyl type III and clearly admit the xKS form (10) with the flat background $\bar{g}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = 2\mathrm{d}u\,\mathrm{d}r + \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ where

$$\mathcal{H} = -H, \quad k_a\,\mathrm{d}x^a = \mathrm{d}u, \quad \mathcal{K} = -\sqrt{W_i W_i}, \quad m_a\,\mathrm{d}x^a = \frac{W_i\,\mathrm{d}x^i}{\sqrt{W_j W_j}}. \tag{33}$$

Table 1. The relation of the class of higher dimensional Ricci-flat *pp*-waves to the classes of KS, xKS, and VSI spacetimes depending on the Weyl types.

Weyl type	KS	xKS	VSI
N	✓	✓	✓
III	×	✓	✓
II	×	only CSI	×

The class of *pp*-wave spacetimes is geometrically defined as metrics admitting a covariantly constant null vector field which can be written in the form [18]

$$\mathrm{d}s^2 = 2\mathrm{d}u\left[\mathrm{d}r + H(u,x^k)\,\mathrm{d}u + W_i(u,x^k)\,\mathrm{d}x^i\right] + g_{ij}(u,x^k)\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{34}$$

It can be shown that Einstein *pp*-waves are Ricci-flat and of the Weyl type II. Type N Ricci-flat *pp*-waves admit the KS form. Type III Ricci-flat *pp*-waves are VSI and therefore belong to the class of xKS spacetimes. Type II Ricci-flat *pp*-waves can be cast to the xKS form only if they have constant scalar invariants (CSI). The situation is summarized in table 1.

Note that four dimensional Ricci-flat *pp*-waves are of the Weyl type N, belong to the VSI class and take the KS form.

4. Example of expanding xKS spacetimes

The Chong–Cvetič–Lü–Pope charged rotating black hole [19] can be cast to the xKS form [20]

$$\begin{aligned}\bar{g}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = &-\left(1-\lambda r^2\right)\frac{\Delta}{\Xi_a\Xi_b}\,\mathrm{d}t^2 - 2\mathrm{d}r\left(\frac{\Delta}{\Xi_a\Xi_b}\,\mathrm{d}t - \frac{a\sin^2\theta}{\Xi_a}\,\mathrm{d}\phi - \frac{b\cos^2\theta}{\Xi_b}\,\mathrm{d}\psi\right)\\ &+\frac{\rho^2}{\Delta}\,\mathrm{d}\theta^2 + \frac{\left(r^2+a^2\right)\sin^2\theta}{\Xi_a}\,\mathrm{d}\phi^2 + \frac{\left(r^2+b^2\right)\cos^2\theta}{\Xi_b}\,\mathrm{d}\psi^2,\end{aligned} \tag{35}$$

$$k_a\,\mathrm{d}x^a = -\frac{\Delta}{\Xi_a\Xi_b}\,\mathrm{d}t + \frac{a\sin^2\theta}{\Xi_a}\,\mathrm{d}\phi + \frac{b\cos^2\theta}{\Xi_b}\,\mathrm{d}\psi, \tag{36}$$

$$\hat{m}_a\,\mathrm{d}x^a = \lambda ab\frac{\Delta}{\Xi_a\Xi_b}\,\mathrm{d}t + \frac{b\sin^2\theta}{\Xi_a}\,\mathrm{d}\phi + \frac{a\cos^2\theta}{\Xi_b}\,\mathrm{d}\psi, \tag{37}$$

$$\mathcal{H} = -\frac{M}{\rho^2} + \frac{Q^2}{2\rho^4}, \qquad \mathcal{K} = -\frac{Q\nu}{r\rho^2}, \qquad A = -\frac{\sqrt{3}Q}{2\rho}\boldsymbol{k} \tag{38}$$

where a and b are spins, M and Q is mass and charge, respectively, $\rho^2 = r^2+\nu^2$, $\Delta = 1+\lambda\nu^2$, $\Xi_a = 1+\lambda a^2$, $\Xi_b = 1+\lambda b^2$, and $\nu = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$. It is a solution of 5D minimal gauged supergravity which is equivalent to the Einstein–Maxwell–Chern–Simons theory with the Chern–Simons coefficient $\chi = 1$ and $\Lambda < 0$

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 2F_{ac}F_b{}^c - \frac{1}{2}F_{cd}F^{cd}g_{ab}, \qquad \nabla_b F^{ab} + \frac{\chi}{2\sqrt{3}\sqrt{-g}}\epsilon^{abcde}F_{bc}F_{de} = 0. \tag{39}$$

It can be shown that the CCLP black hole is of the Weyl type I_i and the vectors $\boldsymbol{k}$ and $\boldsymbol{m}$ satisfy the relation (18). Interestingly, the optical matrix takes the same form as for the 5D Kerr–(A)dS black hole

$$\rho_{ij} = \begin{pmatrix} \frac{1}{r} & 0 & 0 \\ 0 & \frac{r}{\rho^2} & \frac{\nu}{\rho^2} \\ 0 & -\frac{\nu}{\rho^2} & \frac{r}{\rho^2} \end{pmatrix} \tag{40}$$

and therefore the optical constraint (4) is met and the vectors $\boldsymbol{k}$ and $\boldsymbol{m}$ are surface-forming. In the case of the uncharged ($Q = 0$) and static ($\nu = 0$) limit, the metric reduces to the KS form (3) and is of the Weyl type D.

5. Conclusion

We believe that the xKS form (10) may lead to the discovery of new solutions of general relativity in higher dimensions in vacuum and also in the presence of matter fields aligned with the KS vector $\boldsymbol{k}$, such as aligned Maxwell field. Using the xKS ansatz, one could also obtain new vacuum solutions of more general theories of gravity, for instance, the Gauss–Bonnet theory or Lovelock gravities of higher order. We hope that the results of our analysis of xKS spacetimes [15] will be useful for finding such new solutions in a subsequent work.

Acknowledgments

The author acknowledges the support of the Albert Einstein Center for gravitation and astrophysics, Czech Science Foundation 14-37086G.

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