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Tetrahedron equation and quantum R matrices for q-oscillator representations

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Abstract. We review and supplement the recent result by the authors on the reduction of the three dimensional R (3d R) satisfying the tetrahedron equation to the quantum R matrices for the q-oscillator representations of $U_q(D^{(2)}_{n+1})$, $U_q(A^{(2)}_{2n})$ and $U_q(C^{(1)}_n)$. A new formula for the 3d R and a quantum R matrix for $n = 1$ are presented and a proof of the irreducibility of the tensor product of the q-oscillator representations is detailed.

1. Introduction

This paper is a summary and supplement of the recent result [9] by the authors, which is motivated by the earlier works [13, 2, 11]. The tetrahedron equation (1) [14] is a three dimensional generalization of the Yang-Baxter equation [1]. In [11] a new prescription was proposed to reduce it to the Yang-Baxter equation $R_{1,2}R_{1,3}R_{2,3} = R_{2,3}R_{1,3}R_{1,2}$ by using the special boundary vectors defined by (3) and (10). Applied to a particular solution of the tetrahedron equation (3d L operator [2]), the reduction was shown [11] to give the quantum R matrices for the spin representations [12].

In [9] a similar reduction was studied for the distinguished solution of the tetrahedron equation which we call 3d R. The 3d R was obtained as the intertwiner of the quantum coordinate ring $A_q(sl_3)$ [6], (The original formula on p194 therein contains a misprint.) and was found later also in a different setting [2]. They were shown to coincide and to constitute the solution of the 3d reflection equation in [7]. See [9, App. A] for more detail. The main result of [9] was the identification of the reduction of the 3d R with the quantum R matrices for the quantum affine algebras $U_q = U_q(D^{(2)}_{n+1})$, $U_q(A^{(2)}_{2n})$ and $U_q(C^{(1)}_n)$. Their relevant representations turned out to be new infinite dimensional ones which we called the *q-oscillator representations*. There are two kinds of boundary vectors, which curiously correspond to the choices of the above three algebras. See Remark 5.

This paper contains a summary of these results and a few supplements. The formula (9) for the 3d R and (19) for the quantum R matrix for $n = s = t = 1$ case are new. Section 4 recollects a proof of the irreducibility of the tensor product of the q-oscillator representations whose detail was omitted in [9]. The result for $n = 1$ was reported earlier in [8]. More recently it has been shown that the q-oscillator representations [9] quoted in Prop. 1-3 here actually factor through a homomorphism from U_q to the n fold tensor product of the *q-oscillator algebra* [10].

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Throughout the paper we assume that q is generic and use the following notations:

$$(z;q)_m = \prod_{k=1}^{m}(1-zq^{k-1}),\quad (q)_m = (q;q)_m,\quad \binom{m}{k}_q = \frac{(q)_m}{(q)_k(q)_{m-k}},\quad [m]_q = \frac{q^m - q^{-m}}{q-q^{-1}},$$

where the q-binomial is to be understood as zero unless $0 \le k \le m$. $[m]_{q^t}$ with $t=1$ will simply be denoted by $[m]$.

2. Reducing the tetrahedron equation to the Yang-Baxter equation

2.1. General scheme using boundary vectors

Let F be a vector space and $R \in \mathrm{End}(F^{\otimes 3})$. Consider the tetrahedron equation:

$$R_{1,2,4}R_{1,3,5}R_{2,3,6}R_{4,5,6} = R_{4,5,6}R_{2,3,6}R_{1,3,5}R_{1,2,4} \in \mathrm{End}(F^{\otimes 6}) \tag{1}$$

where $R_{i,j,k}$ acts as R on the i,j,k th components from the left in $F^{\otimes 6}$.

We recall the prescription which produces an infinite family of solutions to the Yang-Baxter equation from a solution to the tetrahedron equation based on special boundary vectors [11]. First we regard (1) as a one-site relation, and extend it to the n-site version. Let $\overset{\alpha_i}{F}, \overset{\beta_i}{F}, \overset{\gamma_i}{F}$ be the copies of F, where α_i, β_i and γ_i $(i=1,\ldots,n)$ are just labels. Renaming the spaces $1,2,3$ by them gives $R_{\alpha_i,\beta_i,4}R_{\alpha_i,\gamma_i,5}R_{\beta_i,\gamma_i,6}R_{4,5,6} = R_{4,5,6}R_{\beta_i,\gamma_i,6}R_{\alpha_i,\gamma_i,5}R_{\alpha_i,\beta_i,4}$ for each i. Thus for any i one can carry $R_{4,5,6}$ through $R_{\alpha_i,\beta_i,4}R_{\alpha_i,\gamma_i,5}R_{\beta_i,\gamma_i,6}$ to the left reversing it into $R_{\beta_i,\gamma_i,6}R_{\alpha_i,\gamma_i,5}R_{\alpha_i,\beta_i,4}$. Applying this n times leads to

$$\begin{aligned}&\big(R_{\alpha_1,\beta_1,4}R_{\alpha_1,\gamma_1,5}R_{\beta_1,\gamma_1,6}\big)\cdots\big(R_{\alpha_n,\beta_n,4}R_{\alpha_n,\gamma_n,5}R_{\beta_n,\gamma_n,6}\big)R_{4,5,6}\\ &= R_{4,5,6}\big(R_{\beta_1,\gamma_1,6}R_{\alpha_1,\gamma_1,5}R_{\alpha_1,\beta_1,4}\big)\cdots\big(R_{\beta_n,\gamma_n,6}R_{\alpha_n,\gamma_n,5}R_{\alpha_n,\beta_n,4}\big).\end{aligned} \tag{2}$$

This is an equality in $\mathrm{End}(\overset{\boldsymbol{\alpha}}{F}\otimes\overset{\boldsymbol{\beta}}{F}\otimes\overset{\boldsymbol{\gamma}}{F}\otimes\overset{4}{F}\otimes\overset{5}{F}\otimes\overset{6}{F})$, where $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ is the array of labels and $\overset{\boldsymbol{\alpha}}{F} = \overset{\alpha_1}{F}\otimes\cdots\otimes\overset{\alpha_n}{F}\ (= F^{\otimes n})$. The notations $\overset{\boldsymbol{\beta}}{F}$ and $\overset{\boldsymbol{\gamma}}{F}$ should be understood similarly.

Next we introduce special boundary vectors. Suppose one has a vector $|\chi_s(x)\rangle \in F$ depending on a variable x such that its tensor product

$$|\chi_s(x,y)\rangle = |\chi_s(x)\rangle \otimes |\chi_s(xy)\rangle \otimes |\chi_s(y)\rangle \in F\otimes F\otimes F \tag{3}$$

satisfies the relation

$$R|\chi_s(x,y)\rangle = |\chi_s(x,y)\rangle. \tag{4}$$

The index s is put to distinguish possibly more than one such vectors. Suppose there exist vectors in the dual space

$$\langle\chi_s(x,y)| = \langle\chi_s(x)| \otimes \langle\chi_s(xy)| \otimes \langle\chi_s(y)| \in F^*\otimes F^*\otimes F^*$$

having the similar property

$$\langle\chi_s(x,y)|R = \langle\chi_s(x,y)|. \tag{5}$$

Then evaluating (2) between $\langle\chi_s(x,y)|$ and $|\chi_t(1,1)\rangle$, one obtains

$$S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) = \varrho^{s,t}(z)\langle\chi_s(z)|R_{\alpha_1,\beta_1,3}R_{\alpha_2,\beta_2,3}\cdots R_{\alpha_n,\beta_n,3}|\chi_t(1)\rangle \in \mathrm{End}(\overset{\boldsymbol{\alpha}}{F}\otimes\overset{\boldsymbol{\beta}}{F}), \tag{6}$$

where $\varrho^{s,t}(z)$ is inserted to control the normalization. The composition of R and matrix elements are taken for the space signified by 3. One may simply write it as $S(z) \in \mathrm{End}(F^{\otimes n}\otimes F^{\otimes n})$ dropping the dummy labels. The $S(z)$ depends on s and t although they have been temporarily suppressed. It follows from (2), (4) and (5) that $S(z)$ satisfies the Yang-Baxter equation:

$$S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)S_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(xy)S_{\boldsymbol{\beta},\boldsymbol{\gamma}}(y) = S_{\boldsymbol{\beta},\boldsymbol{\gamma}}(y)S_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(xy)S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x) \in \mathrm{End}(\overset{\boldsymbol{\alpha}}{F}\otimes\overset{\boldsymbol{\beta}}{F}\otimes\overset{\boldsymbol{\gamma}}{F}). \tag{7}$$

2.2. A realization of the scheme

We focus on the solution R of the tetrahedron equation mentioned in the introduction. Take F to be an infinite dimensional space $F = \bigoplus_{m\geq 0} \mathbb{Q}(q)|m\rangle$ with the dual $F^* = \bigoplus_{m\geq 0} \mathbb{Q}(q)\langle m|$ having the bilinear pairing $\langle l|m\rangle = (q^2)_m \delta_{l,m}$. Then the 3d R is given by

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c\geq 0} R^{a,b,c}_{i,j,k} |a\rangle \otimes |b\rangle \otimes |c\rangle, \tag{8}$$

$$R^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k} \sum_{\lambda+\mu=b} (-1)^\lambda q^{ik+b+\lambda(c-a)+\mu(\mu-i-k-1)} \binom{\lambda+a}{a}_{q^2} \binom{i}{\mu}_{q^2}, \tag{9}$$

where $\delta^i_j = 1$ if $i = j$ and 0 otherwise. The sum is over $\lambda, \mu \in \mathbb{Z}_{\geq 0}$ such that $\lambda + \mu = b$ with the further condition $\mu \leq i$. It satisfies $(q^2)_a(q^2)_b(q^2)_c R^{a,b,c}_{i,j,k} = (q^2)_i(q^2)_j(q^2)_k R^{i,j,k}_{a,b,c}$ [9, eq.(A.1)]. The formula (9) is simpler than [9, eq.(2.10)]. Its derivation will be given elsewhere.

The two boundary vectors satisfying (4) and (5) are known [11] and given by

$$\langle \chi_s(z)| = \sum_{m\geq 0} \frac{z^m}{(q^{s^2})_m} \langle sm|, \quad |\chi_s(z)\rangle = \sum_{m\geq 0} \frac{z^m}{(q^{s^2})_m} |sm\rangle \quad (s = 1, 2). \tag{10}$$

Given two boundary vectors, one can construct four families of solutions to the Yang-Baxter equation $S(z) = S^{s,t}(z) = S^{s,t}(z, q)$ $(s, t = 1, 2)$ by (6) by substituting (9) and (10). Each family consists of the solutions labeled with $n \in \mathbb{Z}_{\geq 1}$. They are the matrices acting on $F^{\otimes n} \otimes F^{\otimes n}$ whose elements read

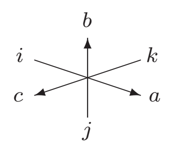
$$S^{s,t}(z)\big(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle\big) = \sum_{\mathbf{a},\mathbf{b}} S^{s,t}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} |\mathbf{a}\rangle \otimes |\mathbf{b}\rangle, \tag{11}$$

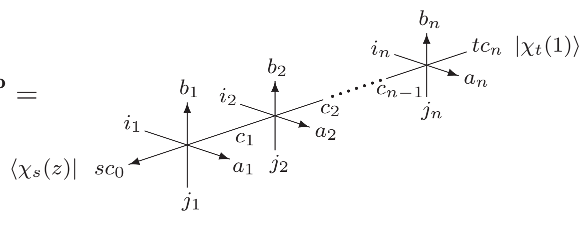
$$S^{s,t}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = \varrho^{s,t}(z) \sum_{c_0,\dots,c_n\geq 0} \frac{z^{c_0}(q^2)_{sc_0}}{(q^{s^2})_{c_0}(q^{t^2})_{c_n}} R^{a_1,b_1,sc_0}_{i_1,j_1,c_1} R^{a_2,b_2,c_1}_{i_2,j_2,c_2} \cdots R^{a_{n-1},b_{n-1},c_{n-2}}_{i_{n-1},j_{n-1},c_{n-1}} R^{a_n,b_n,c_{n-1}}_{i_n,j_n,tc_n}, \tag{12}$$

where $|\mathbf{a}\rangle = |a_1\rangle \otimes \cdots \otimes |a_n\rangle \in F^{\otimes n}$ for $\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{Z}_{\geq 0})^n$, etc. By Applying [9, eq.(A.1)] to (12) it is straightforward to show

$$S^{t,s}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} / \varrho^{t,s}(z) = \left(\prod_{r=1}^{n} \frac{z^{\frac{1}{t}j_r}(q^2)_{i_r}(q^2)_{j_r}}{z^{\frac{1}{t}b_r}(q^2)_{a_r}(q^2)_{b_r}} \right) S^{s,t}(z^{\frac{s}{t}})^{\bar{\mathbf{i}},\bar{\mathbf{j}}}_{\bar{\mathbf{a}},\bar{\mathbf{b}}} / \varrho^{s,t}(z^{\frac{s}{t}}), \tag{13}$$

where $\bar{\mathbf{a}} = (a_n, \dots, a_1)$ is the reverse array of $\mathbf{a} = (a_1, \dots, a_n)$ and similarly for $\bar{\mathbf{b}}, \bar{\mathbf{i}}$ and $\bar{\mathbf{j}}$. Henceforth we shall only consider $S^{1,1}(z), S^{1,2}(z)$ and $S^{2,2}(z)$ in the rest of the paper. The matrix elements $R^{a,b,c}_{i,j,k}$ (9) and $S^{s,t}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}}$ (12) are depicted as follows:

$R^{a,b,c}_{i,j,k} =$ 

$S^{s,t}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} =$ 

Due to δ factors in (9), $S^{s,t}(z)$ obeys the conservation law

$$S^{s,t}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = 0 \ \text{ unless } \ \mathbf{a} + \mathbf{b} = \mathbf{i} + \mathbf{j} \tag{14}$$

and the sum (12) is constrained by the n conditions $b_1 + sc_0 = j_1 + c_1, \dots, b_n + c_{n-1} = j_n + tc_n$ leaving effectively a *single* sum. For $(s,t) = (2,2)$, they further enforce a parity constraint

$$S^{2,2}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = 0 \ \text{ unless } \ |\mathbf{a}| \equiv |\mathbf{i}|, \ \ |\mathbf{b}| \equiv |\mathbf{j}| \mod 2, \tag{15}$$

where $|\mathbf{a}| = a_1 + \cdots + a_n$, etc. Thus we have a direct sum decomposition

$$S^{2,2}(z) = S^{+,+}(z) \oplus S^{+,-}(z) \oplus S^{-,+}(z) \oplus S^{-,-}(z), \tag{16}$$

$$S^{\epsilon_1,\epsilon_2}(z) \in \mathrm{End}\big((F^{\otimes n})_{\epsilon_1} \otimes (F^{\otimes n})_{\epsilon_2}\big), \qquad (F^{\otimes n})_{\pm} = \bigoplus_{\mathbf{a}\in(\mathbb{Z}_{\geq 0})^n,\, (-1)^{|\mathbf{a}|}=\pm 1} \mathbb{Q}(q)|\mathbf{a}\rangle. \tag{17}$$

We dare allow the coexistence of somewhat confusing notations $S^{s,t}(z)$ and $S^{\epsilon_1,\epsilon_2}(z)$ expecting that they can be properly distinguished from the context. (A similar warning applies to $\varrho^{s,t}(z)$ in the sequel.) We choose the normalization factors as

$$\varrho^{1,1}(z) = \frac{(z;q)_\infty}{(-zq;q)_\infty}, \quad \varrho^{1,2}(z) = \frac{(z^2;q^2)_\infty}{(-z^2q;q^2)_\infty}, \quad \varrho^{\epsilon_1,\epsilon_2}(z) = \Big(\frac{(z;q^4)_\infty}{(zq^2;q^4)_\infty}\Big)^{\epsilon_1\epsilon_2}. \tag{18}$$

Then the matrix elements of $S^{1,1}(z), S^{1,2}(z)$ and $S^{\epsilon_1,\epsilon_2}(z)$ are rational functions of q and z.

2.3. Example

Let us present an explicit form of the matrix element (12) for $n = 1$. It was worked out earlier in [8, Prop.2] by using a formula for $R^{a,b,c}_{i,j,k}$ different from (9). For simplicity we concentrate on the case $s = t = 1$ and write $S^{s,t}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}}$ as $S(z)^{a,b}_{i,j}$ with $a,b,i,j \in \mathbb{Z}_{\geq 0}$. A direct calculation using (9) and (18) leads to

$$\begin{aligned} &S(z)(|i\rangle \otimes |j\rangle) = \sum_{a,b\geq 0} S(z)^{a,b}_{i,j}|a\rangle \otimes |b\rangle, \qquad S(z)^{a,b}_{i,j} = z^{a-i}\frac{(q^2)_i(q^2)_j}{(q^2)_a(q^2)_b} S(z)^{i,j}_{a,b}, \\ &S(z)^{a,b}_{i,j} = \delta^{a+b}_{i+j} \sum_{\lambda,\mu} (-1)^\lambda q^{j(1-a)+\mu(\mu-1)} \binom{j}{\lambda}_{q^2} \binom{\lambda+i}{b}_{q^2} \frac{(-q;q)_{i-a}(z;q)_{a+\lambda-\mu}}{(-zq,q)_{i+\lambda-\mu}} \quad (0 \leq a \leq i). \end{aligned} \tag{19}$$

The last sum is over $\lambda, \mu \in \mathbb{Z}_{\geq 0}$ such that $\lambda + \mu = j$ and $\lambda + i \geq b$. Thus it is actually a single sum over $\max(0, b-i) \leq \lambda \leq j$. The formula (19) is simpler than [8, eq.(2.19)]. From our main Theorem 4 it follows that $S^{a,b}_{i,j}(z=1) = \delta^a_j \delta^b_i$, which is consistent with the above result.

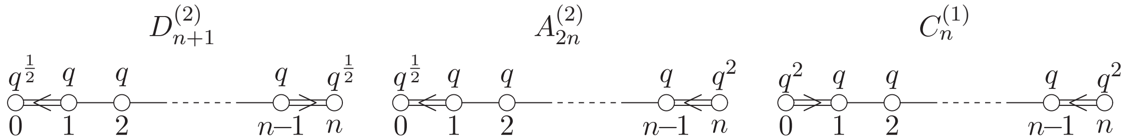
3. Quantum R matrices for q-oscillator representations

The Drinfeld-Jimbo quantum affine algebras without derivation $U_q = U_q(D^{(2)}_{n+1})$, $U_q(A^{(2)}_{2n})$ and $U_q(C^{(1)}_n)$ are the Hopf algebras generated by $e_i, f_i, k^{\pm 1}_i$ $(0 \leq i \leq n)$ satisfying the relations [3, 4]:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \ [k_i, k_j] = 0, \ k_i e_j k_i^{-1} = q_i^{a_{ij}} e_j, \ k_i f_j k_i^{-1} = q_i^{-a_{ij}} f_j, \ [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu e_i^{(1-a_{ij}-\nu)} e_j e_i^{(\nu)} = 0, \qquad \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu f_i^{(1-a_{ij}-\nu)} f_j f_i^{(\nu)} = 0 \ \ (i \neq j),$$

where $e^{(\nu)}_i = e^\nu_i/[\nu]_{q_i}!$, $f^{(\nu)}_i = f^\nu_i/[\nu]_{q_i}!$ with $[\nu]_q = [\nu]_q[\nu-1]_q \cdots [1]_q$. The Cartan matrix $(a_{ij})_{0\leq i,j\leq n}$ [5] is given by $a_{i,j} = 2\delta_{i,j} - \max((\log q_j)/(\log q_i), 1)\delta_{|i-j|,1}$. The data q_i is specified above the corresponding vertex $i\,(0 \le i \le n)$ in the Dynkin diagrams:



We employ the coproduct Δ of the form $\Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}$, $\Delta(e_i) = 1 \otimes e_i + e_i \otimes k_i$ and $\Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i$.

3.1. q-oscillator representations

We introduce representations of U_q on the tensor product of the Fock space $\hat{F}^{\otimes n}$ or $F^{\otimes n}$, where $\hat{F} = \bigoplus_{m\ge 0} \mathbb{C}(q^{\frac{1}{2}})|m\rangle$ is a slight extension of the coefficient field of F. They all factor through an algebra homomorphism from U_q to the q-oscillator algebra as shown in [10, Prop. 2.1]. As in the previous section we write the elements of $\hat{F}^{\otimes n}$ as $|\mathbf{m}\rangle = |m_1\rangle \otimes \cdots \otimes |m_n\rangle \in \hat{F}^{\otimes n}$ for $\mathbf{m} = (m_1, \dots, m_n) \in (\mathbb{Z}_{\ge 0})^n$ and describe the changes in $\mathbf{m}$ by the vectors $\mathbf{e}_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0) \in \mathbb{Z}^n$. In the following propositions $\kappa = \frac{q+1}{q-1}$ and x is a nonzero parameter.

Proposition 1. *The following defines an irreducible $U_q(D^{(2)}_{n+1})$ module structure on $\hat{F}^{\otimes n}$.*

$$\begin{aligned}
e_0|\mathbf{m}\rangle &= x|\mathbf{m}+\mathbf{e}_1\rangle,\\
f_0|\mathbf{m}\rangle &= i\kappa[m_1]x^{-1}|\mathbf{m}-\mathbf{e}_1\rangle,\\
k_0|\mathbf{m}\rangle &= -iq^{m_1+\frac{1}{2}}|\mathbf{m}\rangle,\\
e_j|\mathbf{m}\rangle &= [m_j]|\mathbf{m}-\mathbf{e}_j+\mathbf{e}_{j+1}\rangle \quad (1 \le j \le n-1),\\
f_j|\mathbf{m}\rangle &= [m_{j+1}]|\mathbf{m}+\mathbf{e}_j-\mathbf{e}_{j+1}\rangle \quad (1 \le j \le n-1),\\
k_j|\mathbf{m}\rangle &= q^{-m_j+m_{j+1}}|\mathbf{m}\rangle \quad (1 \le j \le n-1),\\
e_n|\mathbf{m}\rangle &= i\kappa[m_n]|\mathbf{m}-\mathbf{e}_n\rangle,\\
f_n|\mathbf{m}\rangle &= |\mathbf{m}+\mathbf{e}_n\rangle,\\
k_n|\mathbf{m}\rangle &= iq^{-m_n-\frac{1}{2}}|\mathbf{m}\rangle.
\end{aligned}$$

Proposition 2. *The following defines an irreducible $U_q(A^{(2)}_{2n})$ module structure on $\hat{F}^{\otimes n}$.*

$$\begin{aligned}
e_0|\mathbf{m}\rangle &= x|\mathbf{m}+\mathbf{e}_1\rangle,\\
f_0|\mathbf{m}\rangle &= i\kappa[m_1]x^{-1}|\mathbf{m}-\mathbf{e}_1\rangle,\\
k_0|\mathbf{m}\rangle &= -iq^{m_1+\frac{1}{2}}|\mathbf{m}\rangle,\\
e_j|\mathbf{m}\rangle &= [m_j]|\mathbf{m}-\mathbf{e}_j+\mathbf{e}_{j+1}\rangle \quad (1 \le j \le n-1),\\
f_j|\mathbf{m}\rangle &= [m_{j+1}]|\mathbf{m}+\mathbf{e}_j-\mathbf{e}_{j+1}\rangle \quad (1 \le j \le n-1),\\
k_j|\mathbf{m}\rangle &= q^{-m_j+m_{j+1}}|\mathbf{m}\rangle \quad (1 \le j \le n-1),\\
e_n|\mathbf{m}\rangle &= \frac{[m_n][m_n-1]}{[2]^2}|\mathbf{m}-2\mathbf{e}_n\rangle,\\
f_n|\mathbf{m}\rangle &= |\mathbf{m}+2\mathbf{e}_n\rangle,\\
k_n|\mathbf{m}\rangle &= -q^{-2m_n-1}|\mathbf{m}\rangle.
\end{aligned}$$

Proposition 3. *The following defines an irreducible $U_q(C_n^{(1)})$ module structure on $(F^{\otimes n})_+$ and $(F^{\otimes n})_-$ defined in (17).*

$$
\begin{aligned}
e_0|\mathbf{m}\rangle &= x|\mathbf{m}+2\mathbf{e}_1\rangle,\\
f_0|\mathbf{m}\rangle &= \frac{[m_1][m_1-1]}{[2]^2}x^{-1}|\mathbf{m}-2\mathbf{e}_1\rangle,\\
k_0|\mathbf{m}\rangle &= -q^{2m_1+1}|\mathbf{m}\rangle,\\
e_j|\mathbf{m}\rangle &= [m_j]|\mathbf{m}-\mathbf{e}_j+\mathbf{e}_{j+1}\rangle \quad (1 \le j \le n-1),\\
f_j|\mathbf{m}\rangle &= [m_{j+1}]|\mathbf{m}+\mathbf{e}_j-\mathbf{e}_{j+1}\rangle \quad (1 \le j \le n-1),\\
k_j|\mathbf{m}\rangle &= q^{-m_j+m_{j+1}}|\mathbf{m}\rangle \quad (1 \le j \le n-1),\\
e_n|\mathbf{m}\rangle &= \frac{[m_n][m_n-1]}{[2]^2}|\mathbf{m}-2\mathbf{e}_n\rangle,\\
f_n|\mathbf{m}\rangle &= |\mathbf{m}+2\mathbf{e}_n\rangle,\\
k_n|\mathbf{m}\rangle &= -q^{-2m_n-1}|\mathbf{m}\rangle.
\end{aligned}
$$

We call these irreducible representations the *q-oscillator representations* of U_q. For the twisted case $U_q(D_{n+1}^{(2)})$ and $U_q(A_{2n}^{(2)})$, they are *singular* at $q=1$ because of the factor κ.

3.2. Quantum R matrices

Let $V = \hat{F}^{\otimes n}$ for $U_q(D_{n+1}^{(2)}), U_q(A_{2n}^{(2)})$ and $V = F^{\otimes n}$ for $U_q(C_n^{(1)})$. First we consider $U_q(D_{n+1}^{(2)})$ and $U_q(A_{2n}^{(2)})$. Let $V_x = \hat{F}^{\otimes n}[x, x^{-1}]$ be the representation space of U_q in Propositions 1 and 2. By the existence of the universal R matrix [3] there exists an element $R \in \mathrm{End}(V_x \otimes V_y)$ such that

$$\Delta'(g)R = R\Delta(g) \quad \forall g \in U_q \tag{20}$$

up to an overall scalar. Here Δ' is the opposite coproduct defined by $\Delta' = P \circ \Delta$, where $P(u \otimes v) = v \otimes u$ is the exchange of the components. A little inspection of our representations shows that R depends on x and y only through the ratio $z = x/y$. Moreover $V_x \otimes V_y$ is irreducible ([9, Prop. 12] and Sec. 4 of this paper) hence R is determined only by postulating (20) for $g = k_r, e_r$ and f_r with $0 \le r \le n$. Thus denoting the R by $R(z)$, we may claim [4] that it is determined by the conditions

$$(k_r \otimes k_r)R(z) = R(z)(k_r \otimes k_r), \tag{21}$$

$$(e_r \otimes 1 + k_r \otimes e_r)R(z) = R(z)(1 \otimes e_r + e_r \otimes k_r), \tag{22}$$

$$(1 \otimes f_r + f_r \otimes k_r^{-1})R(z) = R(z)(f_r \otimes 1 + k_r^{-1} \otimes f_r) \tag{23}$$

for $0 \le r \le n$ up to an overall scalar. We fix the normalization of $R(z)$ by

$$R(z)(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle) = |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle, \tag{24}$$

where $|\mathbf{0}\rangle \in \hat{F}^{\otimes n}$ is defined in the beginning of Section 3.1 with $\mathbf{0} = (0, \ldots, 0)$. We call the intertwiner $R(z)$ the *quantum R matrix* for q-oscillator representation. It satisfies the Yang-Baxter equation

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x). \tag{25}$$

Next we consider $U_q(C_n^{(1)})$. Denote by $V_x^{\pm} = (F^{\otimes n})_{\pm}[x, x^{-1}]$ the representation spaces in Proposition 3 and set $V_x = V_x^+ \oplus V_x^- = F^{\otimes n}[x, x^{-1}]$. See (17) for the definition of $(F^{\otimes n})_{\pm}$. We define the quantum R matrix $R(z)$ to be the direct sum

$$R(z) = R^{+,+}(z) \oplus R^{+,-}(z) \oplus R^{-,+}(z) \oplus R^{-,-}(z), \tag{26}$$

where each $R^{\epsilon_1,\epsilon_2}(z) \in \mathrm{End}(V_x^{\epsilon_1} \otimes V_y^{\epsilon_2})$ is the quantum R matrix with the normalization condition

$$\begin{aligned} &R^{+,+}(z)(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle) = |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle, && R^{+,-}(z)(|\mathbf{0}\rangle \otimes |\mathbf{e}_1\rangle) = \frac{-iq^{1/2}}{1-z}|\mathbf{0}\rangle \otimes |\mathbf{e}_1\rangle, \\ &R^{-,+}(z)(|\mathbf{e}_1\rangle \otimes |\mathbf{0}\rangle) = \frac{-iq^{1/2}}{1-z}|\mathbf{e}_1\rangle \otimes |\mathbf{0}\rangle, && R^{-,-}(z)(|\mathbf{e}_1\rangle \otimes |\mathbf{e}_1\rangle) = \frac{z-q^2}{1-zq^2}|\mathbf{e}_1\rangle \otimes |\mathbf{e}_1\rangle. \end{aligned} \tag{27}$$

The R matrix $R(z)$ satisfies the Yang-Baxter equation (25). In fact it is decomposed into the finer equalities ($\epsilon_1, \epsilon_2, \epsilon_3 = \pm$)

$$R_{12}^{\epsilon_1,\epsilon_2}(x) R_{13}^{\epsilon_1,\epsilon_3}(xy) R_{23}^{\epsilon_2,\epsilon_3}(y) = R_{23}^{\epsilon_2,\epsilon_3}(y) R_{13}^{\epsilon_1,\epsilon_3}(xy) R_{12}^{\epsilon_1,\epsilon_2}(x).$$

3.3. Main theorem

Define the operator K acting on $\hat{F}^{\otimes n}$ by $K|\mathbf{m}\rangle = (-iq^{\frac{1}{2}})^{m_1+\cdots+m_n}|\mathbf{m}\rangle$. Introduce the gauge transformed quantum R matrix by

$$\tilde{R}(z) = (K^{-1} \otimes 1) R(z) (1 \otimes K). \tag{28}$$

It is easy to see that $\tilde{R}(z)$ also satisfies the Yang-Baxter equation (25).

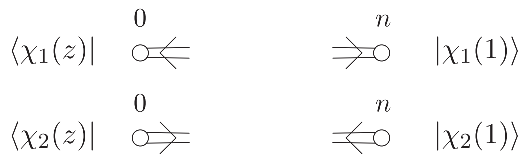
In Section 2.2 the solutions $S^{s,t}(z)$ of the Yang-Baxter equation have been constructed from the 3d R in (11), (12) and (18). In Section 3.2 the quantum R matrices for q-oscillator representations of $U_q(D_{n+1}^{(2)})$, $U_q(A_{2n}^{(2)})$ and $U_q(C_n^{(1)})$ have been defined. The next theorem, which is the main result of [9], states the precise relation between them. (See (13) for $S^{2,1}(z)$.)

Theorem 4. *Denote by $\tilde{R}_{\mathfrak{g}}(z)$ the gauge transformed quantum R matrix (28) for $U_q(\mathfrak{g})$. Then the following equalities hold:*

$$S^{1,1}(z) = \tilde{R}_{D_{n+1}^{(2)}}(z), \quad S^{1,2}(z) = \tilde{R}_{A_{2n}^{(2)}}(z), \quad S^{2,2}(z) = \tilde{R}_{C_n^{(1)}}(z),$$

where the last one means $S^{\epsilon_1,\epsilon_2}(z) = \tilde{R}^{\epsilon_1,\epsilon_2}(z)$ between (16) and (26) with the gauge transformation (28).

Remark 5. Theorem 4 suggests the following correspondence between the boundary vectors (10) with the end shape of the Dynkin diagrams:



Consistently with Remark 5, $S^{2,1}(z)$, which is reducible to $S^{1,2}(z^{1/2})$ by (13), is identified [10] with the quantum R matrix for q-oscillator representation of another $U_q(A_{2n}^{(2)})$ realized as the affinization of the classical part $U_q(B_n)$. (Proposition 2 corresponds to taking the classical part to be $U_q(C_n)$.) As far as $\langle\chi_1(z)|$ and $|\chi_1(1)\rangle$ are concerned, the above correspondence agrees with the observation made in [11, Remark 7.2] on the similar result concerning a 3d L operator. With regard to $\langle\chi_2(z)|$ and $|\chi_2(1)\rangle$, the relevant affine Lie algebras $A_{2n}^{(2)}$ and $C_n^{(1)}$ in this paper are the subalgebras of $B_{n+1}^{(1)}$ and $D_{n+2}^{(1)}$ in [11, Theorem 7.1] obtained by folding their Dynkin diagrams.

4. Proof of the irreducibility of the tensor product

In [9] we gave a proof of the following proposition.

Proposition 6 (Prop. 12 of [9]). *As a $U_q(D^{(2)}_{n+1})$ or $U_q(A^{(2)}_{2n})$ module $V_x \otimes V_y$ is irreducible. As a $U_q(C^{(1)}_n)$ module each $V^{\epsilon_1}_x \otimes V^{\epsilon_2}_y$ $(\epsilon_1, \epsilon_2 = \pm)$ is irreducible.*

Since the explanation there was not sufficient, we give the detailed proof here. Let $\mathfrak{g} = D^{(2)}_{n+1}, A^{(2)}_{2n}$ or $C^{(1)}_n$, $I = \{0, 1, \dots, n\}$, and for a subset J of I let $U_q(\mathfrak{g}_J)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{e_j, f_j, k^{\pm 1}_j \mid j \in J\}$. Recall the vector v_l [9, Prop. 4] for $\mathfrak{g} = D^{(2)}_{n+1}$ and v^{ϵ}_l [9, Prop. 5] for $A^{(2)}_{2n}, C^{(1)}_n$.

Proposition 7. *For $\mathfrak{g} = D^{(2)}_{n+1}$*

$$V^{\otimes 2} = \sum_{l=0}^{\infty} U_q(\mathfrak{g}_{I \setminus \{0\}}) v_l,$$

and for $\mathfrak{g} = A^{(2)}_{2n}, C^{(1)}_n$

$$V^{\epsilon_1} \otimes V^{\epsilon_2} = \sum_{\substack{l=0 \\ (-1)^l = \epsilon_1 \epsilon_2}}^{\infty} U_q(\mathfrak{g}_{I \setminus \{0\}}) v^{\epsilon_1}_l.$$

This is an immediate consequence of the following two lemmas. Set $w_{l,k} = |k\mathbf{e}_{n-1}\rangle \otimes |(l-k)\mathbf{e}_n\rangle \in V^{\otimes 2}$ $(l \geq 0, 0 \leq k \leq l)$.

Lemma 8. *For $\mathfrak{g} = D^{(2)}_{n+1}$*

$$w_{l,k} \in \sum_{j=0}^{l} U_q(\mathfrak{g}_{\{n-1,n\}}) v_j,$$

and for $\mathfrak{g} = A^{(2)}_{2n}, C^{(1)}_n$

$$w_{l,k} \in \sum_{\substack{j=0 \\ j \equiv l \pmod 2}}^{l} U_q(\mathfrak{g}_{\{n-1,n\}}) v^{\epsilon}_j \quad \textit{where } \epsilon = (-1)^k.$$

Proof. We treat the $\mathfrak{g} = D^{(2)}_{n+1}$ case first. Note that the set of vectors $B = \{v_l, f_n v_{l-1}, \dots, f^l_n v_0\}$ is linearly independent in the vector subspace spanned by $\{|k\mathbf{e}_n\rangle \otimes |(l-k)\mathbf{e}_n\rangle \mid 0 \leq k \leq l\}$. Hence, B is also a basis and

$$w_{l,0} = |\mathbf{0}\rangle \otimes |l\mathbf{e}_n\rangle \in \sum_{j=0}^{l} U_q(\mathfrak{g}_{\{n\}}) v_j. \tag{29}$$

Next note that

$$w_{l,k} = (f_{n-1} f_n - q^{-1} f_n f_{n-1}) w_{l-1,k-1} + i \frac{q^{l-k+1/2}}{[l-k+1]} f_{n-1} w_{l,k-1} \quad (1 \leq k \leq l).$$

This relation together with (29) shows the result.

Suppose now $\mathfrak{g} = A^{(2)}_{2n}, C^{(1)}_n$. We compare $B' = \{v^{\epsilon}_l, f_n v^{\epsilon}_{l-1}, \dots, f^{l'}_n v^{\epsilon}_{l-2l'}\}$ $(l' = \lfloor l/2 \rfloor\ (\epsilon = +), = \lfloor (l-1)2 \rfloor\ (\epsilon = -))$ and the subspace spanned by $\{|k\mathbf{e}_n\rangle \otimes |(l-k)\mathbf{e}_n\rangle \mid 0 \leq k \leq l, (-1)^k = \epsilon\}$.

We have

$$w_{l,0}=|\mathbf{0}\rangle\otimes|l\mathbf{e}_n\rangle\in\sum_{\substack{j=0\\ j\equiv l\ (\mathrm{mod}\ 2)}}^{l}U_q(\mathfrak{g}_{\{n\}})v_j^+,$$

$$|\mathbf{e}_n\rangle\otimes|(l-1)\mathbf{e}_n\rangle\in\sum_{\substack{j=0\\ j\equiv l\ (\mathrm{mod}\ 2)}}^{l}U_q(\mathfrak{g}_{\{n\}})v_j^-\quad(l\geq 1).$$

From

$$w_{1,1}=f_{n-1}v_1^-,\quad w_{2,1}=\frac{q^{-1}}{[2]}v_0^-+\frac{1}{[2]^2}f_{n-1}v_2^-,$$
$$w_{l,1}=-\frac{[l-1]}{[l]}f_nw_{l-2,1}+\frac{q^{-1}[l-1]}{[2][l]}(f_{n-1}f_n-q^{-2}f_nf_{n-1})|\mathbf{e}_n\rangle\otimes|(l-3)\mathbf{e}_n\rangle$$
$$+\frac{q^{l-1}}{[l]}f_{n-1}|\mathbf{e_n}\rangle\otimes|(l-1)\mathbf{e}_n\rangle\quad(l\geq 3),$$
$$w_{l,k}=(\frac{1}{[2]}f_{n-1}^2f_n-q^{-1}f_{n-1}f_nf_{n-1}+q^{-2}f_nf_{n-1}^2)w_{l-2,k-2}+\frac{q^{2l-2k+1}}{[l-k+1][l-k+2]}f_{n-1}^2w_{l,k-2}$$
$$(2\leq k\leq l),$$

we obtain the result. □

Lemma 9. *Let W_l be the vector subspace of $V^{\otimes 2}$ spanned by $|\sum_{j=1}^n k_j\mathbf{e}_j\rangle\otimes|\sum_{j=1}^n k'_j\mathbf{e}_j\rangle$ such that $\sum_{j=1}^n(k_j+k'_j)=l$. Then we have*

$$W_l=\sum_{0\leq k\leq l}U_q(\mathfrak{g}_{I\setminus\{0,n\}})w_{l,k}.$$

Proof. As a $U_q(\mathfrak{g}_{I\setminus\{0,n\}})(=U_q(A_{n-1}))$-module W_l is isomorphic to $\bigoplus_{k=0}^l L(k\mathbf{e}_n)\otimes L((l-k)\mathbf{e}_n)$, where $L(\lambda)$ stands for the irreducible highest weight $U_q(A_{n-1})$-module with highest weight λ. By representation theory of $U_q(A_{n-1})$, $L(k\mathbf{e}_n)\otimes L((l-k)\mathbf{e}_n)$ is generated by the highest weight vectors of weight of the form $j\mathbf{e}_{n-1}+(l-j)\mathbf{e}_n$ for some $0\leq j\leq\min(k,l-k)$. Hence, it is enough to show that any vector in W_l of weight of the form $(l-j)\mathbf{e}_n+j\mathbf{e}_{n-1}$ is generated by $w_{l,j}$ over $U_q(\mathfrak{g}_{\{n-1\}})(=U_q(sl_2))$. But it is a well-known fact from representation theory of $U_q(sl_2)$, namely, $L(a\mathbf{e}_n)\otimes L(b\mathbf{e}_n)$ is generated by $|a\mathbf{e}_{n-1}\rangle\otimes|b\mathbf{e}_n\rangle$, where $|a\mathbf{e}_{n-1}\rangle$ (resp. $|b\mathbf{e}_n\rangle$) is the lowest (resp. highest) weight vector. □

Proof of Prop. 6. Suppose $\mathfrak{g}=D_{n+1}^{(2)}$ and let W be a nonzero submodule of $V^{\otimes 2}$. In the proof of Prop. 12 of [9], we have shown that W contains v_l for any $l\geq 0$. Similarly, for $\mathfrak{g}=A_{2n}^{(2)}$ (resp. $C_n^{(1)}$), using Lemma 8 (resp. 10) of [9], we can show a nonzero submodule W of $V^{\otimes 2}$ (resp. $V^{\epsilon_1}\otimes V^{\epsilon_2}$) contains v_l^ϵ for any $l\geq 0,\epsilon=\pm$ (resp. $v_l^{\epsilon_1}$ for any l such that $(-1)^l=\epsilon_1\epsilon_2$). The claim now follows from Prop. 7. □

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