Non-additive probabilities and quantum logic in finite quantum systems

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Abstract. A quantum system $\Sigma(d)$ with variables in $\mathbb{Z}(d)$ and with Hilbert space $H(d)$, is considered. It is shown that the additivity relation of Kolmogorov probabilities, is not valid in the Birkhoff-von Neumann orthocomplemented modular lattice of subspaces $\mathcal{L}(d)$. A second lattice $\Lambda(d)$ which is distributive and contains the subsystems of $\Sigma(d)$ is also considered. It is shown that in this case also, the additivity relation of Kolmogorov probabilities is not valid. This suggests that a more general (than Kolmogorov) probability theory is needed, and here we adopt the Dempster-Shafer probability theory. In both of these lattices, there are sublattices which are Boolean algebras, and within these 'islands' quantum probabilities are additive.

1. Introduction

Quantum mechanics is a probabilistic theory, but quantum probabilities are more general than the standard (Kolmogorov) probabilities. Many authors have discussed various aspects of quantum probabilities[1, 2, 3]. Operational and convex geometry methods have been studied in [4, 5, 6, 7]. Test spaces have been studied in [8, 9], and fuzzy phase spaces in [10, 11]. Category theory methods have been studied in [12, 13]. We are interested in quantum probabilities as non-additive probabilities, and in particular in their link to logic. Non-additive probabilities describe aggregations, where 'the whole is greater than the sum of its parts'. In this general area, the Dempster-Shafer theory has been used extensively in Artificial Intelligence, Operations Research, Economics, etc[14, 15, 16, 17], and here we use it for quantum probabilities and link it to quantum logic. Quantum logic is based on the Birkhoff-von Neumann orthomodular lattice of closed subspaces of a Hilbert space and has been studied extensively in the literature[18, 19, 20, 21, 22].

Kolmogorov's probability theory is intimately related to set theory and Boolean algebra. It is a map from the powerset 2^Ω of some set Ω, to $[0, 1]$. The powerset 2^Ω is a Boolean algebra with subset ($\subseteq$), intersection ($\cap$) union ($\cup$), complement ($\overline{A} = \Omega - A$), as the logical connectives partial order ($\prec$), conjuction ($\wedge$), disjunction ($\vee$), negation ($\neg$), correspondingly. The additivity property of Kolmogorov probabilities is

$$\delta(A, B) = 0; \qquad \delta(A, B) = p(A \vee B) - p(A) - p(B) + p(A \wedge B), \tag{1}$$

where $A, B \subseteq \Omega$, and $p(A)$ are probabilities. From this follows that

$$p(A) + p(\overline{A}) = 1. \tag{2}$$

In recent work we have shown that the analogues of these additivity relations, are not valid in general, for quantum probabilities.

We consider a quantum system $\Sigma(d)$, with variables in $\mathbb{Z}(d)$ (the integers modulo d), described with the Hilbert space $H(d)$. We then consider two different lattices. The first lattice $\mathcal{L}(d)$ [23] is the Birkhoff-von Neumann orthomodular lattice of closed subspaces of $H(d)$, which in the finite case considered here, is actually the modular orthocomplemented lattice of subspaces. The second lattice $\Lambda(d)$ [24] is the distributive lattice (Heyting algebra) of subsystems of $\Sigma(d)$. A subsystem is a smaller system with variables in a subgroup of $\mathbb{Z}(d)$. The subgroups of $\mathbb{Z}(d)$ are $\mathbb{Z}(e)$ where $e|d$ (e is divisor of d), and therefore the subsystems of $\Sigma(d)$ are $\Sigma(e)$ with $e|d$. We will show explicitly how the states of a subsystem are embedded into the larger system. $\mathcal{L}(d)$ is a different lattice from $\Lambda(d)$. $\mathcal{L}(d)$ has an infinite number of elements, while $\Lambda(d)$ has a finite number of elements (corresponding to the divisors of d). Furthermore, the logical connectives in the two lattices are different.

In both of these lattices we show that the analogue of the additivity property $\delta(A, B) = 0$ is not valid in general. It is valid only within sublattices of $\mathcal{L}(d)$, and also sublattices of $\Lambda(d)$, which are Boolean algebras. As soon as we work with structures which are not Boolean algebras, the additivity property $\delta(A, B) = 0$ is not valid.

The Birkhoff-von Neumann lattice $\mathcal{L}(d)$ is not distributive, and this is a big deviation from Boolean algebras. Somebody might conjecture that the lack of distributivity is the only factor responsible for the non-additivity of quantum probabilities. For this reason we considered the second lattice which is distributive. But even in this case, quantum probabilities are non-additive.

There is a long history of the study of non-additive probabilities, and a practical scheme is the Dempster-Shafer theory[14, 15, 16, 17]. We adopt this scheme and interpret quantum probabilities as Dempster-Shafer probabilities. In the present paper we review and expand the work in [23, 24]. The emphasis is on the physical implications, and the interplay between quantum physics and logic (the formal proof of some statements, involves lattice theory techniques and has been given in [23, 24]). We also compare and contrast the two lattices, because there is different reason for the non-additivity of probability, in these two cases. In $\mathcal{L}(d)$ is non-commutativity, and in the Heyting algebra $\Lambda(d)$ the non-validity of the 'law of the exclusive middle' ($\neg a \vee a \prec \mathcal{I}$) and the fact that two negations of a statement are not equivalent to the statement ($a \prec \neg\neg a$).

2. The Dempster-Shafer theory: lower and upper probabilities
The Dempster-Shafer theory assigns two probabilities to a subset A of Ω. The lower probability (or belief) $\ell(A)$ and the upper probability (or plausibility) $u(A) = 1 - \ell(\overline{A})$. In Kolmogorov probability theory these two quantities are the same, due to Eq.(2). Kolmogorov probability theory is based on Boolean logic, according to which an element of Ω, belongs to either A or to $\overline{A}$. In Dempster-Shafer theory there are three categories: 'belongs to A', 'belongs to $\overline{A}$' and 'don't know'. In this case

$$\ell(A) + \ell(\overline{A}) \leq 1, \tag{3}$$

and the $1 - \ell(A) - \ell(\overline{A})$), corresponds to the 'don't know' category. The upper probability combines the 'belong to A' and the 'don't know'.

In his original work Dempster[14] considered Kolmogorov probabilities associated to subsets of a space X. He then considered a multivalued map from X to another space Ω. There is uncertainty related to probabilities in the space X, but there is an extra level of uncertainty due to the multivaluedness in the space Ω, which is sometimes called ambiguity in order to distinguish it from the original uncertainty. In a quantum context, the Dempster multivaluedness is the fact that a product of two classical quantities becomes the product of two operators which can be ordered in various ways.

An example similar to one in ref[25], is a company which does not know the exact age of its employees. It knows that n_1 employees are under 30 years old, n_2 employees are between 30 and 50, and n_3 are over 50. It wants to find the probability that a random employee is under 40 years old. The set A contains employees whose age is under 40. In this example, n_1 employees belong to A, n_2 employees belong to $\overline{A}$, and n_3 employees belong to Dempster's 'don't know' category. The lower probability is $n_1/(n_1 + n_2 + n_3)$ ('safe choice') and the upper probability $(n_1 + n_2)/(n_1 + n_2 + n_3)$.

The lower and upper probabilities obey the inequalities

$$\ell(A \cup B) - \ell(A) - \ell(B) + \ell(A \cap B) \geq 0$$
$$u(A \cup B) - u(A) - u(B) + u(A \cap B) \leq 0 \tag{4}$$

3. Non-commutativity in $\mathcal{L}(d)$ and the non-additivity operator $\mathfrak{D}(H_1, H_2)$

Let H_1, H_2 be subspaces of $H(d)$. We define the conjunction and disjunction as

$$H_1 \wedge H_2 = H_1 \cap H_2; \quad H_1 \vee H_2 = \mathrm{span}(H_1 \cup H_2). \tag{5}$$

We also use the notation $\mathcal{O} = H(0)$ (the space which has only the zero vector) and $\mathcal{I} = H(d)$, for the smallest and greatest elements in the lattice. We denote as $\Pi(H_1)$ the projector to the subspace H_1.

In the lattice $\mathcal{L}(d)$, we define the following 'non-additivity operator':

$$\mathfrak{D}(H_1, H_2) = \Pi(H_1 \vee H_2) + \Pi(H_1 \wedge H_2) - \Pi(H_1) - \Pi(H_2). \tag{6}$$

The $\mathrm{Tr}[\rho\mathfrak{D}(H_1, H_2)]$ with a density matrix ρ, is analogous to $\delta(A, B)$ and it indicates deviations from additivity by quantum probabilities. The disjunctions describe 'aggregations' of subsystems, and the non-additivity shows that an aggregation (or clustering or coalition) is different than a naive sum of its parts. The commutator $[\Pi(H_1), \Pi(H_2)]$ is related to $\mathfrak{D}(H_1, H_2)$, through the relation[23]:

$$[\Pi(H_1), \Pi(H_2)] = \mathfrak{D}(H_1, H_2)[\Pi(H_1) - \Pi(H_2)]. \tag{7}$$

This shows a strong link between non-commutativity and non-additivity of quantum probabilities. Within the lattice $\mathcal{L}(d)$, there are sublattices which are Boolean algebras, and there the projectors commute, the $\mathfrak{D}(H_1, H_2) = 0$ and quantum probabilities obey the additivity relation $\mathrm{Tr}[\rho\mathfrak{D}(H_1, H_2)] = 0$ which is analogous to $\delta(A, B) = 0$ for Kolmogorov probabilities. But in the full lattice both $[\Pi(H_1), \Pi(H_2)]$ and $\mathfrak{D}(H_1, H_2)$ are in general non-zero and and therefore $\mathrm{Tr}[\rho\mathfrak{D}(H_1, H_2)]$ is in general non-zero. Taking into account Eq.(4), we interpret a pair $\mathrm{Tr}[\Pi(H_1)\rho]$ and $\mathrm{Tr}[\Pi(H_2)\rho]$ of quantum probabilities, as upper or lower probabilities according to whether $\mathrm{Tr}[\rho\mathfrak{D}(H_1, H_2)]$ is negative or positive correspondingly.

3.1. Example

We consider the 3-dimensional space $H(3)$ and its subspaces

$$H_1 = \{a(1, 0, 0)\}; \quad H_2 = \{a(1, 1, 0)\}. \tag{8}$$

Here we give the general vector in these subspaces. In this case

$$H_1 \vee H_2 = \{(a, b, 0)\}; \quad H_1 \wedge H_2 = \mathcal{O}. \tag{9}$$

The corresponding projectors are

$$\Pi(H_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \Pi(H_2) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \Pi(H_1 \vee H_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (10)$$

Therefore

$$\mathfrak{D}(H_1, H_2) = \frac{1}{2}\begin{pmatrix} -1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (11)$$

We also consider the density matrix

$$\rho = \begin{pmatrix} a & 0 & 0 \\ 0 & 1-a & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad 0 \leq a \leq 1 \quad (12)$$

In this case $\text{Tr}[\rho\mathfrak{D}(H_1, H_2)] = (1/2) - a$ and the $\text{Tr}[\Pi(H_1)\rho]$ and $\text{Tr}[\Pi(H_2)\rho]$ are upper or lower probabilities in the cases that $a > (1/2)$ or $a < (1/2)$, correspondingly.

4. The Heyting algebra $\Lambda(d)$
In this section we study the lattice $\Lambda(d)$ of subsystems of $\Sigma(d)$. This is a distributive lattice, which is a Heyting algebra (every finite distributive lattice is a Heyting algebra). We show that the analogue of Eq.(1) is not valid, and we interpret the corresponding quantum probabilities as Dempster-Shafer probabilities. There are sublattices of $\Lambda(d)$ which are Boolean algebras (all chains) and in these 'islands' the probabilities obey Eq.(1), and they are additive (Kolmogorov) probabilities. But in the full lattice they cannot be interpreted as Kolmogorov probabilities. This shows that the distributivity property in a lattice, is not enough to make the corresponding probabilities to be Kolmogorov probabilities. We need to have a Boolean algebra. Probabilities in structures which are not Boolean algebras, are not Kolmogorov probabilities, and in this paper we interpret them as Dempster-Shafer probabilities.

For simplicity, we use the same symbols for the logical operations in the two lattices $\mathcal{L}(d)$ and $\Lambda(d)$, although they are different.

4.1. The lattice $\Lambda_1(d)$ of divisors of d
We consider the set of all divisors of d. This set with

$$a \wedge b = \text{GCD}(a,b); \quad a \vee b = \text{LCM}(a,b), \quad (13)$$

where GCD and LCM are the greatest common divisor and least common multiplier correspondingly, is a lattice which we denote as $\Lambda_1(d)$. The negation $\neg a$ is the largest divisor of d, which is coprime with a. The partial order in this lattice is divisibility ($e \prec d$ means $e|d$), and $\mathcal{I} = d$ and $\mathcal{O} = 1$ are the greatest and smallest elements. This lattice is a Heyting algebra and it is isomorphic to the lattice of subsystems below.

4.2. Subsystems of $\Sigma(d)$
A subsystem of $\Sigma(d)$, is a smaller system with variables in a subgroup of $\mathbb{Z}(d)$. The subgroups of $\mathbb{Z}(d)$, are the $\mathbb{Z}(e)$ with $e|d$. Therefore if $e|d$, the $\Sigma(e)$ is a subsystem of $\Sigma(d)$.

Let $|X_d; r\rangle$ where $r \in \mathbb{Z}(d)$, be an orthonormal basis in $H(d)$ which we call 'basis of position states'. The X_d in this notation is not a variable, but it indicates that they are position states in $H(d)$. $\Sigma(e)$ can be embedded into $\Sigma(d)$ as follows:

$$\sum_{r=0}^{e-1} a_r |X_e; r\rangle \;\; \rightarrow \;\; \sum_{r=0}^{d-1} a_r |X_d; \frac{rd}{e}\rangle; \quad e|d. \tag{14}$$

The set of subsystems of $\Sigma(d)$ with the logical operations

$$\begin{aligned}
\Sigma(e) \wedge \Sigma(f) &= \Sigma(e \wedge f) \\
\Sigma(e) \vee \Sigma(f) &= \Sigma(e \vee f) \\
\neg \Sigma(e) &= \Sigma(\neg e)
\end{aligned} \tag{15}$$

is a Heyting algebra with $\mathcal{O} = \Sigma(1)$ and $\mathcal{I} = \Sigma(n)$. It is isomorphic to $\Lambda_1(d)$ and we denote it as $\Lambda(d)$. The physical meaning of the logical operations is discussed in ref[26]).

For $e \prec f \prec d$, we define the projector from the subsystem $\Sigma(f)$ to a smaller subsystem $\Sigma(e)$, as

$$\varpi(f \rightarrow e) = \sum_{r=0}^{e-1} |X_f; \frac{fr}{e}\rangle \langle X_f; \frac{fr}{e}|; \quad e \prec f \prec d. \tag{16}$$

We also denote the $\varpi(d \rightarrow e)$ simply as $\varpi(e)$. Taking into account Eq.(14), we find the following compatibility condition:

$$\varpi(f \rightarrow e)\varpi(f) = \varpi(e); \quad e \prec f \prec d. \tag{17}$$

All these projectors commute with each other. Logical operations analogous to Eq.(15) can also be defined for the corresponding projectors. For example, $\varpi(e_1) \vee \varpi(e_2) = \varpi(e_1 \vee e_2)$ and $\varpi(e_1) \wedge \varpi(e_2) = \varpi(e_1 \wedge e_2)$. Also $\neg \varpi(e) = \varpi(\neg e)$ is projector to the space of the system $\neg \Sigma(e)$.

4.3. The non-additivity operator $\varpi_S(e_1, e_2)$
In [26] we have proved that

$$H(e_1 \vee e_2) = \text{span}[H(e_1) \cup H(e_2)] \oplus S(e_1, e_2). \tag{18}$$

where $S(e_1, e_2)$ is a s-dimensional space (where $s = e_1 \vee e_2 - e_1 - e_2 + e_1 \wedge e_2$), which is orthogonal to $\text{span}[H(e_1) \cup H(e_2)]$. Comparison of Eqs(5),(18) shows that the disjunction in the two lattices $\mathcal{L}(d)$ and $\Lambda(d)$ is different, and it proves again that the two lattices are different. The projector to the space $S(e_1, e_2)$ is

$$\varpi_S(e_1, e_2) = \varpi(e_1 \vee e_2) - \varpi(e_1) - \varpi(e_2) + \varpi(e_1 \wedge e_2). \tag{19}$$

Taking the trace of these projectors with a density matrix ρ we get probabilities:

$$\ell(e_1 \vee e_2) - \ell(e_1) - \ell(e_2) + \ell(e_1 \wedge e_2) = \text{Tr}[\rho \varpi_S(e_1, e_2)]. \tag{20}$$

This shows that these probabilities are not Kolmogorov probabilities. They are lower probabilities (the $\text{Tr}[\rho \varpi_S(e_1, e_2)]$ is non-negative) and the letter ℓ in the notation, indicates this. In the special case that $e_1 \prec e_2$ then $\varpi_S(e_1, e_2) = 0$. Therefore in chains within the lattice $\Lambda(d)$, the probabilities obey Eq.(1) and they can be interpreted as Kolmogorov probabilities.

We next insert $e_2 = \neg e_1$ in Eq.(19) and we get

$$\varpi_S(e, \neg e) = \varpi(e \vee \neg e) - \varpi(e) - \varpi(\neg e) + \varpi(1). \tag{21}$$

We use the notation $\neg \widetilde{\Sigma}(e)$ for the system $\neg \Sigma(e) = \Sigma(\neg e)$ without its lowest state $|X; 0\rangle$ (the corresponding projector is $\varpi(\neg e) - \varpi(1)$). In set theory (which is a Boolean algebra) 'do not belong to $\overline{A}$' is equivalent to 'belong to A'. Also $A \cup \overline{A} = \Omega$ and this leads to Eq.(2). In the present formalism (which is a Heyting algebra) 'does not belong in $\neg \widetilde{\Sigma}(e)$' is not equivalent to 'belongs in $\Sigma(e)$'. Firstly, $e \vee \neg e \neq \mathcal{I}$ in general (here $\mathcal{I} = d$), and secondly $\varpi_S(e, \neg e)$ is in general non-zero.

4.4. Example:
We consider the $\Lambda(12)$ which comprises of the subsystems of $\Sigma(12)$. The projectors to these subsystems are

$$\varpi(2) = |X_{12}; 0\rangle \langle X_{12}; 0| + |X_{12}; 6\rangle \langle X_{12}; 6|$$
$$\varpi(3) = |X_{12}; 0\rangle \langle X_{12}; 0| + |X_{12}; 4\rangle \langle X_{12}; 4| + |X_{12}; 8\rangle \langle X_{12}; 8|$$
$$\varpi(2) \vee \varpi(3) = \varpi(6) = \sum_{\nu=0}^{5} |X_{12}; 2\nu\rangle \langle X_{12}; 2\nu|$$
$$\varpi(2) \wedge \varpi(3) = \varpi(1) = |X_{12}; 0\rangle \langle X_{12}; 0| \tag{22}$$

In this case

$$\varpi_S(2, 3) = \varpi(2 \vee 3) - \varpi(2) - \varpi(3) + \varpi(2 \wedge 3) = |X_{12}; 2\rangle \langle X_{12}; 2| + |X_{12}; 10\rangle \langle X_{12}; 10|. \tag{23}$$

The heart of Boolean algebras is the 'law of the excluded middle' $\neg a \vee a = \mathcal{I}$, i.e., something is either true or false. This is not valid in Heyting algebras, where $\neg a \vee a \prec \mathcal{I}$. Equivalently, in Boolean algebras $\neg \neg a = a$, i.e., two negations are equivalent to the positive statement. In Heyting algebras this is not true ($a \prec \neg \neg a$).

We next exemplify the fact that 'does not belong in $\neg \widetilde{\Sigma}(e)$' is not equivalent to 'belongs in $\Sigma(e)$'. In $\Lambda(12)$, the spaces of subsystems $\Sigma(2)$ and $\neg \widetilde{\Sigma}(2) = \widetilde{\Sigma}(3)$ are span$\{|X_{12}; 0\rangle, |X_{12}; 6\rangle\}$ and span$\{|X_{12}; 4\rangle, |X_{12}; 8\rangle\}$, correspondingly. The state $|X_{12}; 5\rangle$ does not belong to either $\neg \widetilde{\Sigma}(2)$ or $\Sigma(2)$.

5. Discussion
Kolmogorov probabilities are intimately connected to Boolean logic, and their additivity property is that $\delta(A, B) = 0$ in Eq.(1). Quantum logic is based on the orthomodular lattice $\mathcal{L}(d)$ of Birkhoff and von Neumann. We have introduced the non-additivity operator $\mathfrak{D}(H_1, H_2)$ in Eq.(6), which quantifies the difference between an aggregation and the sum of its parts, and we have shown in Eq.(7) that it is related to the commutator of two projectors. Therefore non-commutativity is reponsible for the non-additivity of quantum probabilities. Within $\mathcal{L}(d)$ there are sublattices which are Boolean algebras and there the projectors commute and the corresponding probabilities are Kolmogorov probabilities. But in the full lattice $\mathcal{L}(d)$, we have interpreted quantum probabilities as Dempster-Shafer probabilities.

The lattice $\mathcal{L}(d)$ is not distributive (it is modular which is a weak version of distributivity). Lack of distributivity is a strong deviation from Boolean algebras. Therefore the question arises, if the lack of distributivity alone, is what makes quantum probabilities to be non-additive. In order to answer this question, we have considered the lattice $\Lambda(d)$ of subsystems, which is distributive (it is a Heyting algebra), and which is associated to commuting projectors. In

this case also the quantum probabilities are non-additive, and they can be interpreted as lower probabilities within the Dempster-Shafer theory. This is because 'does not belong in $\neg\widetilde{\Sigma}(e)$' is not equivalent to 'belongs in $\Sigma(e)$' (in Heyting algebras two negations of a statement is not the same as the original statement). In chains within $\Lambda(d)$ quantum probabilities are additive, i.e., they are Kolmogorov probabilities.

The work shows the non-additive nature of quantum probabilities, and their interpretation in the context of the Dempster-Shafer theory. We note that without additivity the concept of integration needs revision. Choquet integrals [27] is the appropriate concept in this case, and further work is needed in this direction.

References

[1] Lahti P and Maczynski J 1992 *J. Math. Phys.* **33** 4133
[2] Caticha A 1998 *Phys. Rev. A* **57** 1572
[3] Schack R, Brun T A and Caves M C 2001 *Phys. Rev. A* **64** 014305
[4] Ludwig G 1967 *Comm. Math. Phys.* **4** 331
[5] Davies E B and Lewis J T 1970 *Commun. Math. Phys.* **17** 239
[6] Mielnik B 1974 *Comm. Math. Phys.* **37** 221
[7] Busch P, Grabowski M and Lahti P J 1995 *Operational Quantum Physics* (Berlin: Springer)
[8] Foulis D J and Randall C H 1972 *J. Math. Phys.* **13** 1667
[9] Randall C H and Foulis D J 1973 *J. Math. Phys.* **14** 1472
[10] Twareque Ali S and Emch G G 1974 *J. Math. Phys.* **15** 176
[11] Twareque Ali S and Doebner H D 1976 *J. Math. Phys.* **17** 1105
[12] Abramsky S and Coecke B 2004 *Proc. 19th Annual IEEE Symposium on Logic in Computer Science* 415
[13] Coecke B and Ross D 2011 *New J. Phys.* **13** 043016
[14] Dempster A P 1967 *Annal. Math. Stat.* **38** 325
[15] Shafer G 1976 *A Mathematical Theory of Evidence* (Princeton: Princeton Univ. Press)
[16] Halpern J Y and Fagin R 1992 *Artificial Intelligence* **54** 275
[17] Yager R and Liu L (Ed.) 2008 *Classic works of the Dempster-Shafer theory of belief functions* (Berlin: Springer)
[18] Birkhoff G and von Neumann J 1936 *Ann. Math.* **37** 823
[19] Piron C 1976 *Foundations of quantum physics* (New York: Benjamin)
[20] Mackey G W 1963 *Mathematical foundations of quantum mechanics* (New York: Benjamin)
[21] Jauch J 1968 *Foundations of quantum mechanics* (Reading: Addison-Wesley)
[22] Beltrametti E and Cassinelli G 1981 *The Logic of Quantum Mechanics* (Reading: Addison-Wesley)
[23] Vourdas A 2014 *J. Math. Phys.* **55** 082107
[24] Vourdas A 2014 *J. Phys. A: Math. Theor.* **47** 345203
[25] Zadeh L A 1986 *Artificial Intelligence Magazine* **7** 85
[26] Vourdas A 2013 *J. Math. Phys* **54** 082105
[27] Choquet G 1953 *Ann. Inst. Fourier* **5** 131