Richardson–Gaudin form of Bethe Ansatz solutions for an atomic-molecular Bose-Einstein condensate model

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Abstract. An integrable model describing the interconversion between Bose-Einstein condensates of atomic and molecular degrees of freedom is studied. An exact solution for this model has been previously derived via the algebraic Bethe Ansatz formalism. Here we derive an alternative solution of the Richardson–Gaudin form, and highlight some of the implications of the result.

1. Introduction

In 2001 Vardi, Yurovsky, and Anglin [1] presented one of the earliest and simplest models for an atomic-molecular Bose–Einstein condensate. It was subsequently reported in [2] that this model admits an exact solution via the algebraic Bethe Ansatz, which was used to investigate ground-state and dynamical properties. A feature of the exact solution is that it is obtained in the *quasi-classical limit*. The connection between Yang–Baxter integrability, which underlies the algebraic Bethe Ansatz approach, and Richardson-Gaudin systems via the quasi-classical limit, was observed by Sklyanin in 1989 [3]. Strictly speaking Skylanin's work made connection to Gaudin's algebraic approach [4], which itself referenced Richardson's earlier work [5,6]. It is now commonplace to refer to a broad class of models as Richardson–Gaudin systems. Specifically, we say that a Hamiltonian admits a Bethe Ansatz solution of Richardson–Gaudin form if the eigenvalues of the Hamiltonian are expressible as a function of the roots of the system of coupled non-linear equations

$$f(v_j) = \sum_{k\neq j}^{M} \frac{2}{v_k - v_j}, \qquad j = 1, ..., M, \tag{1}$$

where $f(v)$ is a rational function that is model dependent. See [7] for examples. We mention that the form (1) also accommodates *trigonometric* solutions through an appropriate variable change. See [4, 8] for a discussion of this point.

A later study established that there exists a more general integrable atomic-molecular Bose–Einstein condensate model, which contains that of [1] as a limiting case, involving atom-atom, atom-molecule, and molecule-molecule scattering interactions. This is discussed in [9] and details the derivation of the algebraic Bethe Ansatz solution, not of the form (1), based on use of the Yang–Baxter equation. Subsequently there have been several studies of the extended model described in [9] including the calculation of quantum dynamics [10], molecular conversion efficiency [11], quantum phase transitions [12,13], and the effects of particle losses [14]. However it seems that no studies have been made which exploit the existence of an exact solution. The objective of the present work is to aid progress in this direction by deriving an exact solution of the extended model of [9] which retains the Richardson-Gaudin form of exact solution (1). The motivation for achieving this is that it opens up an avenue to make exact quantitative calculations following the methods described in [15], which were guided by the qualitative observations of [16].

The approach used to derive this exact solution in the Richardson–Gaudin form does not rely on a solution of the Yang-Baxter equation [3], nor the use of Gaudin's algebra [4]. We instead exploit a correspondence between Richardson–Gaudin systems and ordinary differential equations. This correspondence has received significant attention as a tool for numerically solving the Bethe Ansatz equations associated with Richardson–Gaudin systems [2,17–23]. Here we invert this correspondence to *derive* the Bethe Ansatz equations for the Hamiltonian under consideration. The method, which provides a simple and direct derivation, is based on the following observation. A polynomial

$$Q(x) = \prod_{k=1}^{M} (x - v_k),$$

where the roots v_k are distinct, satisfies the equations

$$-\frac{Q''(v_j)}{Q'(v_j)} = \sum_{k\neq j}^{M} \frac{2}{v_k - v_j}, \qquad j = 1, ..., M, \tag{2}$$

where the prime denotes differentiation. Eq. (2) has the same generic form as Eq. (1).

2. The Hamiltonian

The model for condensed, interacting, atomic and molecular bosons is given by the Hamiltonian [9–14]

$$H = U_{aa}N_a^2 + U_{bb}N_b^2 + U_{ab}N_aN_b + \mu_aN_a + \mu_bN_b + \Omega\left(a^\dagger a^\dagger b + b^\dagger aa\right) \tag{3}$$

where a, $a^\dagger$ are the annihilation and creation operators for an atomic mode, and b, $b^\dagger$ are the annihilation and creation operators for a diatomic, homonuclear molecular mode. These satisfy the canonical commutation relations

$$\left[a,\, a^\dagger\right] = \left[b,\, b^\dagger\right] = I$$

where I denotes the identity operator. Moreover

$$\left[a,\ b\right] = \left[a,\, b^\dagger\right] = \left[a^\dagger, b\right] = \left[a^\dagger,\, b^\dagger\right] = 0.$$

The parameters μ_i are chemical potentials for species i and Ω is the amplitude for the interconversion of atoms and molecules. The parameters U_j are scattering couplings, taking into account atom-atom (U_{aa}), atom-molecule (U_{ab}), and molecule-molecule (U_{bb}) interactions. The Hamiltonian commutes with the total atom number $N = N_a + 2N_b$ where $N_a = a^\dagger a$ and $N_b = b^\dagger b$. The limiting case $U_{aa} = U_{ab} = U_{bb} = 0$ is the model introduced in [1].

The Bethe Ansatz solution as derived in [9] provides energy eigenvalues given by

$$\begin{aligned} E &= \sigma + \delta(M+\kappa) + \gamma(M+\kappa)^2 \\ &\quad + 2\eta^{-1}\kappa\Omega\left[(1-\eta\left(\alpha(M+\kappa)+\beta\right))\prod_{i=1}^{M}\frac{u_i-\eta}{u_i} - \prod_{i=1}^{M}\frac{u_i+\eta}{u_i}\right], \end{aligned}$$

where the parameters $\{u_i\}$ satisfy the Bethe Ansatz equations

$$[1-\eta u_i - \eta(\alpha(M+\kappa)+\beta)]\left(\frac{u_i+\eta\kappa}{u_i-\eta\kappa}\right) = \prod_{j\neq i}^{M}\frac{u_i-u_j-\eta}{u_i-u_j+\eta}. \tag{4}$$

The parameters above are related to the coupling parameters of the Hamiltonian through

$$\begin{aligned}
\eta &= \frac{4U_{aa}+U_{bb}-2U_{ab}}{2\Omega},\\
\alpha &= \frac{U_{bb}-4U_{aa}}{2\Omega},\\
\beta &= \frac{2\mu_b-4\mu_a+4U_{aa}-U_{ab}}{4\Omega},\\
\sigma &= \frac{U_{aa}-2\mu_a}{4},\\
\delta &= \frac{2\mu_b-U_{ab}}{2},\\
\gamma &= U_{bb}.
\end{aligned}$$

The allowed values for κ are $\kappa = 1/4,\ 3/4$, through which the total particle number is given by $N = 2M + 2\kappa - 1/2$.

Setting $U_{aa} = U_{ab} = U_{bb} = 0$ imposes $\eta = 0$, with which (4) becomes an identity. Furthermore setting $\mu_b = 0$ and expanding (4) in η yields the Bethe Ansatz solution in the quasi-classical limit:

$$E = 2\mu_a(M+\kappa-1/4) - 2\Omega\sum_{i=1}^{M} u_i, \tag{5}$$

where the parameters $\{u_i\}$ satisfy the Bethe Ansatz equations

$$\frac{\mu_a}{\Omega} - u_i + \frac{2\kappa}{u_i} = \sum_{j\neq i}^{M}\frac{2}{u_j-u_i}. \tag{6}$$

The above solution has the Richardson–Gaudin form (1). Taking into account different conventions used, this is equivalent to the solution provided in [2].

3. The Bethe Ansatz solution of the full Hamiltonian in Richardson–Gaudin form

Our objective now is to provide an alternative derivation of Bethe Ansatz solution for (3) which has the Richardson–Gaudin form (1). Let $|\mathrm{vac}\rangle$ denote the vacuum state. Since N is conserved, for each subspace of fixed N we have the basis states

$$|j\rangle = \left(a^\dagger\right)^{k+2j}\left(b^\dagger\right)^{M-j}|\mathrm{vac}\rangle \tag{7}$$

for $j = 0, ..., M$, $M = (N-k)/2$ where $k = 1$ for odd N and $k = 0$ for even N. This implies the identity

$$k(k-1) = 0 \quad \Leftrightarrow \quad k^2 = k$$

which will be assumed below. Let

$$U = U_{aa}N_a^2 + U_{ab}N_aN_b + U_{bb}N_b^2 + \mu_aN_a + \mu_bN_b.$$

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Then

$$U|j\rangle = \mathcal{U}_j|j\rangle$$

where

$$\begin{aligned}\mathcal{U}_j &= U_{aa}(k+2j)^2 + U_{ab}(k+2j)(M-j) + U_{bb}(M-j)^2 + \mu_a(k+2j) + \mu_b(M-j)\\ &= (4U_{aa} - 2U_{ab} + U_{bb})j(j-1) + (4(k+1)U_{aa} + (2M-k-2)U_{ab} + (1-2M)U_{bb} + 2\mu_a - \mu_b)j\\ &\quad + k^2U_{aa} + kMU_{ab} + M^2U_{bb} + k\mu_a + M\mu_b.\end{aligned}$$

We set

$$\begin{aligned}A &= 4U_{aa} - 2U_{ab} + U_{bb},\\ B &= 4(k+1)U_{aa} + (2M-k-2)U_{ab} + (1-2M)U_{bb} + 2\mu_a - \mu_b,\\ C &= k^2U_{aa} + kMU_{ab} + M^2U_{bb} + k\mu_a + M\mu_b.\end{aligned}$$

We also have

$$\begin{aligned}b^\dagger aa|j\rangle &= a\left(aa^\dagger\right)\left(a^\dagger\right)^{k+2j-1}\left(b^\dagger\right)^{M-j+1}|\mathrm{vac}\rangle\\ &= a\left(I+N_a\right)\left(a^\dagger\right)^{k+2j-1}\left(b^\dagger\right)^{M-j+1}|\mathrm{vac}\rangle\\ &= (4j(j-1) + (4k+2)j)|j-1\rangle,\\ a^\dagger a^\dagger b|j\rangle &= \left(a^\dagger\right)^{k+2(j+1)} bb^\dagger \left(b^\dagger\right)^{M-j-1}|\mathrm{vac}\rangle\\ &= \left(a^\dagger\right)^{k+2(j+1)}(I+N_b)\left(b^\dagger\right)^{M-j-1}|\mathrm{vac}\rangle\\ &= (M-j)|j+1\rangle.\end{aligned}$$

It is now seen that the action of (3) on the basis elements (7) is equivalent to the action of the differential operator

$$\mathbf{H} = \left(Ax^2 + 4\Omega x\right)\frac{d^2}{dx^2} + \left(Bx + \Omega(4k+2-x^2)\right)\frac{d}{dx} + (C + \Omega Mx)$$

on the basis of monomials through the identification $x^j \equiv |j\rangle$. Consider

$$\mathbf{H}Q(x) = EQ(x),$$

or equivalently

$$\left(Ax^2 + 4\Omega x\right)Q''(x) + \left(Bx + \Omega(4k+2-x^2)\right)Q'(x) + (C + \Omega Mx)\,Q(x) = EQ(x), \tag{8}$$

where

$$Q(x) = \prod_{j=1}^{M}(x - v_j).$$

Then

$$\begin{aligned}\frac{Bv_j + \Omega(4k+2-v_j^2)}{Av_j^2 + 4\Omega v_j} &= -\frac{Q''(v_j)}{Q'(v_j)}\\ &= \sum_{k\neq j}^{M}\frac{2}{v_k - v_j}.\end{aligned} \tag{9}$$

Finally, by equating the Mth-order terms in (8), with

$$Q(x) \sim x^M - x^{M-1} \sum_{j=1}^{M} v_j,$$

we obtain

$$E = AM(M-1) + BM + C - \Omega \sum_{j=1}^{M} v_j. \tag{10}$$

Eqs. (9,10) provide the Bethe Ansatz solution of (3) in Richardson–Gaudin form. It can be verified that setting $U_{aa} = U_{ab} = U_{bb} = \mu_b = 0$ leads to (9) and (10) reducing to (6) and (5) respectively, with the change of variables $v_j = 2u_j$ and $k = 2\kappa - 1/2$.

An alternative formulation is to instead adopt the basis states

$$|l\rangle = \left(a^\dagger\right)^{N-2l} \left(b^\dagger\right)^{l} |\mathrm{vac}\rangle \tag{11}$$

for $l = 0, ..., M$, $M = (N-k)/2$, and again $k = 1$ for odd N and $k = 0$ for even N. We set

$$\begin{aligned}
A &= 4U_{aa} - 2U_{ab} + U_{bb}, \\
\tilde{B} &= 4(N-1)U_{aa} - (N-2)U_{ab} - U_{bb} + 2\mu_a - \mu_b, \\
\tilde{C} &= N^2 U_{aa} + N\mu_a.
\end{aligned}$$

Now it can be shown that the action of (3) on the basis elements (11) is equivalent to the action of the differential operator

$$\tilde{\mathbf{H}} = \left(Ax^2 + 4\Omega x^3\right) \frac{d^2}{dx^2} + \left(-\tilde{B}x + \Omega(1 + (6-4N)x^2)\right) \frac{d}{dx} + \left(\tilde{C} + \Omega N(N-1)x\right) \tag{12}$$

on the basis of monomials through $x^l \equiv |l\rangle$. Consider

$$\tilde{\mathbf{H}} Q(x) = E Q(x)$$

where

$$Q(x) = \prod_{j=1}^{M} (x - w_j).$$

Then by the same considerations as above we find

$$\begin{aligned}
\frac{-\tilde{B}w_j + \Omega(1 + (6-4N)w_j^2)}{Aw_j^2 + 4\Omega w_j^3} &= -\frac{Q''(w_j)}{Q'(w_j)} \\
&= \sum_{k \neq j}^{M} \frac{2}{w_k - w_j}
\end{aligned} \tag{13}$$

and

$$E = AM(M-1) - \tilde{B}M + \tilde{C} - (4k+2)\Omega \sum_{j=1}^{M} w_j. \tag{14}$$

Set $w_j = v_j^{-1}$. Through standard algebraic manipulations it can be shown that (9) and (13) are equivalent by noting that

$$B = \tilde{B} - A(N-k-2),$$
$$C = AM(M-1) - \tilde{B}M + \tilde{C}.$$

Using (13) we may write

$$-\tilde{B} + \Omega(w_j^{-1} + (6-4N)w_j) = (Aw_j + 4\Omega w_j^2)\sum_{k\neq j}^{M} \frac{2}{w_k - w_j},$$

from which it follows that

$$-\tilde{B}M + \Omega\sum_{j=1}^{M} v_j - (4k+2)\Omega\sum_{j=1}^{M} w_j = -AM(M-1).$$

This leads to the conclusion that (10) and (14) are equivalent.

Eqs. (9) and (10), or equivalently Eqs. (13) and (14), provide the Bethe Ansatz solution of the Hamiltonian (3) in Richardson–Gaudin form. As mentioned earlier, the motivation for obtaining this form is that it opens up an avenue to make exact quantitative calculations for the ground-state energy and particular expectation values. This can be achieved following the methods described in [15] used for similar models. We look forward to reporting on developments in this direction in future work.

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