

# Second order symmetries of the conformal laplacian and $R$ -separation

Jean-Philippe Michel<sup>1</sup>, Fabian Radoux<sup>2</sup> and Josef Silhan<sup>3</sup>

<sup>1</sup> Institut de recherche en mathématique et physique (IRMP), Université Catholique de Louvain (UCL), Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

<sup>2</sup> Department of Mathematics, University of Liège, Grande Traverse 12, 4000 Liège, Belgium

<sup>3</sup> Department of Algebra and Geometry of the Masaryk University in Brno, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic

E-mail: jean-philippe.michel@ulg.ac.be, fabian.radoux@ulg.ac.be, silhan@math.muni.cz

**Abstract.** Let  $(M, g)$  be an arbitrary pseudo-Riemannian manifold of dimension at least 3, let  $\Delta := \nabla_a g^{ab} \nabla_b$  be the Laplace-Beltrami operator and let  $\Delta_Y$  be the conformal Laplacian. In some references, Kalnins and Miller provide an intrinsic characterization for  $R$ -separation of the Laplace equation  $\Delta\Psi = 0$  in terms of second order conformal symmetries of  $\Delta$ . The main goal of this paper is to generalize this result and to explain how the (resp. conformal) symmetries of  $\Delta_Y + V$  (where  $V$  is an arbitrary potential) can be used to characterize the  $R$ -separation of the Schrödinger equation  $(\Delta_Y + V)\Psi = E\Psi$  (resp. the Schrödinger equation at zero energy  $(\Delta_Y + V)\Psi = 0$ ). Using a result exposed in our previous paper, we obtain characterizations of the  $R$ -separation of the equations  $\Delta_Y\Psi = 0$  and  $\Delta_Y\Psi = E\Psi$  uniquely in terms of (conformal) Killing tensors pertaining to (conformal) Killing-Stäckel algebras.

## 1. Introduction

We work over a pseudo-Riemannian manifold  $(M, g)$  of dimension  $n \geq 3$ , with Levi-Civita connection  $\nabla$  and scalar curvature  $Sc$ . The main result of [1] was the classification of all the second order differential operators  $D_1$  such that the relation

$$\Delta_Y D_1 = D_2 \Delta_Y \quad (1)$$

holds for some differential operator  $D_2$ , where  $\Delta_Y := \nabla_a g^{ab} \nabla_b - \frac{n-2}{4(n-1)} Sc$  is the conformal Laplacian. Such operators  $D_1$  are called conformal symmetries of order 2 of  $\Delta_Y$ . They preserve the kernel of  $\Delta_Y$ , i.e. the solution space of the equation  $\Delta_Y \psi = 0$ ,  $\psi \in \mathcal{C}^\infty(M)$ .

The main goal of this paper is to explain how the second-order (resp. conformal) symmetries of  $\Delta_Y + V$  can be used to characterize the existence of  $R$ -separating coordinates systems for the Schrödinger equation  $(\Delta_Y + V)\Psi = E\Psi$  (resp. the Schrödinger equation at zero energy  $(\Delta_Y + V)\Psi = 0$ ), extending in this way the results in [2] and [3], where the authors characterized the  $R$ -separation of the Laplace equation  $\Delta\Psi = 0$  and the Helmholtz equation  $\Delta\Psi = E\Psi$  using (conformal) symmetries of the Laplace-Beltrami operator  $\Delta$ .

The paper is organized as follows.

In the first section, in a first step, we recall briefly the basic notions that are necessary to understand the paper: the conformal Laplacian  $\Delta_Y$ , (conformal) symmetries of  $\Delta_Y$ , (conformal)



Killing tensors. In a second step, we recall the characterization of the existence of (conformal) symmetries of  $\Delta_Y$  obtained in [1]. This characterization uses an obstruction operator defined in [1] from curvature tensors. Then, we recall the structure of (conformal) symmetries of  $\Delta_Y$  given in [1] thanks to the natural and conformally invariant quantization.

In the second section, we recall first the definition of the  $R$ -separability of the equations  $(\Delta_Y + V)\Psi = E\Psi$  and  $(\Delta_Y + V)\Psi = 0$ . Next, we give the definitions of (conformal) Killing-Stäckel algebras which can be used to characterize the additive separation of the Hamilton-Jacobi equation. We give then the characterizations of the  $R$ -separation of the equations  $(\Delta_Y + V)\Psi = E\Psi$  and  $(\Delta_Y + V)\Psi = 0$  in terms of these (conformal) Killing-Stäckel algebras and of (conformal) symmetries of  $\Delta_Y + V$ . Using a result in [1], we obtain then characterizations of the  $R$ -separation of the equations  $\Delta_Y\Psi = 0$  and  $\Delta_Y\Psi = E\Psi$  uniquely in terms of (conformal) Killing tensors belonging to (conformal) Killing-Stäckel algebras. Finally, we give a sufficient (but not necessary) condition which ensures the  $R$ -separation of the equations  $\Delta_Y\Psi = E\Psi$  and  $\Delta_Y\Psi = 0$ . This condition is expressed uniquely in terms of components of curvature tensors and is similar to the Robertson condition which ensures the separation of the Schrödinger equation  $\Delta\Psi = E\Psi$ .

In the appendices, we give the proof of the main theorem giving the characterization of the  $R$ -separation of the equation  $(\Delta_Y + V)\Psi = 0$  in terms of conformal symmetries of  $\Delta_Y + V$  (Theorem 5). The proof is divided in two steps: first, we characterize the  $R$ -separation of  $(\Delta_Y + V)\Psi = 0$  in terms of the notions of conformal Stäckel metric and pseudo-Stäckel multiplier. We can notice that this characterization was already obtained by the authors of [4] by means of a definition of  $R$ -separation different from ours (but equivalent). In a second step, we prove Theorem 5 by using the previous characterization and by adapting the proof given in [2] of the characterization of the  $R$ -separation of the equation  $\Delta\psi = 0$ .

## 2. Second order symmetries of the conformal Laplacian

We are going to recall the fundamental notions that are necessary to understand the paper. The reader who would to know more details about them is invited to consult [1].

### 2.1. The conformal Laplacian

Starting from a pseudo-Riemannian manifold  $(M, g)$  of dimension  $n$ , the conformal Laplacian  $\Delta_Y$  is defined in the following way:

$$\Delta_Y := \nabla_a g^{ab} \nabla_b - \frac{n-2}{4(n-1)} \text{Sc},$$

where  $\nabla$  denotes the Levi-Civita connection of  $g$  and  $\text{Sc}$  the scalar curvature.

### 2.2. The algebra of symmetries of the conformal Laplacian

The symmetries of  $\Delta_Y$  are defined as differential operators that commute with  $\Delta_Y$ . Hence, they preserve the eigenspaces of  $\Delta_Y$ .

More generally, conformal symmetries  $D_1$  are defined by the weaker algebraic condition

$$\Delta_Y \circ D_1 = D_2 \circ \Delta_Y, \quad (2)$$

for some differential operator  $D_2$ , so that they only preserve the kernel of  $\Delta_Y$ .

### 2.3. Killing tensors and conformal Killing tensors

**Definition 1.** Let  $(M, g)$  be a pseudo-Riemannian manifold and  $H = g^{ij} p_i p_j$  the function on  $T^*M$  associated with  $g$ , where  $(x^i, p_i)$  denotes the canonical coordinates on  $T^*M$ . If we denote

by  $\mathcal{S}_0$  the algebra of polynomial functions on  $T^*M$ , the algebra of symmetries of the null geodesic flow of  $g$ , denoted by  $\mathcal{K}$ , is given by the following subalgebra of  $\mathcal{S}_0$ :

$$\mathcal{K} = \{K \in \mathcal{S}_0; \{H, K\} \in (H)\}.$$

A function in  $\mathcal{K}$  can be viewed as a conformal Killing tensor, when we view it as a symmetric contravariant tensor.

**Definition 2.** The algebra of symmetries of the geodesic flow of  $g$  is given by the set of functions in  $\mathcal{S}_0$  that Poisson commute with  $H$ .

An element of this algebra can be viewed as a Killing tensor, when we view it as a symmetric contravariant tensor.

#### 2.4. Structure of the second order symmetries of $\Delta_Y$

In the sequel,  $\mathcal{S}_\delta^k$  denotes the space of polynomial functions of degree  $k$  on  $T^*M$  with values in the space of  $\delta$ -densities whereas  $\mathcal{D}_{\lambda,\mu}^k$  denotes the space of differential operators of order  $k$  acting between  $\lambda$  and  $\mu$ -densities, with  $\mu - \lambda = \delta$ .

Let us recall the formula giving the natural and conformally equivariant quantization at the order two (see [5]):

**Theorem 1.** [5] Let  $\delta \notin \{\frac{2}{n}, \frac{n+2}{2n}, 1, \frac{n+1}{n}, \frac{n+2}{n}\}$ . A natural and conformally invariant quantization  $\mathcal{Q}_{\lambda,\mu} : \mathcal{S}_\delta^{\leq 2} \rightarrow \mathcal{D}_{\lambda,\mu}^2$  is provided, on a pseudo-Riemannian manifold  $(M, g)$  of dimension  $n$ , by the formulas

$$\begin{aligned} \mathcal{Q}_{\lambda,\mu}(f) &= f \\ \mathcal{Q}_{\lambda,\mu}(X) &= X^a \nabla_a + \frac{\lambda}{1-\delta} (\nabla_a X^a) \\ \mathcal{Q}_{\lambda,\mu}(S) &= S^{ab} \nabla_a \nabla_b + \beta_1 (\nabla_a S^{ab}) \nabla_b + \beta_2 g^{ab} (\nabla_a \text{Tr} S) \nabla_b \\ &\quad + \beta_3 (\nabla_a \nabla_b S^{ab}) + \beta_4 g^{ab} \nabla_a \nabla_b (\text{Tr} S) + \beta_5 \text{Ric}_{ab} S^{ab} + \beta_6 \text{Sc} (\text{Tr} S) \end{aligned} \quad (3)$$

where  $f, X, S$  are symbols of degrees 0, 1, 2 respectively and  $\text{Tr} S = g_{ab} S^{ab}$ . The coefficients  $\beta_i$  entering the last formula are given e.g. in [1] (see Theorem 2.4).

The principal symbol of a (conformal) symmetry of  $\Delta_Y$  has to be a (conformal) Killing tensor. On an arbitrary pseudo-Riemannian manifold, a (conformal) Killing tensor has to satisfy some condition to be the principal symbol of a (conformal) symmetry. This condition can be expressed by means of a natural and conformally invariant operator which is denoted here by **Obs** and which is defined below. In the following definition,  $C$  and  $A$  denote respectively the Weyl tensor and the Cotton-York tensor.

**Definition 3.** The operator **Obs** is defined as follows:

$$\mathbf{Obs} : \mathcal{S}_0^2 \rightarrow \mathcal{S}_{2/n}^1 : S \mapsto \frac{2(n-2)}{3(n+1)} \mathbf{F}(S),$$

where  $(\mathbf{F}(S))^a = C^r_{st}{}^a \nabla_r S^{st} - 3A_{rs}{}^a S^{rs}$ .

In the following statement,  $\mathcal{Q}_{\lambda_0, \lambda_0}$  denotes the natural and conformally invariant quantization introduced in Theorem 1. If we denote by  $F_{\frac{2}{n}}$  the fiber bundle of  $\frac{2}{n}$ -densities, the isomorphism  $\Gamma(TM \otimes F_{\frac{2}{n}}) \cong \Gamma(T^*M)$  provided by the metric is denoted by  $^b$  (see [1] for more details).

**Theorem 2.** The second order (conformal) symmetries of  $\Delta_Y$  are exactly the operators

$$\mathcal{Q}_{\lambda_0, \lambda_0}(K + X) + f,$$

where  $X$  is a (conformal) Killing vector field,  $K$  is a (conformal) Killing 2-tensor such that  $\mathbf{Obs}(K)^b$  is an exact one-form and  $f \in C^\infty(M)$  is defined up to a constant by  $\mathbf{Obs}(K)^b = -2df$ .

### 3. Application to the $R$ -separation of the Schrödinger equations

Following [2, 3], we provide an intrinsic characterization for  $R$ -separation of the Schrödinger equation and of the Schrödinger equation at fixed energy in terms of second order (conformal) symmetries of the conformal Laplacian. Resorting to our previous results, this leads to a new criterion for having  $R$ -separation of these equations. Proofs are deferred to appendices.

#### 3.1. Definition of $R$ -separation

The Schrödinger equation, with fixed potential  $V \in C^\infty(M)$ , reads as

$$(\Delta_Y + V)\psi = E\psi, \quad (4)$$

where  $\psi \in C^\infty(M)$  is the unknown and  $E \in \mathbb{R}$  is called the energy. Solving the Schrödinger equation means to determine the solutions for all  $E \in \mathbb{R}$ .

We consider also the Schrödinger equation at fixed energy, i.e., up to changing  $V$ ,

$$(\Delta_Y + V)\psi = 0. \quad (5)$$

Up to modifying  $V$  by the curvature term  $\frac{(n-2)}{4(n-1)}\text{Sc}$ , one can replace the conformal Laplacian by the Laplace-Beltrami one, as done usually. Multiplicative  $R$ -separation for one of these equations is usually understood [4, 6] as the search for a coordinate system  $(x^i)$  and a family of solutions of the form

$$\psi(x) = R(x) \prod_{i=1}^n \phi_i(x^i, c_\alpha), \quad (6)$$

parametrized by  $c_\alpha \in \mathbb{R}$  with  $\alpha = 1, \dots, 2n-1$ , and satisfying the completeness condition

$$\text{rank} \left[ \frac{\partial}{\partial c_\alpha} \left( \frac{\phi_i'}{\phi} \right) \middle| \frac{\partial}{\partial c_\alpha} \left( \frac{\phi_i''}{\phi} \right) \right] = 2n-1, \quad \text{with } \alpha = 1, \dots, 2n-1 \text{ and } i = 1, \dots, n.$$

We restrict ourself to separation along orthogonal coordinates  $(x^i)$ , i.e. such that  $g_{ij} = 0$  if  $i \neq j$ . We use an alternative working definition which is equivalent, see e.g. Remark 1.

**Definition 4.** [2, 3] Equation (5) is  $R$ -separable in an orthogonal coordinate system  $(x^i)$  if there exist  $n+1$  functions  $R, h_i \in C^\infty(M)$  and  $n$  differential operators  $L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i)$  such that

$$R^{-1}(\Delta_Y + V)R = \sum_{i=1}^n h_i L_i.$$

The Schrödinger equation (4) is also  $R$ -separable in the coordinate system  $(x^i)$  if, for all  $E \in \mathbb{R}$ , there exist  $n+1$  functions  $R, h_i \in C^\infty(M)$  and  $n$  differential operators  $L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i)$  such that

$$R^{-1}(\Delta_Y + V)R - E = \sum_{i=1}^n h_i L_i.$$

In the  $R$ -separation for the Schrödinger equation, it is clear that only the zero order coefficients of the operators  $L_i$  depend on  $E$ . More precisely, the functions  $m_i$  are then affine functions in  $E$ . Looking at the principal symbol of the Schrödinger equation or of the equation (5), we deduce that the functions  $h_i$  are equal to  $g^{ii}$  for all  $i = 1, \dots, n$ . One can show that  $\psi$ , defined in (6), is a solution of the Schrödinger equation or of the equation (5) if and only if  $L_i \phi_i = 0$  for all  $i$ . One has the following easy but crucial fact:

**Proposition 1.** [6] The  $R$ -separability of the equation  $\Delta_Y \psi = 0$  is a conformally invariant property.

### 3.2. Intrinsic characterizations for $R$ -separation

Let  $(M, g)$  be a pseudo-Riemannian manifold,  $(x^i, p_i)$  a canonical coordinate system on  $T^*M$  and  $H = g^{ij}p_i p_j \in \mathcal{S}^2$ . Remark that a quadratic symmetric tensor  $K \in \mathcal{S}^2$  identifies via the metric as a symmetric endomorphism of  $TM$ . In that way,  $H$  identifies with the identity. The geodesic Hamilton-Jacobi equation is the coordinate independent equation

$$g^{ij}(\partial_i W)(\partial_j W) = E,$$

where  $W \in \mathcal{C}^\infty(M)$  is the unknown and  $E \in \mathbb{R}$  is the energy. If  $E = 0$ , this is the null geodesic Hamilton-Jacobi equation.

We need the following notions, borrowed from [7, 8] and [9, 10] respectively.

**Definition 5.** A Killing-Stäckel algebra is an  $n$ -dimensional linear space  $\mathcal{I}$  of Killing 2-tensors that satisfy the three following properties:

- (i) they commute as linear operators,
- (ii) they are diagonalizable as linear operators,
- (iii) they are in involution:  $\{K_1, K_2\} = 0$  for all  $K_1, K_2 \in \mathcal{K}$ .

Note that the metric Hamiltonian  $H = g^{ij}p_i p_j$  belongs to any Killing-Stäckel algebra (for more details on this point, see [8], Theorem 7.8).

**Definition 6.** A conformal Killing-Stäckel algebra is an  $n$ -dimensional linear space  $\mathcal{I}$  of conformal Killing 2-tensors that satisfy the following conditions:

- (i) they commute as linear operators,
- (ii) they are diagonalizable as linear operators,
- (iii)  $\{K_1, K_2\} \in (H)$  for all  $K_1$  and  $K_2$  in  $\mathcal{I}$ .

As pointed out in [10], each conformal Killing-Stäckel space contains a tensor of the kind  $fH$ , for some function  $f$ .

In the Riemannian case, the hypothesis (ii) in Definitions 5 and 6 is not necessary: indeed, in this situation, the 2-tensors in the (conformal) Killing-Stäckel algebra are automatically diagonalizable as linear operators because they are symmetric with respect to a Riemannian metric.

Kalnins and Miller provide in [7] (resp. [9]) an intrinsic characterization for additive separation of the (resp. null) geodesic Hamilton-Jacobi equation, in terms of (resp. conformal) Killing-Stäckel algebra. These classical results, completing those of Stäckel [11] and Eisenhart [12] can be summarized up as follows, see [7, 9] or [8, 10].

**Theorem 3.** If  $\mathcal{I}$  is a (conformal) Killing-Stäckel algebra and if  $K \in \mathcal{I}$ , then  $K$ , viewed as an endomorphism, admits  $n$  principal directions which integrate in a local coordinate system  $(x^i)$  on  $M$ . In this coordinate system, the tensors in  $\mathcal{I}$  are diagonal.

**Theorem 4.** The (resp. null) geodesic Hamilton-Jacobi equation admits additive separation in an orthogonal coordinate system if and only if there exists a (resp. conformal) Killing-Stäckel algebra.

Actually, it turns out that the coordinate system  $(x^i)$  in Theorem 3 is a coordinate system in which the (null) geodesic Hamilton-Jacobi equation is separable.

In [2, 3], Kalnins and Miller provide an analogue of the latter theorem for  $R$ -separation of the equations  $\Delta\psi = 0$  and  $\Delta\psi = E\psi$ . In the presence of potentials, their results extend easily, leading to the two following theorems. We provide a proof for the first one in Appendix B, it relies on [2] and on the material in Appendix A. The proof of the second one is similar and is left to the reader.

**Theorem 5.** Equation (5),  $(\Delta_Y + V)\psi = 0$ ,  $R$ -separates in an orthogonal coordinate system if and only if:

- (a) there exists a conformal Killing-Stäckel algebra  $\mathcal{I}$ ,
- (b) for all  $K \in \mathcal{I}$ , there exists  $D \in \mathcal{D}^2(M)$  with principal symbol  $\sigma_2(D) = K$  such that  $[\Delta_Y + V, D] = A \circ (\Delta_Y + V)$ , for some  $A \in \mathcal{D}(M)$ .

**Theorem 6.** The Schrödinger equation,  $(\Delta_Y + V)\psi = E\psi$ ,  $R$ -separates in an orthogonal coordinate system if and only if:

- (a) there exists a Killing-Stäckel algebra  $\mathcal{I}$ ,
- (b) for all  $K \in \mathcal{I}$ , there exists  $D \in \mathcal{D}^2(M)$  with principal symbol  $\sigma_2(D) = K$  such that  $[\Delta_Y + V, D] = 0$ .

Remark that, compared to the analogous theorems in [2, 3], we relax the hypothesis of commutation of the symmetries of  $\Delta_Y + V$  between themselves. From our results on the second order (resp. conformal) symmetries of the conformal Laplacian we deduce:

**Corollary 1.** The equation  $\Delta_Y\psi = 0$  separates in orthogonal coordinates if and only if there exists a conformal Killing-Stäckel algebra  $\mathcal{I}$  such that  $\mathbf{Obs}(K)^b$  is exact for all  $K \in \mathcal{I}$ .

**Corollary 2.** The Schrödinger equation separates in orthogonal coordinates if and only if there exists a Killing-Stäckel algebra  $\mathcal{I}$  such that  $\mathbf{Obs}(K)^b$  is exact for all  $K \in \mathcal{I}$ .

There exists a convenient and geometrical condition which ensures separation for the usual Schrödinger equation  $\Delta\psi = E\psi$ , where  $\Delta = \nabla_i g^{ij} \nabla_j$  is the Laplace-Beltrami Laplacian. Namely, if the geodesic Hamilton-Jacobi equation separates in a coordinate system  $(x^i)$ , then the Schrödinger equation separates in the same coordinate system if and only if the Ricci tensor satisfies the Robertson condition:  $\text{Ric}_{ij} = 0$  if  $i \neq j$  [12]. In particular, this condition implies the  $R$ -separation for the Schrödinger equation in the coordinate system  $(x^i)$ . From the preceding corollaries and the explicit form of the operator  $\mathbf{Obs}$ , we get an analogous but distinct condition for  $R$ -separation.

**Proposition 2.** Let  $(M, g)$  be a pseudo-Riemannian manifold of dimension at least 4. If there exists a (resp. conformal) Killing-Stäckel algebra such that  $C_{ijjk} = 0$  in the corresponding coordinate system, then the Schrödinger equation  $\Delta_Y\psi = E\psi$  (resp. the equation  $\Delta_Y\psi = 0$ ) admits  $R$ -separation in the same coordinate system. If  $M$  is of dimension 3, the condition  $C_{ijjk} = 0$  should be replaced by  $A_{jji} = 0$ .

*Proof.* Indeed, on one hand, the tensors belonging to the (conformal) Killing-Stäckel algebra ensuring the separation of the Hamilton-Jacobi equation are diagonal in the coordinate system  $(x^i)$  (see Theorem 3). On the other hand, recall that the operator  $\mathbf{Obs}$  is equal to

$$\frac{2(n-2)}{3(n+1)} g^{im} p_i \partial_{p_j} \partial_{p_l} \left( C^k_{jlm} \nabla_k - 3A_{jlm} \right).$$

If  $(M, g)$  has a dimension greater than or equal to 4, then the fact that  $C_{ijjk} = 0$  implies that  $A_{jji} = 0$  thanks to the relation

$$(n-3)A_{abc} = \nabla_r C_{bc}^r{}_a.$$

We can then conclude that  $\mathbf{Obs}(K) = 0$  for all  $K \in \mathcal{I}$  using the fact that the tensors in  $\mathcal{I}$  are diagonal.

If  $M$  is of dimension 3, then  $C$  vanishes and the condition  $A_{jji} = 0$  is then sufficient to conclude that  $\mathbf{Obs}(K) = 0$  for all  $K \in \mathcal{I}$ .

Eventually, Corollaries 1 and 2 yield the result.  $\square$

However, neither the Robertson's condition nor the one in Proposition 2 is necessary to get  $R$ -separation of the Schrödinger equation. This is illustrated thanks to the DiPirro metrics given in [1] (section 5, page 23). Let us recall that the Hamiltonians associated with these metrics are given by

$$H = \frac{1}{2(\gamma(x_1, x_2) + c(x_3))} (a(x_1, x_2)p_1^2 + b(x_1, x_2)p_2^2 + p_3^2) \quad (7)$$

where  $a, b, c$  and  $\gamma$  are arbitrary functions and  $(x^i, p_i)$  are canonical coordinates on  $T^*\mathbb{R}^3$ . The DiPirro metrics admit diagonal Killing tensors  $K$  given by

$$K = \frac{1}{\gamma(x_1, x_2) + c(x_3)} (c(x_3)a(x_1, x_2)p_1^2 + c(x_3)b(x_1, x_2)p_2^2 - \gamma(x_1, x_2)p_3^2).$$

We know thanks to Proposition 5.1 in [1] that there exists a symmetry of  $\Delta_Y$  given by the Killing tensor  $K$  defined above. If  $c(x^3) = 1$ , DiPirro metrics admit a third Killing tensor  $K' = p_3^2$ , with a corresponding symmetry operator given by  $D' = \frac{\partial^2}{\partial (x^3)^2}$ . Thus, the 3-dimensional linear space generated by  $H, K$  and  $K'$  constitutes a Killing-Stäckel algebra whose elements satisfy the condition (b) in Theorem 6 because this condition is verified for each generator of this algebra. Then, the Schrödinger equation  $R$ -separates in the coordinate system  $(x^1, x^2, x^3)$ . However, a direct computation, performed e.g. with Mathematica, leads to  $\text{Ric}_{12} \neq 0$  and  $A_{112} \neq 0$ .

Let  $(x^i)$  be a coordinate system which separates the (resp. null) geodesic Hamilton-Jacobi equation. This is an open question to find a Robertson's like condition, which exactly determines if the coordinate system  $(x^i)$   $R$ -separates the Schrödinger equation (resp. the equation  $\Delta_Y \psi = 0$ ). Such a condition should be ideally written only in terms of the curvature tensors, in the coordinate system  $(x^i)$ .

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## Appendix A. Stäckel metrics

### Appendix A.1. Stäckel metrics and Stäckel multipliers

Let  $(x^i)$  be an orthogonal coordinate system for the metric  $g$ . We introduce two coordinates dependent notions. First, the Stäckel operators are given by

$$\text{St}_{ij} = \partial_i \partial_j - (\partial_i \log |g^{jj}|) \partial_j - (\partial_j \log |g^{ii}|) \partial_i, \quad i \neq j, \quad (\text{A.1})$$

with no summation on the repeated indices. Second, a Stäckel matrix is an invertible matrix of functions  $(S_{ij}(x^i))$ , whose  $i^{\text{th}}$ -row depends on the coordinate  $x^i$  only.

**Proposition 3.** *A metric  $g$  is called a Stäckel metric if there exist orthogonal coordinates  $(x^i)$ , called Stäckel coordinates, such that one of the four following equivalent conditions is satisfied:*

- (i)  $g^{jj} = \frac{S^{j1}}{S}$ , where  $S^{j1}$  is the cofactor in  $(j, 1)$  of a Stäckel matrix  $(S_{ij})$  and  $S$  its determinant,
- (ii)  $\text{St}_{ij}(g^{kk}) = 0$ , for all  $i, j, k = 1, \dots, n$  and  $i \neq j$ ,
- (iii) there exists a Killing-Stäckel algebra with diagonal elements in the coordinates  $(x^i)$ ,
- (iv) the geodesic Hamilton-Jacobi equation separates in the coordinates  $(x^i)$ .

The notion of Stäckel metric goes back to Stäckel himself [11] and he proves the equivalence between (i) and (iv). The remaining equivalences are proved in [12]. In particular, (i) and (iii) are linked through the fact that  $\mathcal{I}$  is generated by the tensors  $K_i = \frac{S^{ji}}{S} p_j^2$ , for  $i = 1, \dots, n$ . Remark that a given Stäckel metric can admit several inequivalent Stäckel coordinate systems. They have been classified in the flat case in low dimensions, see e.g. [13] for the 3-dimensional case.

**Proposition 4.** *A metric  $g$  is called a conformal Stäckel metric if there exist orthogonal coordinates  $(x^i)$ , called conformal Stäckel coordinates, such that one of the four following equivalent conditions is satisfied:*

- (i) *there exists a Stäckel metric  $\hat{g}$ , with Stäckel coordinates  $(x^i)$ , conformally related to  $g$ ,*
- (ii)  *$\frac{\text{St}_{ij}(g^{kk})}{g^{kk}} = \frac{\text{St}_{ij}(g^{ll})}{g^{ll}}$ , for all  $i, j, k, l = 1, \dots, n$  and  $i \neq j$ ,*
- (iii) *there exists a conformal Killing-Stäckel algebra with diagonal elements in the coordinates  $(x^i)$ ,*
- (iv) *the null geodesic Hamilton-Jacobi equation separates in the coordinates  $(x^i)$ .*

The equivalence between (i) and (iv) goes back to Stäckel [11]. The proof of the equivalence between the remaining statements can be found in [9]. We introduce now Stäckel multipliers. They relate different Stäckel metrics in a conformal class, which admit common Stäckel coordinates.

**Proposition 5.** *Let  $g$  be a Stäckel metric with Stäckel coordinates  $(x^i)$ . A Stäckel multiplier is a function  $Q$  satisfying one of the three following equivalent conditions:*

- (i) *the metric  $Qg$  is a Stäckel metric, with Stäckel coordinates  $(x^i)$ ,*
- (ii)  *$\text{St}_{ij}(Q) = 0$ , for all  $i, j = 1, \dots, n$  and  $i \neq j$ ,*
- (iii) *there exist functions  $k_j(x^j)$  such that  $Q(x) = \sum_{j=1}^n g^{jj} k_j(x^j)$ .*

The equivalence of (i), (ii) and (iii) is established in [14]. The latter notion can be straightforwardly extended to the case of conformal Stäckel metrics.

**Proposition 6.** *Let  $g$  be a conformal Stäckel metric with conformal Stäckel coordinates  $(x^i)$ . A pseudo-Stäckel multiplier is a function  $Q$  satisfying one of the three following equivalent conditions:*

- (i) *the metric  $Qg$  is a Stäckel metric, with Stäckel coordinates  $(x^i)$ ,*
- (ii)  *$\frac{\text{St}_{ij}(Q)}{Q} = \frac{\text{St}_{ij}(g^{kk})}{g^{kk}}$ , for all  $i, j, k = 1, \dots, n$  and  $i \neq j$ ,*
- (iii) *there exist functions  $k_j(x^j)$  such that  $Q(x) = \sum_{j=1}^n g^{jj} k_j(x^j)$ .*

Remark that, by the very definition of a conformal Stäckel metric, there always exist pseudo-Stäckel multipliers.

#### Appendix A.2. Necessary and sufficient conditions for $R$ -separation

It is well-known that multiplicative  $R$ -separation for the Schrödinger equation (4) implies additive separation for the geodesic Hamilton-Jacobi equation, and that multiplicative  $R$ -separation of the equation (5) implies additive separation of the null geodesic Hamilton-Jacobi equation. The converse statement is not true and suppose an extra condition involving the potential and the metric. The following theorem is a straightforward generalization of a result in [2], see also [4].

**Theorem 7.** *Equation (5),  $(\Delta_Y + V)\psi = 0$ ,  $R$ -separates in the orthogonal coordinate system  $(x^i)$  if and only if the two following conditions hold:*

- (i)  *$g$  is a conformal Stäckel metric with conformal Stäckel coordinates  $(x^i)$ ,*
- (ii)  *$V + \frac{\Delta_Y R}{R}$  is a pseudo-Stäckel multiplier.*

Moreover, the function  $R$  is then any function of the form  $\frac{S}{Q(\det g)^{1/2}} \prod_j a_j$ , where  $\hat{g} = Qg$  is a Stäckel metric,  $S$  is the determinant of a Stäckel matrix  $(S_{ij})$  of  $\hat{g}$  and where the functions  $a_j$  are such that their derivatives  $\partial_j a_j$  depend only on the coordinate  $x^j$ .



*Proof.* The  $R$ -separation of Equation (5) occurs in coordinates  $(x^i)$  if and only if there exists a function  $R \in \mathcal{C}^\infty(M)$  such that

$$R^{-1}(\Delta_Y + V)R = \sum_{j=1}^n g^{jj} L_j, \quad (\text{A.2})$$

where  $L_j = \partial_j^2 + l_j \partial_j + m_j$ , with the functions  $l_j$  and  $m_j$  which depend only on the coordinate  $x^j$ . Since  $g$  is diagonal in the coordinates  $(x^i)$ , we have the formula  $\nabla_i g^{ij} \nabla_j = g^{jj} \partial_j^2 + \Gamma^j \partial_j$ , where  $\Gamma^j = (\det g)^{-1/2} \partial_j (g^{jj} (\det g)^{1/2})$  is the trace of the Christoffel symbol of  $\nabla$ . A direct computation shows that

$$R^{-1} \Delta_Y R = g^{jj} \partial_j^2 + 2g^{jj} (\partial_j \log R) \partial_j + \Gamma^j \partial_j + \frac{\Delta_Y R}{R}.$$

In consequence, Equation (A.2) holds if and only if:

$$\begin{cases} \partial_j (\log g^{jj}) = l_j - \partial_j (2 \log R + \log (\det g)^{1/2}), & \forall j = 1, \dots, n, \\ V + \frac{\Delta_Y R}{R} = \sum_i g^{ii} m_i. \end{cases} \quad (\text{A.3})$$

These two equations hold if and only if the metric  $g$  is a conformal Stäckel metric and  $V + \frac{\Delta_Y R}{R}$  is a pseudo-Stäckel multiplier. We determine now the form of the function  $R$ . Since  $g$  is a conformal Stäckel metric, there exists a pseudo-Stäckel multiplier  $Q$  and a Stäckel metric  $\hat{g} = Qg$ , with Stäckel matrix  $(S_{ij})$ . Denoting by  $S$  its determinant, one sets  $f = \log \frac{Q(\det g)^{1/2}}{S}$ . By definition of a Stäckel matrix we have  $\partial_j S^{j1} = 0$ , and the equality  $g^{jj} \partial_j f = (\det g)^{-1/2} \partial_j (g^{jj} (\det g)^{1/2})$  follows. In consequence, we deduce that

$$2\partial_j \log R = l_j - \partial_j f, \quad \forall j = 1, \dots, n,$$

whose solutions are  $R = \left( \frac{S}{Q(\det g)^{1/2}} \right)^{1/2} \prod_j a_j$  with  $\partial_j a_j = -\frac{1}{2} l_j$ , and the functions  $l_j$  are free.  $\square$

**Remark 1.** Theorem 7 is proved in [4, Proposition 23], but resorting to the definition of  $R$ -separation of variables provided by Equation (6) rather than the one given in Definition 4. This shows that these two definitions of  $R$ -separation of variables are indeed equivalent.

The following theorem is easily deduced from the preceding one.

**Theorem 8.** The Schrödinger equation (4)  $R$ -separates in the orthogonal coordinate system  $(x^i)$  if and only if the two following conditions hold:

- (i)  $g$  is a Stäckel metric with Stäckel coordinates  $(x^i)$ ,
- (ii)  $V + \frac{\Delta_Y R}{R}$  is a Stäckel multiplier.

*Proof.* If the  $R$ -separation condition (A.3) holds for all  $E \in \mathbb{R}$ , we deduce that constants are pseudo-Stäckel multipliers. Hence, the metric  $g$  is Stäckel and the pseudo-Stäckel multiplier  $V + \frac{\Delta_Y R}{R}$  is then a Stäckel multiplier.  $\square$

## Appendix B. Proof of Theorem 5

First, we suppose that  $(x^i)$  is an  $R$ -separable coordinate system for Equation (5) and prove conditions (a) and (b). By the very definition of  $R$ -separation, there exists a function  $R \in \mathcal{C}^\infty(M)$  such that

$$R^{-1}(\Delta_Y + V)R = \sum_{j=1}^n g^{jj} L_j, \quad (\text{B.1})$$

where  $L_j = \partial_j^2 + l_j \partial_j + m_j$ , with the functions  $l_j$  and  $m_j$  which depend only on the coordinate  $x^j$ . Moreover, Theorem 7 applies and  $g$  is then a conformal Stäckel metric. In consequence, there exists a pseudo-Stäckel multiplier  $Q$  and a Stäckel metric  $\hat{g} = Qg$ .

We denote by  $(S_{ij})$  a Stäckel matrix attached to the metric  $\hat{g}$  and by  $(S^{ij})$  the matrix of cofactors, so that  $S^{ij} S_{jk} = S \delta_k^i$  with  $S$  the determinant of  $(S_{ij})$ . Hence, we obtain  $n$  independent diagonal Killing tensors  $K_i = \frac{S^{ji}}{S} p_j^2$  for  $\hat{g}$ , with  $K_1 = \hat{g}^{ij} p_i p_j$ . They are automatically conformal Killing tensors for  $g$  and they generate a conformal Killing-Stäckel algebra  $\mathcal{I}$ , so the condition (a) is satisfied. Following [2], we set

$$A_i = \sum_j \frac{S^{ji}}{S} (\partial_j^2 + (\partial_j f) \partial_j + a_j),$$

where  $f = \log \frac{Q(\det g)^{1/2}}{S}$  and  $a_j = m_j + \frac{1}{2} \partial_j (\partial_j f - l_j) + \frac{1}{4} ((\partial_j f)^2 - l_j^2)$ . The principal symbol of  $A_i$  is then  $K_i$ . Setting  $\rho_{(i)}^j = \hat{g}_{jj} \frac{S^{ji}}{S}$ , the Poisson commutation relations,  $\{K_1, K_i\} = 0$ , translate as  $\partial_k (\rho_{(i)}^j \hat{g}^{jj}) = \rho_{(i)}^k (\partial_k \hat{g}^{jj})$  for all indices  $i, j, k$ . Then a direct computation shows that (see [2]),

$$[A_1, A_i] = 0, \quad \forall i = 1, \dots, n. \quad (\text{B.2})$$

The next step is to prove that  $QA_1 = \Delta_Y + V$ . By definition of a Stäckel matrix we have  $g^{jj} = Q \frac{S^{j1}}{S}$  and  $\partial_j S^{j1} = 0$ . The equality  $g^{jj} \partial_j f = (\det g)^{-1/2} \partial_j (g^{jj} (\det g)^{1/2})$  follows. In particular, one has  $\Delta := \nabla_i g^{ij} \nabla_j = g^{jj} (\partial_j^2 + (\partial_j f) \partial_j)$ , which leads to

$$QA_1 = \Delta + \sum_j g^{jj} a_j.$$

According to the proof of Theorem 7, the equation (B.1) is equivalent to the system (A.3), which reads now as

$$\begin{cases} 2\partial_j(\log R) = l_j - \partial_j f, & \forall j = 1, \dots, n, \\ V + \frac{\Delta_Y R}{R} = \sum_i g^{ii} m_i. \end{cases}$$

Combining these two equations with the expression of  $a_j$  and the equality  $\frac{\Delta_Y R}{R} = g^{jj} (R^{-1} \partial_j^2 (R) + (\partial_j f)(\partial_j \log R))$ , we get  $QA_1 = \Delta_Y + V$ . Therefore, Equation (B.2) implies that

$$[\Delta_Y + V, A_i] = [Q, A_i] \circ A_1$$

for  $i = 1, \dots, n$ . Since  $\sigma_2(A_i) = K_i$  for  $i = 1, \dots, n$  and since the 2-tensors  $K_i$  ( $i = 1, \dots, n$ ) generate  $\mathcal{I}$ , the condition (b) is satisfied.

We prove now that the conditions (a) and (b) imply that  $g$  is a conformal Stäckel metric and  $V + \frac{\Delta_Y R}{R}$  is a pseudo-Stäckel multiplier. This leads to the conclusion thanks to Theorem 7. We straightforwardly get that  $g$  is a conformal Stäckel metric by Proposition 4. Its conformally related Stäckel metric  $\hat{g} = Qg$  admits a Stäckel matrix  $(S_{ij})$  so that the tensors  $K_i = \frac{S^{ji}}{S} p_j^2$

( $i = 1, \dots, n$ ) are generators of the conformal Killing-Stäckel algebra  $\mathcal{I}$ . Hence, we can introduce  $f = \log \frac{Q(\det g)^{1/2}}{S}$  as above and we set  $\rho_{(i)}^j = \hat{g}_{jj} \frac{S^{ji}}{S}$ . Setting  $V' = V - \frac{n-2}{4(n-1)} \text{Sc}$ , we get that

$$\Delta_Y + V = g^{jj}(\partial_j^2 + (\partial_j f)\partial_j) + V'.$$

Let  $D_1 = Q^{-1}(\Delta_Y + V)$  and  $D_i$ , ( $i = 2, \dots, n$ ) be the conformal symmetries corresponding to the tensors  $K_i$  ( $i = 1, \dots, n$ ) defined above. If we set  $\tilde{D}_i = e^{f/2} D_i e^{-f/2}$ , the operators  $\tilde{D}_i$  read as

$$\sum_j \rho_{(i)}^j \hat{g}^{jj} \partial_j^2 + \xi_{(i)}^j \partial_j + \mu_{(i)},$$

where  $\rho_{(1)}^j = 1$ ,  $\xi_1^j = 0$  and  $\mu_{(1)} = Q^{-1}(V + \frac{\Delta_Y R}{R})$  where  $2 \log R = -f$ . According to [2], the equations

$$[\tilde{D}_1, \tilde{D}_j] \in (\tilde{D}_1), \quad \forall j = 2, \dots, n,$$

imply that  $\mu_{(1)} = V + \frac{\Delta_Y R}{R}$  is a pseudo-Stäckel multiplier. Moreover,  $R = e^{-f/2}$  is one of the admissible functions  $R$  for  $R$ -separation and Theorem 5 is then proved.

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