

Hermite polynomials and representations of the unitary group

A Strasburger, D Dziewa-Dawidczyk

Department of Applied Mathematics, Warsaw University of Life Sciences (SGGW),
Nowoursynowska 159, 02-787 Warsaw, Poland

E-mail: aleksander_strasburger@sggw.pl, diana.dziewa-dawidczyk@sggw.pl

Abstract. Spaces of homogeneous complex polynomials in D variables form carrier spaces for representations of the unitary group $U(D)$. These representations are well understood and their connections with certain families of classical orthogonal polynomials (Gegenbauer, Jacobi, and other) are widely studied. However, there is another realization for the action of the unitary group $U(D)$ on polynomials, not necessarily homogeneous, in which Hermite polynomials in D variables play an important role. This action is related to the metaplectic (oscillator) representation, and was studied some time ago by one of the present authors (A. S.) and, independently, by A. Wünsche for $D = 2$. In this note we want to concentrate on the latter realization and describe its properties in a more comprehensible way.

1. Introduction

This note pursues the common lines of investigations started some time ago by one of the present authors, [1, 2], and about the same time approached from a rather different perspective by A. Wünsche, [4, 5].

2. Hermite polynomials in one and several variables

2.1. The one variable case

Due to their exceptionally wide applicability Hermite polynomials in one variable belong to the toolkit of practically every theoretical physicist. They may be approached from a number of different directions — eigenvalue problem of the quantum mechanical oscillator Hamiltonian, recurrence formulas or as an instance of a family of orthogonal polynomials, to mention just a few most popular viewpoints. For the sake of brevity, we quote here in addition to the basic explicit representation,

$$H_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n!}{k!(n-2k)!} (2x)^{n-2k}.$$

just two expressions, which are perhaps not so often remembered. An expression via a differential polynomial

$$H_n(x) = 2^n \left(\sum_{k=0}^{[n/2]} (-1)^k \frac{1}{2^{2k} k!} \frac{d^{2k}}{dx^{2k}} \right) x^n = \exp \left[-\frac{d^2}{2^2 dx^2} \right] x^n \quad (2.1)$$



and via the conjugation with the Gaussian

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx} \right)^n (e^{-x^2}) \quad (2.2)$$

This later one is basic for deriving the extension to several variables.

2.2. The case of several variables

We work in the space \mathbb{R}^d of real d dimension whose elements are denoted $x = (x_1, \dots, x_d)$, and use the standard multi-index notation. Given a d -tuple of non-negative integers $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d$, to be called a multi-index, we write

$$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \quad \text{and} \quad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}.$$

The sum $|\alpha| = \sum_{j=1}^d \alpha_j$ is called the height of the multi-index α . For any polynomial $P(x) = \sum p_\alpha x^\alpha$ on \mathbb{R}^d we shall denote by $P(\partial) = \sum p_\alpha \partial^\alpha$ the corresponding differential polynomial. In particular, if $r^2(x) = |x|^2 = \sum_{j=1}^d x_j^2$ is the square of the euclidean length, the corresponding differential polynomial $r^2(\partial) = \sum_{j=1}^d \partial_j^2 = \Delta$ is the usual Laplacian. A polynomial $P(x) = \sum p_\alpha x^\alpha$ is said to be a homogeneous of degree l if all its coefficients p_α are equal 0 unless $|\alpha| = l$. In this case the corresponding differential operator $P(\partial)$ is called homogeneous of degree l .

Finally, the algebra of complex valued polynomials in d variables is denoted by $\mathcal{P} = \mathcal{P}(\mathbb{R}^d)$ and its subspace of homogeneous of degree k elements by \mathcal{P}^k . Obviously each polynomial can be written as a sum of homogeneous polynomials in a unique way, hence

$$\mathcal{P} = \bigoplus_{k=0}^{\infty} \mathcal{P}^k.$$

Since the polynomial algebra $\mathcal{P}(\mathbb{R}^d)$ can be naturally identified with the algebra of symmetric tensors, it carries a natural representation of the full linear group $\mathbf{GL}(d, \mathbb{C})$. The subspaces \mathcal{P}^k are invariant and irreducible with respect to this action. Formally, it is induced by the natural action of $\mathbf{GL}(d, \mathbb{C})$ on the generators of the algebra, i.e. the first degree polynomial functions (coordinates) x_j , $j = 1, \dots, d$. It should be stressed, however, that it does not straightforwardly come from an action of $\mathbf{GL}(d, \mathbb{C})$ on the domain of polynomials, which in this case is \mathbb{R}^d .

The following property of polynomials in several variables is crucial in what follows. Recall that a function $f(x)$ (a polynomial) is called harmonic if it is annihilated by the Laplacian, $\Delta f = 0$. Now, any homogeneous polynomial $P \in \mathcal{P}^l(\mathbb{R}^d)$ can be uniquely decomposed as a sum of products of powers r^{2k} with homogeneous harmonic polynomials of degree $l - 2k$,

$$P = \sum_{k=0}^{[l/2]} r^{2k} h_{l-2k}(P), \quad \text{where} \quad \Delta(h_{l-2k}(P)) = 0. \quad (2.3)$$

This is called a canonical decomposition of P , cf. [14, 7].

In the common approach to Hermite polynomials in several variables, cf. [6], one views them as polynomial eigenfunctions of the operator

$$\Delta - 2E, \quad \text{where } E \text{ is the Euler operator} \quad E = \sum_{j=1}^d x_j \partial_j$$

By separation of variables it is immediately clear that they are linear combinations of products of Hermite polynomials in each coordinate separately,

$$H_\alpha(x) = H_{\alpha_1}(x_1) \cdots H_{\alpha_d}(x_d), \quad \text{where } \alpha = (\alpha_1, \dots, \alpha_d), \quad x = (x_1, \dots, x_d).$$

However it is not easy to make apparent properties of symmetry of so defined polynomials, e.g. for cases involving orthogonal or unitary symmetry. Thus another approach is desirable, which is better suited for expressing the action of symmetry groups. For that purposes the so called *Hermite map*, see [9, 7]), which is a suitable extension of the formula (2.2) is more appropriate.

Definition 1 (The Hermite map) Given a homogeneous polynomial $P(x)$ of degree l we assign it the polynomial H_P , which will be called Hermite polynomial associated with P , by means of the formula

$$H_P(x) = (-1)^l 2^{-l} e^{r^2} P(\partial) e^{-r^2}. \quad (2.4)$$

The Hermite map $P \mapsto H_P$ is a linear automorphism of the polynomial algebra. However, it does not preserve the grading with respect to the degree, nor the multiplication. In fact, using the Hobson formula one can derive the following formula (cf. [3])

$$H_P(x) = \sum_{k=0}^{[l/2]} (-1)^k \frac{1}{2^{2k} k!} \Delta^k P(x), \quad (2.5)$$

hence H_P is not necessarily homogeneous. We point out explicitly two special cases of the above:

For $P(x) = x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ there is $H_P(x) = H_\alpha(x) = H_{\alpha_1}(x_1) \cdots H_{\alpha_d}(x_d)$;

For $P(x) = r^{2l}(x)$ there is $H_P(x) = c_{l,d} L_l^{(d-2)/2}(r^2)$,

where $L_l^{(d-2)/2}(r^2) = \frac{1}{l!} e^{r^2} r^{2-d} \frac{d}{d(r^2)} e^{-r^2} r^{d-2+2l}$

is the classical Laguerre polynomial of degree l and index $(d-2)/2$.

2.3. Harmonic decomposition

In general there seem to be no simple and direct way to describe the elements of the space

$$\text{span} \{ H_{\alpha_1}(x_1) \cdots H_{\alpha_d}(x_d) \mid \alpha_1 + \dots + \alpha_d = l \} = \{ H_P \mid P \in \mathcal{P}^l \},$$

avoiding the use of factorization. However, one can accomplish it by relation to harmonicity.

Proposition 1 For a homogeneous polynomial $P \in \mathcal{P}^l$ with the harmonic decomposition (2.3) of the form

$$P = \sum_{k=0}^{[l/2]} r^{2k} h_{l-2k}(P),$$

the Hermite polynomial H_P associated to P has a decomposition

$$H_P = \sum_{k=0}^{[l/2]} (-1)^k k! L_k^{(\alpha+l-2k)}(r^2) h_{l-2k}(P). \quad (2.6)$$

In particular, the Hermite map equals the identity on the space of harmonic polynomials.

3. Defining an action of the linear group $\mathbf{GL}(d, \mathbb{C})$ on Hermite polynomials

3.1. Creation and annihilation operators

Let us briefly recall the formalism of creation and annihilation operators. We use a_j^+ , a_j to denote the standard creation and annihilation operators

$$a_j^+ = \frac{1}{\sqrt{2}}(x_j - \partial_j), \quad a_j = \frac{1}{\sqrt{2}}(x_j + \partial_j), \quad j = 1, \dots, d$$

The associative algebra \mathcal{W} they generate is called the *Weyl algebra* and is characterized by the Canonical Commutation Relations, thereafter referred to as CCR,

$$[a_j, a_k^+] = \delta_{jk} \mathbf{1}, \quad [a_j, a_k] = 0 = [a_j^+, a_k^+], \quad j, k = 1, \dots, d,$$

where δ_{jk} denotes the Kronecker delta. Moreover, a_j^+ and a_j are conjugate to each other with respect to the usual inner product in $L^2(\mathbb{R}^d)$, the space of functions in \mathbb{R}^d , which are square-integrable with respect to the Lebesgue measure. Explicitly, \mathcal{W} can be viewed as a algebra of partial differential operators with polynomial coefficients, cf. [11, 12]

3.2. The action of $\mathbf{GL}(d, \mathbb{C})$

Let $\mathcal{M}_{\mathbb{C}}$ be the span over \mathbb{C} of $\{a_j^+, a_j \mid j = 1, \dots, d\}$. The CCR impose on it a symplectic form given by

$$[X, Y] = iB(X, Y) \mathbf{1}, \quad X, Y \in \mathcal{M}_{\mathbb{C}}.$$

By virtue of general principles, every symplectic automorphism of $\mathcal{M}_{\mathbb{C}}$ can be uniquely extended to an automorphism of the Weyl algebra \mathcal{W} . Within the Weyl algebra we may form the symmetrized products of creation and annihilation operators,

$$q_{jk} = \frac{1}{2}(a_j a_k^+ + a_k^+ a_j) = a_k^+ a_j + \frac{1}{2} \delta_{jk}, \quad j, k = 1, \dots, d$$

By straightforward computation we see that

$$\begin{aligned} \text{ad } q_{jk}(a_l^+) &= [q_{jk}, a_l^+] = \delta_{jl} a_k^+ \\ \text{ad } q_{jk}(a_l) &= [q_{jk}, a_l] = -\delta_{kl} a_j. \end{aligned}$$

Thus we may view the elements q_{jk} as giving rise via the adjoint representation to the action of the general linear group $\mathbf{GL}(d, \mathbb{C})$ on $\mathcal{M}_{\mathbb{C}}$.

Proceeding more formally, we let $Q = (q_{jk})$ denote the $d \times d$ matrix with entries q_{jk} .

Proposition 2 *The elements q_{jk} , $j, k = 1, \dots, d$, are linearly independent and span over \mathbb{C} a Lie subalgebra \mathfrak{g} of \mathcal{W} isomorphic to the Lie algebra $\mathfrak{gl}(d, \mathbb{C})$ via the map*

$$\eta : \mathfrak{gl}(d, \mathbb{C}) \ni P \mapsto \text{tr}(PQ) = \sum_{j,k=1}^d p_{jk} q_{kj} \in \mathfrak{g}, \quad P = (p_{jk}).$$

It is worth noting that the Hermite operator — the (anti-hermitian) Hamiltonian of the d -dimensional quantum mechanical harmonic oscillator given by

$$K = \frac{i}{2}(r^2 - \Delta) = \frac{i}{2} \sum_{j=1}^d (a_j a_j^+ + a_j^+ a_j),$$

is a generator of the center of \mathfrak{g} .

To get the global, integrated form of the above action we proceed in the following way. Given a matrix $g = (g_{jk}) \in \mathbf{GL}(d, \mathbb{C})$ we denote its inverse by $g^{-1} = (G_{jk})$ and define $\rho(g) : \mathcal{M}_{\mathbb{C}} \rightarrow \mathcal{M}_{\mathbb{C}}$ by

$$\rho(g)a_j^+ = \sum_{k=1}^d g_{kj}a_k^+, \quad \rho(g)a_j = \sum_{k=1}^d G_{jk}a_k, \quad j = 1, \dots, d.$$

This is a symplectic automorphism of $\mathcal{M}_{\mathbb{C}}$, hence it extends to an automorphism of \mathcal{W} , which we also denote by $\rho(g)$. Thus $g \mapsto \rho(g)$ is a representation of the $\mathbf{GL}(d, \mathbb{C})$ by automorphisms of \mathcal{W} . Moreover, the mapping $P \mapsto \text{ad}(\eta(P))$ coincides with the differential $d\rho$ of the representation ρ defined above.

The representation ρ of $\mathbf{GL}(d, \mathbb{C})$ on \mathcal{W} constructed above leaves invariant (separately) the subspaces $\mathcal{P}[a_1^+, \dots, a_d^+]$ and $\mathcal{P}[a_1, \dots, a_d]$ of polynomials in $\{a_j^+ \mid j = 1, \dots, d\}$ and, resp. in $\{a_j \mid j = 1, \dots, d\}$ and acts by the standard tensorial (fully symmetrized) representation on $\mathcal{P}[a_1^+, \dots, a_d^+]$:

$$\rho(g)P(a_1^+, \dots, a_d^+) = P(\rho(g^{-1})a_1^+, \dots, \rho(g^{-1})a_d^+),$$

and its contragredient on $\mathcal{P}[a_1, \dots, a_d]$.

Remark 1 In the papers [4, 5] Wünsche attempted to define an action of the group $\mathbf{GL}(2, \mathbb{C})$ on polynomials and differential operators on \mathbb{R}^2 by acting formally on them with matrices $U \in \mathbf{GL}(2, \mathbb{C})$, disregarding the fact that there is no *bona-fide* representation of $\mathbf{GL}(2, \mathbb{C})$ in the space of real vectors \mathbb{R}^2 . The above definition from that point of view is more satisfactory, and allows to reproduce the formal computation from his papers in a more comprehensive way.

3.3. The conjugation with the Gaussian

Taking the clue from the construction of the Hermite map (2.4) we investigate the role played by the Gaussian. Since it is a smooth, rapidly vanishing function, it may be viewed as a multiplication operator on the space $L^2(\mathbb{R}^d)$ and thus it makes sense to compose it with other operators acting in that space.

Proposition 3 *Let us denote by G the map of \mathcal{W} into \mathcal{W} obtained by conjugation with the Gaussian, i.e.*

$$G : \mathcal{W} \ni D \mapsto G(D) \in \mathcal{W}, \quad G(D) = e^{r^2/2} \circ D \circ e^{-r^2/2}. \quad (3.1)$$

This map is an automorphism of the Weyl algebra such that for each homogeneous polynomial $P \in \mathcal{P}^l \subset \mathcal{W}$ the following holds

$$G(P(\partial)) = (-1)^l 2^{l/2} P(a^+), \quad G^{-1}(P(\partial)) = 2^{l/2} P(a).$$

Clearly, this follows from

$$\partial_j(e^{-r^2/2}f(x)) = e^{-r^2/2}(x_j - \partial_j)f(x) = e^{-r^2/2}2^{1/2}a_j^+f(x).$$

by taking into account that conjugation is an automorphism of the Weyl algebra. Similar reasoning holds for the annihilation operators a_j .

3.4. The decomposition of $L^2(\mathbb{R}^d)$ into eigenspaces of the harmonic oscillator Hamiltonian

Let us denote by $\phi_0 = e^{-r^2/2}$ the (non normalized) ground state of the d -dimensional harmonic oscillator and for any non-negative integer l denote by \mathcal{K}^l the space generated from ϕ_0 by the polynomials of degree l in creation operators,

$$\mathcal{K}^l = \mathcal{P}^l[a_1^+, \dots, a_d^+]\phi_0 \subset L(\mathbb{R}^d).$$

By the above constructions, the elements of that space are products of Hermite polynomials H_P with the Gaussian,

$$P(a^+)\phi_0 = 2^{l/2}H_P(x)\phi_0(x),$$

and more explicitly, in virtue of (2.6), they may be written in the form

$$\sum_{k=0}^{\lfloor l/2 \rfloor} L_k^{(\alpha+l-2k)}(r^2) Y_{l-2k}(x)$$

where $Y_{l-2k}(x)$ is a homogeneous harmonic polynomial of degree $l - 2k$. They are called Hermite–Weber functions — in physical terms they are eigenfunctions of the harmonic oscillator Hamiltonian K corresponding to the eigenvalue $l + d/2$. In fact, up to a factor $2^{-l/2}$

$$\frac{1}{2}(r^2 - \Delta)H_P(x)\phi_0(x) = (-i)KP(a^+)\phi_0 = (l + d/2)H_P(x)\phi_0(x).$$

Since $d\rho(A)$ for $A \in \mathfrak{gl}(d, \mathbb{C})$ acts by differentiations,

$$(d\rho(A)P)(a^+)\phi_0 + P(a^+)\eta(A)\phi_0 = \eta(A)P(a^+)\phi_0,$$

and in virtue of the fact that the ground state ϕ_0 is annihilated by all $\eta(A) \in \mathfrak{sl}(d, \mathbb{C})$, the map

$$\mathcal{P}^l \ni P \mapsto \Psi(P) = P(a^+)\phi_0 \in \mathcal{S}(\mathbb{R}^d)$$

satisfies

$$\Psi(d\rho(A)P) = \eta(A)\Psi(P), \quad A \in \mathfrak{sl}(d, \mathbb{C}), \quad P \in \mathcal{P}^l.$$

i.e. it intertwines the actions of $\mathfrak{sl}(d, \mathbb{C})$ — the natural action on polynomials $d\rho$ with the restriction of the *metaplectic* representation $\eta(A)$.

This implies that each eigenspace of the Hamiltonian is invariant under the action of the unitary group — a somewhat mysterious fact apparently observed for the first time by Jauch and Hill in [13].

3.5. Connection with Wünsche's approach

The role of the factorized Hermite polynomials $H_\alpha(x)$ is a consequence of the following

Corollary 1 *The Hermite–Weber functions*

$$H_\alpha(x)e^{-r^2/2} = H_{\alpha_1}(x_1) \cdots H_{\alpha_d}(x_d)e^{-r^2/2}, \quad |\alpha| = l$$

corresponding to the monomials $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ constitute the set of weight vectors of \mathcal{K}^l with respect to the standard maximal torus of $\mathbf{SU}(d)$ consisting of diagonal matrices.

Through some explicit computations for $d = 2$ Wünsche was able to express the transforms of $H_{\alpha_1}(x_1)H_{\alpha_2}(x_2)$ with the help of Jacobi polynomials. His results may be viewed in the perspective of deriving matrix elements for representations of the unitary group $\mathbf{SU}(d)$ and will be examined at another place.

References

- [1] Strasburger A 1998 Weyl algebra and a realization of the unitary symmetry *Rend. del Circ. Matem. di Palermo* serie II **Suppl. 54** 101–111
- [2] Strasburger A 1999 On algebras of creation and annihilation operators, in: *Algebraic Methods and q-special functions* (Vinet L and van Diejen J F, eds.) (Providence, Rhode Island: Amer. Math. Soc.) 267–276
- [3] Strasburger A 1991 Differential operators of gradient type associated with spherical harmonics *Annal. Polon. Math.* **53** 2 161–183
- [4] Wünsche A 2000 General Hermite and Laguerre two-dimensional polynomials *J. Phys. A: Math. Gen.* **33** 1603–1629
- [5] Wünsche A 2003 The role of Jacobi polynomials in the theory of Hermite and Laguerre 2D polynomials *J. Comp. Appl. Math.* **153** 521–529
- [6] Ch F Dunkl and Y Xu 2001 *Orthogonal polynomials of several variables* (Cambridge: Cambridge University Press)
- [7] Faraut J 1987 Analyse harmonique et fonctions spéciales in: Faraut J and Harzallah K *Deux Cours d'Analyse Harmonique* Ecole d'Été d'Analyse Harmonique de Tunis 1984 (Basel: Birkhäuser Verlag)
- [8] Folland G B 1989 *Harmonic analysis in the phase space* (Princeton: Princeton University Press)
- [9] Herz C 1965 *Analyse harmonique en plusieurs variables* (Orsay: Lecture notes)
- [10] Hobson E W 1955 *The theory of spherical and ellipsoidal harmonics* (Second ed.) (Cambridge: Cambridge University Press)
- [11] Howe R 1980 Quantum mechanics and partial differential equations *J. Funct. Anal.* **38** 188–254
- [12] Howe R 1985 Dual Pairs in Physics: harmonic oscillators, photons, electrons and singletons 179–207 in: *Applications of Group Theory in Physics and Mathematical Physics* Flato M et al. ed., *Lectures in Applied Mathematics* **21** (Providence, Rhode Island: Amer. Math. Soc.)
- [13] Jauch J M and Hill E L 1940 On the Problem of Degeneracy in Quantum Mechanics *Phys. Rev.* **57**, 641–645
- [14] Stein E M and Weiss G 1971 *Introduction to harmonic analysis on euclidean spaces* (Princeton: Princeton University Press)
- [15] Wawrzyńczyk A 1984 *Group Representations and Special Functions* (Dordrecht–Warszawa: Reidel D and PWN)