Cracking of Anisotropic Spheres in General Relativity Revisited

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Abstract. We have studied the stability in some anisotropic and isotropic matter configurations, by using the concept of cracking in General Relativity, which was conceived as an alternative and different approach to detect instabilities. It is based in the analysis of the radial forces that appear on the system after a perturbation capable of taking it out of its equilibrium state. We have studied the effects of local density perturbations on systems with "barotropic" equations of state. The considerations of the perturbations on the gradient of pressure lead to very different results than the previous works, and have found that not only anisotropic models may present cracking (or overturning), but it can also occur in isotropic matter configurations.

1. Introduction

The concepts of cracking and overturning were developed by Herrera and co-workers [1, 2, 3], with the aim of constituting an alternative way to determine instabilities on anisotropic matter configurations, and is complementary to the traditional formalism of Chandrasekhar [5, 6, 7]. The idea of studying instabilities around the appearance of cracking is very intuitive, it comes from the evaluation of the distribution of radial forces that appear on the system due to perturbations, and if there is a change of sign on this forces then it said that cracking occurs and we have an unstable configuration. The type of perturbation that are considered are capable of taking the system out of is equilibrium state so this distribution of forces may appear.

In the first contributions where the stability was studied with the cracking criterion [1, 2, 3], were analysed independent and simultaneous perturbations on the density and of the anisotropy of some spherical solutions, were it was assumed that the perturbations did not change the gradient of pressure. In a more recent contribution [8], were studied perturbations of the density on systems governed by barotropic equations of state, were the anistotropy was affected though the equations of state and not independently, however the type of perturbations were constant since they did not change the gradient of pressure.

In this paper, we will discuss how a different type of perturbations can be considered, following [8] we will also consider barotropic equations of state and will analyse perturbations of the density. The mean difference is that we want to consider local perturbations of the density, that are localized on a reduced region of the sphere. We will show the difference with the previous works when the effects of a local perturbation are considered. In section 2, we will describe the concepts of cracking and overturning, and will summarize the contributions around this topic. In section 3 will show the implications of local pertubations of the density in the stability of anisotropic and isotropic matter configurations using the concept of cracking.

2. Cracking of anisotropic compact objects

Let us consider an static spherically symmetric metric,

$$\mathrm{d}s^2 = \mathrm{e}^{2\nu(r)}\,\mathrm{d}t^2 - \mathrm{e}^{2\lambda(r)}\,\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{1}$$

and an anisotropic fluid,

$$T_{\mu\nu} = (\rho + P_\perp)u_\mu u_\nu - P_\perp g_{\mu\nu} + (P - P_\perp)v_\mu v_\nu\,, \tag{2}$$

where

$$u_\mu = (\mathrm{e}^{\nu}, 0, 0, 0)\,, \qquad v_\mu = (0, -\mathrm{e}^{\lambda}, 0, 0)\,, \tag{3}$$

ρ represents the energy density, P the radial pressure, and $P_\perp$ the tangential pressure of the fluid. If the system is in equilibrium, the generalization of the Tolman-Oppenheimer-Volkoff equation for an anisotropic fluid is satisfied,

$$\mathcal{R} = \frac{\mathrm{d}P}{\mathrm{d}r} + (\rho + P)\frac{m + 4\pi r^3 P}{r(r - 2m)} - \frac{2(P_\perp - P)}{r}, \tag{4}$$

with $\mathcal{R} = 0$, since this term represents the net radial forces in the system and must vanish if it is in equilibrium. Herrera and co-workers [1, 2, 3] stated that if there is a perturbation in the system capable of taking it out of the equilibrium state, then (4) may not be satisfied and a distribution of radial forces $\delta\mathcal{R}$ might appear. This authors defined cracking or overturning as change of sign within the configuration of this distribution of forces that appear after a perturbation in one or more variables of the system. Cracking occurs when the force is directed inward in the inner part of the configuration and changes sing at some point and then it is directed outward. Overturning occurs in the opposite case.

In the series of papers [1, 2, 3], were analysed simultaneous and independent perturbations of the density and the anisotropy which occurred in all the configuration and leaved the gradient of pressure invariant. As a concluding remark, they found that the perturbations of the anisotropy and not in the density may lead to cracking. By a different approach, [8] studied perturbations of the density $\rho \to \rho + \delta\rho$ in systems with barotropic equations of stated; i.e, $P = P(\rho)$, $P_\perp = P_\perp(\rho)$ which perturbed the anisotropy of the system according to the equations and not independently. This perturbations were constant and as consequence did not affect the pressure gradient. As a result, they obtained the following stability criterion

$$-1 \le v_\perp^2 - v^2 \le 1 \;\Rightarrow\; \begin{cases} -1 \le v_\perp^2 - v^2 \le 0 & \text{Potencially stable} \\ \;\;\,0 \le v_\perp^2 - v^2 \le 1 & \text{Potencially instable} \end{cases} \tag{5}$$

where v^2 and $v_\perp^2$ are defined by

$$v^2 = \frac{\mathrm{d}P}{\mathrm{d}\rho}, \qquad v_\perp^2 = \frac{\mathrm{d}P_\perp}{\mathrm{d}\rho}, \tag{6}$$

and denote the radial sound speed and the tangential sound speed, respectively.

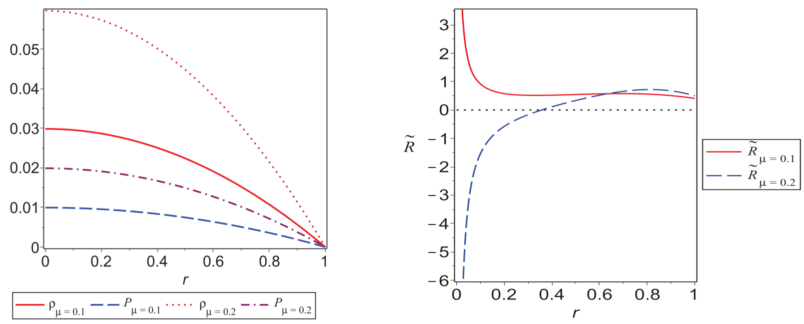

Figure 1. On the left density and pressure and on the right force distribution, $\widetilde{\mathcal{R}} = \delta\mathcal{R}/\delta\rho$, for the isotropic Mehra-Model [9] with $\mu = 0.1$ (solid line) and $\mu = 0.2$ (dashed line). The $\mu = 0.2$ curve presents a cracking point $r_c \approx 0.35$ which illustrates that cracking instability can be found for isotropic matter configurations.

3. Local Perturbations of the density

In this section, we will discuss the effects of local perturbations of the density in barotropic matter configurations and how the results differ from the results of the previous works. Following [8] we will consider systems with equations of state of the form $P = P(\rho)$ and $P_\perp = P_\perp(\rho)$, where a local perturbation of the density occurs; i.e, where the perturbation depends on the radial coordinate,

$$\rho \to \rho + \delta\rho(r), \tag{7}$$

and will be represented by a compact support function with the aims of representing a fluctuation that occurs in a finite and reduced part of the configuration, as any reasonable physical perturbation.

Now, since the pressures are related to the density through the equations of state, as well as the mass of the system,

$$m(\rho) = 4\pi \int_0^r \rho(\bar{r})\bar{r}^2 \mathrm{d}\bar{r}, \tag{8}$$

the effects of the perturbation on the radial pressure and its gradiente, the tangential pressure and the mass can be written as

$$P(\rho + \delta\rho) \approx P(\rho) + \delta P = P(\rho) + \frac{dP}{d\rho}\delta\rho = P(\rho) + v^2\delta\rho\,, \tag{9}$$

$$P_\perp(\rho + \delta\rho) \approx P_\perp(\rho) + \delta P_\perp = P_\perp(\rho) + \frac{dP_\perp}{d\rho}\delta\rho = P_\perp(\rho) + v_\perp^2\delta\rho\,, \tag{10}$$

$$P'(\rho + \delta\rho) \approx P'(\rho) + \delta P' = P'(\rho) + \frac{dP'}{d\rho}\delta\rho = P'(\rho) + \left[(v^2)' + v^2\frac{\rho''}{\rho'}\right]\delta\rho \tag{11}$$

and

$$m(\rho + \delta\rho) \approx m(\rho) + \delta m = m(\rho) + \frac{dm}{d\rho}\delta\rho = m(\rho) + \frac{4\pi r^2\rho}{\rho'}\delta\rho\,, \tag{12}$$

Now, we can find the distribution of radial forces that appear on the system due to the density perturbation by expanding equation (4) to first order,

$$\mathcal{R} \approx \mathcal{R}_0(\rho, P, P_\perp, P', m) + \delta\mathcal{R}, \tag{13}$$

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where

$$\delta\mathcal{R} = \frac{\partial\mathcal{R}}{\partial\rho}\delta\rho + \frac{\partial\mathcal{R}}{\partial P}\delta P + \frac{\partial\mathcal{R}}{\partial P_\perp}\delta P_\perp + \frac{\partial\mathcal{R}}{\partial P'}\delta P' + \frac{\partial\mathcal{R}}{\partial m}\delta m, \tag{14}$$

$$\delta\mathcal{R} = \delta\rho\left\{\frac{\partial\mathcal{R}}{\partial\rho} + \frac{\partial\mathcal{R}}{\partial P}v^2 + \frac{\partial\mathcal{R}}{\partial P_\perp}v_\perp^2 + \frac{\partial\mathcal{R}}{\partial m}\left[\frac{4\pi r^2\rho}{\rho'}\right] + \frac{\partial\mathcal{R}}{\partial P'}\left[(v^2)' + v^2\frac{\rho''}{\rho'}\right]\right\}, \tag{15}$$

where the definition $\rho = \frac{m'}{4\pi r^2}$ has been considered. The derivatives of the force distribution, $\mathcal{R}$, are given by

$$\frac{\partial\mathcal{R}}{\partial\rho} = \frac{4\pi r^3 P + m}{r(r-2m)}, \qquad \frac{\partial\mathcal{R}}{\partial m} = \frac{(\rho+P)(1+8\pi r^2 P)}{(2m-r)^2}, \tag{16}$$

$$\frac{\partial\mathcal{R}}{\partial P} = \frac{m + 4\pi r^3(\rho+2P)}{r(r-2m)} + \frac{2}{r}, \qquad \frac{\partial\mathcal{R}}{\partial P_\perp} = -\frac{2}{r} \quad \text{and} \quad \frac{\partial\mathcal{R}}{\partial P'} = 1\,. \tag{17}$$

Now, it can be seen that the expression for the distribution of forces $\delta\mathcal{R}$ is independent of the form of the perturbation of the density $\delta\rho$, and will be different than zero only in the region where the function of compact support $\delta\rho$ is defined. On the other hand, it can be noticed that other terms appear on the expression for the distribution of forces than the ones that appear on [8], since we have considered local perturbations which affect the gradient of pressure and the function of mass. It can also be noticed that for the case of isotropic configurations a change of sign can occur since there are positive and negative terms.

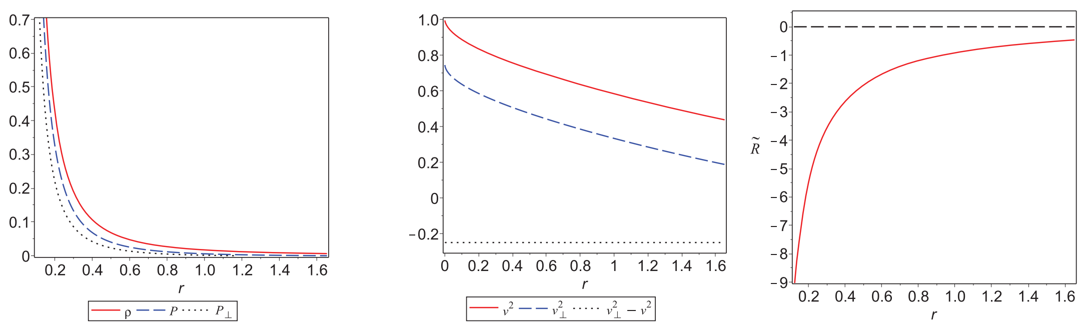

Figure 2. On the left density and pressures, on the middle speeds of sound, and on the right force distribution, $\widetilde{\mathcal{R}} = \delta\mathcal{R}/\delta\rho$, for the anisotropic Herrera-Model [1]. $\widetilde{\mathcal{R}}$ does not change its sign and the model could be considered as potential stable. This picture differs from the one presented in [1].

4. Detection of instabilities due to cracking

To study the effects of density perturbations on the stability of isotropic matter configurations, we examine a model proposed by Mehra [9] and considered physically viable by Delgaty and Lake[10]. Figure 1 displays the force distribution, $\widetilde{\mathcal{R}} = \delta\mathcal{R}/\delta\rho$, for the Mehra-model [9] with two different values of the mass-radius μ. It can be noticed that the curve for $\mu = 0.2$ changes its sign around $r \approx 0.35$, which means that cracking instability can be found for this isotropic matter configurations.

Now, let us examine an anisotropic sphere that was considered unstable in [1] under simultaneous density and anisotropic perturbations. As it can be appreciated from Figure 2

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the total distribution force $\widetilde{\mathcal{R}}$ does not change its sign, thus the model could be considered as potential stable under the present criterion, the same way as it was stable under the criterion of [8] given in equation (5). Another anisotropic solution, derived by Gokhroo and Mehra [11], originally found by Florides[12] and later by Stewart [13], was analysed and found unstable under the criterion of [8]. It is clear from Figure 3 that with the present refinement of local density perturbation - and assuming the same set of parameters: $\mu = 0.42$, $K = 3/56\pi$ and $\gamma = K/4$- this solution does not present any cracking instability.

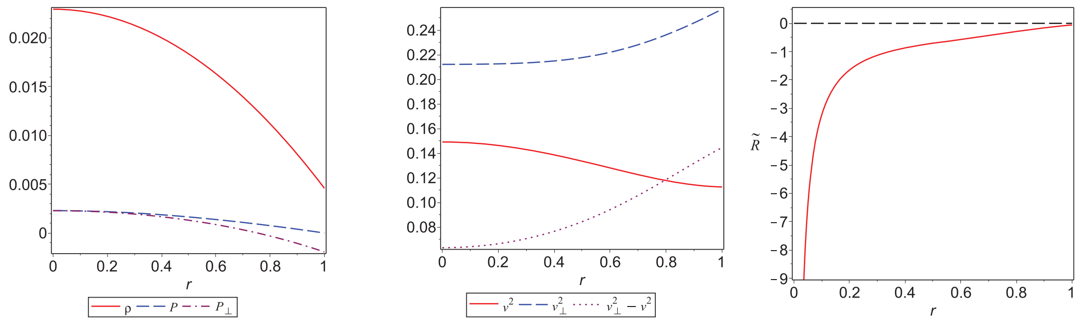

Figure 3. On the left density and pressures, on the middle speeds of sound and on the right force distribution, $\tilde{\mathcal{R}} = \delta\mathcal{R}/\delta\rho$, for the anisotropic Gokhroo/Mehra-Model [11] with: $\mu = 0.42$, $K = 3/56\pi$ and $\gamma = K/4$. Observe that it does not present any cracking instability reported in [8].

5. Remarks and conclusions

We have extended the cracking criterion for anisotropic and isotropic relativistic spheres under local perturbations of the density and have found that isotropic configurations may present cracking, in contrast to the results of the previous works. Also solutions that were unstable under the previous works are found to be stable according to our criterion, this demonstrates that local perturbations introduce changes in the distribution of forces, since they affect the gradient of pressure and the mass function.

It is worth to be mentioned that, the concept of cracking is complementary to the Bondi[4] and Chandrasekhar[5, 6, 7] stability criteria and it refers only to the tendency of the configuration to split (or to compress) at a particular point within the distribution but not to collapse or to expand. The cracking, overturning, expansion or collapse, has to be established from the integration of the full set of Einstein equations. Nevertheless, it should be clear that the occurrence of these phenomena could drastically alter the subsequent evolution of the system. If within a particular configuration no cracking (or overturning) is present, we could identify it as *potentially* stable, because other types of perturbations could lead to its expansions or collapse.

Acknowledgments

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