Integral quantizations with POVM and some applications

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Abstract. We present quantizations of functions (or distributions) on a measure space viewed as a classical set. They are based on positive operator-valued measures. To illustrate the large range of potentialities of the method, we develop examples where the classical sets are the group SU(2), the (projective) Weyl-Heisenberg group, and the affine group. Applications to quantum cosmology and to the quantization of constraints are outlined.

1. General framework for quantization

Quantization of a "classical" object X, such as a phase space in mechanics, can be formulated as a linear map $f \mapsto A_f$ from the space of $\mathcal{C}(X)$ of complex-valued functions $f(x)$ on X to a space $\mathcal{A}(\mathcal{H})$ of linear operators A_f in some complex Hilbert space $\mathcal{H}$ (we omit here the necessary domain considerations). One further requires that

(i) the constant function $f = 1$ be mapped to the identity operator,
(ii) real functions be mapped to (essentially) self-adjoint or at least to symmetric operators A_f in $\mathcal{A}(\mathcal{H})$.

This broad definition can be applied to specific physical problems by adding appropriate structure to X and to $\mathcal{C}(X)$, such as measure, topology, manifold structure, closure under algebraic operations, etc. Moreover, one is free to define which functions or operators are observables, whose spectra have a physical meaning in terms of measurement.

Finally, given an appropriate structure, like a measure or a Poisson structure, one can define an unambiguous classical limit $\check{f} \mapsto f$ of quantum observables through X-labeled expectation values of A_f defining a new function, $\check{f}(x)$ (namely, "lower symbols")

We describe in Section 2 a general quantization procedure based on (normalized) positive operator-valued measures (POVM). We illustrate the method in Section 3 with the cases where the classical set X has a group structure, like SU(2), the projective Weyl-Heisenberg group, and the affine group. Section 4 is an application of the Weyl-Heisenberg integral quantization to simple constraints in the complex plane, showing the difference between two approaches.

Journal of Physics: Conference Series **597** (2015) 012037 doi:10.1088/1742-6596/597/1/012037

2. Integral quantization with POVM

Integral quantization [1, 2, 3, 4, 5] is a generalization of the Berezin-Toeplitz-Klauder (or "anti-Wick" or standard coherent state) quantization. It can be formulated as follows.

One starts from a measure space $(X, \mathfrak{B}, \nu)$ (or (X, ν) for short), where $\mathfrak{B}$ is the σ-algebra of ν-measurable subsets, a separable Hilbert space $\mathcal{H}$, and an X-labelled family of positive semi-definite and unit trace operators, i.e. density operators, on $\mathcal{H}$,

$$X \ni x \mapsto \rho(x) \in \mathcal{L}(\mathcal{H}), \quad \rho(x) > 0, \quad \mathrm{tr}(\rho(x)) = 1, \tag{1}$$

resolving the identity I on $\mathcal{H}$,

$$\int_X \rho(x)\, \mathrm{d}\nu(x) = I, \quad \text{in a weak sense.} \tag{2}$$

One then equips X with a suitable topology, so the normalized positive operator-valued measure (POVM) $\mathfrak{m}_\rho$ on the corresponding σ-algebra $\mathfrak{B}(X)$ of Borel sets is defined through the following map of

$$\mathfrak{B}(X) \ni \Delta \mapsto \mathfrak{m}_\rho(\Delta) = \int_\Delta \rho(x)\, \mathrm{d}\nu(x) \tag{3}$$

into the convex cone $\mathcal{L}^+(\mathcal{H})$ of positive bounded operators on $\mathcal{H}$. As a result, the resolution of the identity produces the X-labeled family of probability distributions

$$X \ni x_0, x \mapsto p_{x_0}(x) = \mathrm{tr}\left(\rho(x_0)\, \rho(x)\right) \tag{4}$$

on (X, ν). The expression $p_{x_0}(x)$ measures the degree of localization of x w.r.t. x_0, in a sense determined by the family $\rho(x)$, and vice versa due to the symmetry $p_{x_0}(x) = p_x(x_0)$ on the measure space (X, ν). In the particular case where $\rho(x)$ is rank-one projector operator (i.e. "pure coherent state"),

$$\rho(x) = |x\rangle \langle x|\,, \quad \langle x\, |x\rangle = 1\,, \tag{5}$$

one has

$$p_{x_0}(x) = |\, \langle x_0\, |x\rangle\, |^2\,, \tag{6}$$

From the fact that any density operator ρ is Hilbert-Schmidt, with norm $\|\rho\| = \sqrt{\mathrm{tr}\rho\rho^\dagger} = \sqrt{\mathrm{tr}\rho^2}$, one can introduce the associated distance

$$d_{\mathrm{HS}}(x, x') = \|\rho(x) - \rho(x')\| = \sqrt{\mathrm{tr}(\rho(x) - \rho(x'))^2}\,. \tag{7}$$

This object forces any pair of points in X to be finitely separated since we have

$$d_{\mathrm{HS}}(x, x') = \sqrt{\mathrm{tr}\left((\rho(x))^2 + (\rho(x'))^2 - 2\rho(x)\rho(x')\right)} \leq \sqrt{2}\sqrt{1 - \mathrm{tr}(\rho(x)\rho(x'))} \leq \sqrt{2}\,. \tag{8}$$

The integral POVM quantization is the linear map

$$f \mapsto A_f = \int_X \rho(x)\, f(x)\, \mathrm{d}\nu(x)\,. \tag{9}$$

This should be understood in terms of the sesquilinear form

$$B_f(\psi_1, \psi_2) = \int_X \langle \psi_1 |\, \rho(x)\, |\psi_2\rangle\; f(x)\, \mathrm{d}\nu(x)\,, \tag{10}$$

defined on a dense subspace of $\mathcal{H}$. There is a caveat: if f is real and at least semi-bounded, Friedrich extension of B_f univocally defines a self-adjoint operator. However, if f is not semi-bounded, there is no natural choice of a self-adjoint operator associated with B_f. We must give more details about the Hilbert space and operator domains, see for instance [6].

Self-adjointness is essential in the Copenhagen-Dirac-von Neumann interpretation of quantum physics, since the spectral theorem for bounded or unbounded self-adjoint operators is the key for (sharp) quantum measurement. Hence, if A_f is self-adjoint, its spectrum, which is determined by its projector-valued (PV) spectral measure E_f corresponding to the integral representation

$$A_f = \int_{\mathbb{R}} \lambda \, \mathrm{d}E_f(\lambda)\,, \tag{11}$$

might have a remote connection with its classical spectrum $\{f(x)\,,\, x \in X\}$ appearing in the integral representation (9). While the POVM used for quantization - and built from a family $\rho(x)$ resolving the identity with respect to the fixed measure ν - should be considered as a frame to analyse such functions on X, the PV measure in the integral representation (11) is proper to the quantum observable and to functions of it. However, there are simple examples (see below for position and momentum in the Weyl-Heisenberg case) where classical and quantum spectra can be considered as identical regardless of the difference between their respective PV and POVM. Moreover, the frame $x \mapsto \rho(x)$ itself may be associated to a specific system to be quantized. A nice pedagogical example (the seastar) is presented in Chap. 11 of [2].

2.1. Semi-classical aspects

Some of the properties of the operator A_f can be grasped by examining the lower (Lieb) or covariant (Berezin) symbol, which in the present generalization reads as

$$A_f \mapsto \check{f}(x) := \mathrm{tr}(\rho(x)\, A_f)\,. \tag{12}$$

This function could be viewed as a sort of Wigner function [1]. However, it has a real probabilistic content, since it is the local averaging of the classical function f with respect to the probability distribution $\mathrm{tr}(\rho(x)\rho(x'))$

$$f(x) \mapsto \check{f}(x) = \int_X f(x')\, \mathrm{tr}(\rho(x)\rho(x'))\, \mathrm{d}\nu(x')\,. \tag{13}$$

The Bargmann-Segal-like map $f \mapsto \check{f}$ above is in general a regularization of the function f. Indeed, depending on the topology on X, the original f can be singular. It can be even a distribution (see below).

2.2. Quantum measurement: sharp or unsharp?

In the quantum context for which A_f is a self-adjoint operator viewed as an observable for a given system, with spectral decomposition (11), and given a density operator $\rho_m = \sum_i q_i \left|\phi_i\right\rangle \left\langle\phi_i\right|$ describing the mixed state of an ensemble, one interprets the quantity

$$\mathrm{tr}\left(\rho_m \mathrm{d}E_f(\lambda)\right) \tag{14}$$

à la Copenhagen-Dirac-von Neumann, that is, as the probability of measuring the spectral value λ. Now, the expectation value of the measurement of the observable A_f when the system is in the state ρ_m is given by the alternative expressions

$$\mathrm{tr}\left(\rho_m A_f\right) = \int_{\mathbb{R}} \lambda \, \mathrm{tr}\left(\mathrm{d}E_f(\lambda)\right) \qquad \text{(PV, sharp)} \tag{15}$$

$$= \int_X f(x)\, \mathrm{tr}(\rho_m \rho(x))\, \mathrm{d}\nu(x) \qquad \text{(POVM, unsharp)}\,. \tag{16}$$

Journal of Physics: Conference Series **597** (2015) 012037 doi:10.1088/1742-6596/597/1/012037

The latter is the average of the original f with respect to the probability density $p_m(x) := \mathrm{tr}(\rho_m \rho(x))$. Hence, it might be viewed as an unsharp measurement, possibly through some marginal integration [7, 8]. Note that when the ρ_m is chosen as the element $\rho_m = \rho(x_0)$ of the family of density operators (1), the measurement (15) has the averaging interpretation given by (13).

3. Covariant integral quantization with UIR of a group

3.1. The material

Let G be a Lie group with left Haar measure $\mathrm{d}\mu(g)$, and let $g \mapsto U(g)$ be a unitary irreducible representation (UIR) of G in a Hilbert space $\mathcal{H}$. Consider a bounded operator M on $\mathcal{H}$ such that the operator

$$R := \int_G \mathsf{M}(g)\, \mathrm{d}\mu(g)\,, \quad \mathsf{M}(g) := U(g)\, \mathsf{M}\, U^\dagger(g)\,, \tag{17}$$

is defined in a weak sense. From the left invariance of $\mathrm{d}\mu(g)$, R commutes with all operators $U(g)$, $g \in G$, and so from Schur's Lemma, $R = c_{\mathsf{M}} I$ with

$$c_{\mathsf{M}} = \int_G \mathrm{tr}\left(\rho_0\, \mathsf{M}(g)\right)\, \mathrm{d}\mu(g)\,, \tag{18}$$

where the unit trace positive operator ρ_0 is chosen in order to make the integral convergent. The resolution of the identity follows:

$$\int_G \mathsf{M}(g)\, \mathrm{d}\nu(g) = I\,, \quad \mathrm{d}\nu(g) := \mathrm{d}\mu(g)/c_{\mathsf{M}}\,. \tag{19}$$

For a square-integrable UIR U for which ρ is an "admissible" density operator, $c(\eta) = \int_G \mathrm{d}\mu(g)\, |\mathrm{tr}(\rho U(g))|^2 < \infty$, the resolution of the identity is then obeyed by the family

$$\rho(g) = U(g) \rho U^\dagger(g) \tag{20}$$

This allows for a *covariant* integral quantization of complex-valued functions on the group

$$f \mapsto A_f = \int_G \rho(g)\, f(g)\, \mathrm{d}\nu(g)\,, \tag{21}$$

which is covariant in the sense that

$$U(g) A_f U^\dagger(g) = A_{\mathsf{U}(g)f}\,, \tag{22}$$

where $g \mapsto \mathsf{U}(g)$ is the regular representation of G for $f \in L^2(G, \mathrm{d}\mu(g))$

$$(\mathsf{U}(g)f)(g') := f(g^{-1}g')\,. \tag{23}$$

The semiclassical portrait and the corresponding classical limit are given via the lower symbol

$$\check{f}(g) := \int_G tr(\rho(g)\, \rho(g'))\, f(g') d\nu\left(g'\right)\,, \tag{24}$$

a generalization of the Berezin transform on G.

Journal of Physics: Conference Series **597** (2015) 012037 doi:10.1088/1742-6596/597/1/012037

3.2. Illustration with SU(2)

A proper rotation in space $\mathcal{R}(\omega,\hat{n}) : \vec{r} \mapsto \vec{r'}$, where $\hat{n}$ a unit vector $\hat{n}$ and ω a rotation angle $0 \leq \omega < 2\pi$, can be defined as

$$\vec{r'} = \mathcal{R}(\omega,\hat{n}) \cdot \vec{r} = \vec{r}\cdot\hat{n}\,\hat{n} + \cos\omega\,\hat{n}\times(\vec{r}\times\hat{n}) + \sin\omega\,(\hat{n}\times\vec{r})\,. \tag{25}$$

In scalar-vector quaternionic form,

$$(0,\vec{r'}) = \xi(0,\vec{r})\bar{\xi}\,,\ \xi := \left(\cos\frac{\omega}{2}, \sin\frac{\omega}{2}\,\hat{n}\right)\,,$$

with $q = (q_0,\vec{q}) \in \mathbb{H}$ and

$$qq' := (q_0q_0' - \vec{q}\cdot\vec{q'}, q_0'\vec{q} + q_0\vec{q'} + \vec{q}\times\vec{q'})\,.$$

The (quaternionic) conjugate of $q = (q_0,\vec{q})$ is $\bar{q} = (q_0,-\vec{q})$, the squared norm is $\|q\|^2 = q\bar{q}$, and the inverse of a nonzero quaternion is $q^{-1} = \bar{q}/\|q\|^2$. Unit quaternions, i.e., quaternions with norm 1, form the multiplicative subgroup isomorphic to SU(2).

The correspondence between the canonical basis of $\mathbb{H} \simeq \mathbb{R}^4$, $(e_0 \equiv 1, e_1, e_2, e_3)$, and the Pauli matrices is

$$e_0 \leftrightarrow \sigma_0\,,\ \ e_a \leftrightarrow (-1)^{a+1} i\sigma_a\,,\ a = 1,2,3\,. \tag{26}$$

Unit quaternions constitute the three-sphere S^3. In hypersherical coordinates (α,θ,ϕ) of S^3 one has

$$\begin{aligned}&\mathrm{S}^3 \ni \xi = \|q\|\,(\cos\alpha, \sin\alpha\,\hat{n})\,, \quad 0 \leq \alpha,\theta \leq \pi\,,\ 0 \leq \phi < 2\pi\,,\\ &\dot{\xi} = \sin^2\alpha\,\sin\theta\,\mathrm{d}\alpha\,\mathrm{d}\theta\,\mathrm{d}\phi\,, \qquad \int_{\mathrm{SU}(2)} \dot{\xi} = 2\pi^2\,.\end{aligned}$$

The unit ball $\mathbb{B}$ in $\mathbb{R}^3$ parametrizes the set of 2×2 complex density matrices ρ. For general 2×2 complex density matrices, $\|\vec{a}\| \leq 1$, this becomes

$$\rho = \frac{1}{2}(1 + \vec{a}\cdot\vec{\sigma}) = \frac{1}{2}(1 - i(0,\vec{r})) \equiv \frac{1}{2}(1 - i\undergroup{\mathbf{r}}) \equiv \rho_{\undergroup{\mathbf{r}}}\,,\ \ r_i = (-1)^{i+1}a_i\,. \tag{27}$$

If $\|\vec{a}\| = 1$, i.e. $\vec{a} \in S^2$ ("Bloch sphere" in this context), with spherical coordinates (θ,ϕ), then ρ is the pure state $\rho = |\theta,\phi\rangle\,\langle\theta,\phi|$. For $\xi \in \mathrm{SU}(2)$, one defines the family of density matrices labeled by ξ:

$$\rho(\xi) := \xi\rho\bar{\xi} = \frac{1}{2}(1 - i\xi\undergroup{\mathbf{r}}\bar{\xi})\,. \tag{28}$$

It is then straightforward to prove the resolution of the identity

$$\int_{\mathrm{SU}(2)} \rho(\xi)\,\frac{\dot{\xi}}{\pi^2} = I\,, \tag{29}$$

which allows us to quantize functions on $SU(2)$ or S^3 along the linear map

$$f(\xi) \mapsto A_f = \int_{\mathrm{SU}(2)} f(\xi)\,\rho(\xi)\,\frac{\dot{\xi}}{\pi^2} \in \mathrm{M}(2,\mathbb{C})\,. \tag{30}$$

The symbols corresponding to the Pauli matrices can be obtained via the Hopf map $S^3 \to S^2$, where the Hopf fibration is understood in terms of the transitive action of rotations on S^2.

Journal of Physics: Conference Series **597** (2015) 012037 doi:10.1088/1742-6596/597/1/012037

Let us fix the North pole unit vector $\left(0,\hat{k}\right) \in S^2$. Then $\chi = \xi \left(0,\hat{k}\right) \bar{\xi}$ is the image of the rotation by the unit quaternion ξ. The components of χ correspond to the operators

$$A_{\chi_1} = \frac{a_3}{3}\sigma_1\,,\ A_{\chi_2} = -\frac{a_3}{3}\sigma_2\,,\ A_{\chi_3} = \frac{a_3}{3}\sigma_3\,. \tag{31}$$

With the redefinitions $2\omega = \beta\,,\ \psi_2 - \psi_1 = \gamma + \frac{\pi}{2}\,,\ \psi_1 + \psi_2 - 2\pi = -\alpha + \frac{\pi}{2}$, the algebra of Euler angle operators becomes

$$[A_\beta, A_\alpha] = iA_\gamma\,,\ [A_\gamma, A_\beta] = iA_\alpha\,,\ [A_\alpha, A_\gamma] = iA_\beta\,, \tag{32}$$

i.e., the Lie algebra of SU(2), as we also had with the quantization of the Hopf map. The Euler parametrization of SU(2) seems to be privileged from an algebraic point of view.

3.3. Weyl-Heisenberg covariant integral quantization

In the absence of square-integrability over G, there exists a definition of square-integrable covariant coherent states with respect to a left coset manifold $X = G/H$, with H a closed subgroup of G, equipped with a quasi-invariant measure ν [2]. For a global Borel section $\sigma : X \to G$ of the group, let ν_σ be the unique quasi-invariant measure defined by

$$\mathrm{d}\nu_\sigma(x) = \lambda(\sigma(x), x)\mathrm{d}\nu(x)\,, \tag{33}$$

where $\lambda(g,x)\mathrm{d}\nu(x) = \mathrm{d}\nu(g^{-1}x)$, $(\forall g \in G)$. Let U be a UIR which is square integrable mod(H) and ρ mod(H) an admissible density operator, i.e. $c_\eta := \int_X \mathrm{tr}\left(\rho\,\rho_\sigma(x)\right)\mathrm{d}\nu_\sigma(x) < \infty$. With $\rho_\sigma(x) = U(\sigma(x))\rho U(\sigma(x))^\dagger$, we have the resolution of the identity and the resulting quantization

$$f \mapsto A_f = \frac{1}{c_\eta}\int_X f(x)\,\rho_\sigma(x)\,\mathrm{d}\nu_\sigma(x) = I\,. \tag{34}$$

In the case of the Weyl-Heisenberg group, X is the coset space $G_{HW}/(\text{phase subgroup}) \sim \mathbb{C}$, while the measure is the Lebesgue measure in the complex plane $\mathrm{d}^2z = \mathrm{d}z \wedge \mathrm{d}\bar{z}/2i$. The triplet $\{a, a^\dagger, I\}$ generates the Weyl-Heisenberg algebra characterized by the canonical commutation rule $\left[a, a^\dagger\right] = I$. To each $z \in \mathbb{C}$ corresponds the (unitary) displacement operator $D(z)$, encoding the CCR

$$\mathbb{C} \ni z \mapsto D(z) = e^{za^\dagger - \bar{z}a}\,,\ D(z)D(z') = e^{\frac{1}{2}(z\bar{z}' - \bar{z}z')}D(z+z')\,.$$

Let $\varpi(z)$ be a function on the complex plane defining a bounded operator $\mathsf{M} := \int_{\mathbb{C}} D(z)\,\varpi(z)\frac{\mathrm{d}^2z}{\pi}$ and such that $\varpi(0) = 1$. The family of displaced operators $\mathsf{M}(z) := D(z)\mathsf{M}D(z)^\dagger$ resolves the identity

$$\int_{\mathbb{C}} \mathsf{M}(z)\,\frac{\mathrm{d}^2z}{\pi} = I\,, \tag{35}$$

and the resulting quantization map is given by

$$f \mapsto A_f = \int_{\mathbb{C}} f(z)\,\mathsf{M}(z)\,\frac{\mathrm{d}^2z}{\pi}\,, \quad A_{f(z-z_0)} = D(z_0)A_{f(z)}D(z_0)^\dagger\,. \tag{36}$$

For $q = \frac{\sqrt{2}}{2}\left(z + \bar{z}\right)$ and $p = \frac{\sqrt{2}}{2i}\left(z - \bar{z}\right)$, one has

$$A_qA_p - A_pA_q = i\left[a, a^+\right] = I\,, \tag{37}$$

i.e., the canonical commutation rule is preserved by the set of quantizations parametrized by functions $\varpi(z)$. In particular, the Cahill-Glauber function [9] $\varpi_s(z) = e^{s|z|^2/2}$ for $s = 0$ corresponds to the Wigner-Weyl integral quantization, while the cases $s = -1, 1$ correspond, respectively, to the CS (anti-normal) and normal quantization ($s = 1$ should be understood in an asymptotic sense). Only for $s \leq -1$ is the operator $\mathsf{M} = \int_{\mathbb{C}} D(z)\,\varpi(z)\frac{\mathrm{d}^2 z}{\pi}$ a density operator, which gives rise to a POVM.

The choice of weight function $\varpi(z)$ determines different orderings. For the simplest one involving the classical q and p, we have

$$A_{qp} = A_q A_p - \frac{i}{2} + \partial^2_{\bar{z}}\varpi\big|_{z=0} - \partial^2_{z}\varpi\big|_{z=0} - \left(\partial_{\bar{z}}\varpi|_{z=0}\right)^2 + \left(\partial_{z}\varpi|_{z=0}\right)^2 . \tag{38}$$

For instance, with $\varpi(z) = (cz+1)e^{s|z|^2}$, $c,\, s \in \mathbb{C}$, $\mathrm{Re}\, s < 1$, we have $A_{qp} = A_q A_p - \frac{i}{2} + c^2$, and putting $c = e^{i\pi/4}/\sqrt{2}$ leads to the so-called xp-quantization $A_{qp} = A_q A_p$. For ϖ chosen *real* and *even*, one has $A_z = a\,,\quad A_{\overline{f(z)}} = A^\dagger_{f(z)}$. Only for real even $\varpi(z)$ one arrives at the correct energy spectrum of the harmonic oscillator. In general,

$$A_{|z|^2} = a^\dagger a + \frac{1}{2} - \partial_z \partial_{\bar{z}}\varpi|_{z=0} + a\,\partial_z\varpi|_{z=0} - a^\dagger\,\partial_{\bar{z}}\varpi|_{z=0} . \tag{39}$$

3.4. Affine integral quantization

3.4.1. The material As the Wigner-Weyl integral quantization, based on the Weyl-Heisenberg group, is natural for a classical motion on the line, affine or wavelet quantization, based on the affine group, is natural for a classical motion on the half-line.

Set X is the upper half-plane $\Pi_+ := \{(q,p)\,|\, p \in \mathbb{R}\,,\, q > 0\}$ equipped with measure $\mathrm{d}q\mathrm{d}p$. It is the phase space for the motion on the half-line $q > 0$. Equipped with the multiplication $(q,p)(q_0,p_0) = (qq_0, p_0/q + p)$, $q \in \mathbb{R}^*_+$, $p \in \mathbb{R}$, X is viewed as the affine group $\mathrm{Aff}_+(\mathbb{R})$ of the real line. The affine group $\mathrm{Aff}_+(\mathbb{R})$ has two non-equivalent UIR, $U_\pm$. Both are square integrable. The UIR $U_+ \equiv U$ is carried by the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^*_+, dx)$ in the form

$$U(q,p)\psi(x) = (e^{ipx}/\sqrt{q})\psi(x/q)\,. \tag{40}$$

The unit-norm state $\psi \in L^2(\mathbb{R}^\dagger_+, \mathrm{d}x) \cap L^2(\mathbb{R}^\dagger_+, \mathrm{d}x/x)$ ("fiducial vector") produces all *wavelets*, or equivalently, CS defined as $|q,p\rangle\ = U(q,p)|\psi\rangle$:

$$\rho = |\psi\rangle\,\langle\psi| \to U(q,p)MU^\dagger(q,p) \equiv \rho(q,p)\,. \tag{41}$$

This yields the crucial resolution of the identity

$$\int_{\Pi_+} \frac{\mathrm{d}q\mathrm{d}p}{2\pi c_{-1}}|q,p\rangle\langle q,p| = I\,, \qquad c_\gamma := \int_0^\infty \mathrm{d}x|\psi(x)|^2/x^{2+\gamma}\,. \tag{42}$$

The covariant integral that arises from the resolution of the identity is

$$A_f = \int_{\Pi_+} \frac{\mathrm{d}q\mathrm{d}p}{2\pi c_{-1}} f(q,p)|q,p\rangle\langle q,p|\,, \tag{43}$$

and $U(g)A_f U^\dagger(g) = A_{\mathsf{U}(g)f}\,,\ \mathsf{U}(g)f)(g') := f(g^{-1}g')$. Quantization is canonical (up to a multiplicative constant) for q and p:

$$A_p = P = -i\partial/\partial x\,,\ A_{q^\beta} = \left(c_{\beta-1}/c_{-1}\right) Q^\beta\,, \tag{44}$$

where $Qf(x) = xf(x)$.

The quantization of the kinetic energy yields

$$A_{p^2} = P^2 + KQ^{-2}\,, \quad K = K(\psi) = \int_0^\infty u\mathrm{d}u(\psi'(u))^2/c_{-1}\,. \tag{45}$$

Thus affine integral quantization forbids a quantum free particle moving on the positive line to reach the origin. Furthermore, the operator $P^2 = -d^2/dx^2$ in $L^2(\mathbb{R}_+^*, \mathrm{d}x)$ is not essentially self-adjoint, whereas the above regularized operator, defined on the domain of smooth compactly supported functions, is essentially self-adjoint for $K \geq 3/4$. Therefore, quantum dynamics of the free motion is consistent.

3.4.2. Affine Integral Quantization for FLRW Quantum Cosmology FLRW models filled with a barotropic fluid with equation of state $\mathrm{p} = w\rho$ and resolving the Hamiltonian constraint lead to a model of a singular universe, or equivalently, of a particle moving on the half-line $(0, \infty)$ with reduced variable Hamiltonian.

$$h(q,p) = \alpha(w)p^2 + 6\tilde{k}q^{\beta(w)}\,, \quad q > 0\,, \quad \{q,p\} = 1\,, \tag{46}$$

with $\tilde{k} = (\int \mathrm{d}\omega)^{2/3}k$, $\alpha(w) = 3(1-w)^2/32$ and $\beta(w) = 2(3w+1)/(3(1-w))$. The constant k is 0, -1 or 1 (in suitable units of inverse area) depending on whether the universe is flat, open or closed.

Assume a closed universe with radiation content : $w = 1/3$ and $k = +1$. Affine quantization on $\mathbb{R}_+^*$ with a fiducial vector like $\psi(x) \propto \exp(-(\alpha(\nu)x + \beta(\nu)/x)$, with parameter $\nu > 0$, yields the quantum Hamiltonian [10]

$$A_h = H = \frac{1}{24}P^2 + \frac{a_P^2 K(\nu)}{24}\frac{1}{Q^2} + 6\frac{a_P^2}{\sigma^2}\frac{c_1}{c_{-1}}Q^2\,, \tag{47}$$

where a_P is a Planck area. For $K(\nu) \geq 3/4$, wavelet quantization removes the quantum singularity and then the quantum evolution is well-defined, contrary to what happens in canonical quantization

The quantum states and their dynamics have a phase space representation through lower symbols. For a state $|\phi\rangle$:

$$\Phi(q,p) = \langle q,p|\phi\rangle/\sqrt{2\pi}\,. \tag{48}$$

The associated probability distribution on phase space is

$$\varrho_\phi(q,p) = \frac{1}{2\pi c_{-1}}|\langle q,p|\phi\rangle|^2\,. \tag{49}$$

With (energy) eigenstates of some quantum Hamiltonian H at our disposal, we can compute the time evolution

$$\varrho_\phi(q,p,t) := \frac{1}{2\pi c_{-1}}|\langle q,p|e^{-iHt}|\phi\rangle|^2 \tag{50}$$

for any state ϕ.

In general the lower symbol $\check{f}(q,p)$ differs from its classical counterpart $f(q,p)$: it is a quantum-corrected effective observable. Thus, computing the lower symbol of the Hamiltonian leads to the semiclassical Friedmann equation for the scale factor $a(t)$:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} + c^2 a_P^2(1-w)^2\frac{\nu}{128}\frac{1}{V^2} = \frac{8\pi G}{3c^2}\rho\,. \tag{51}$$

Note that the repulsive potential depends explicitly on the volume. This excludes non-compact universes from quantum modelling.

Finally, the singularity resolution is confirmed: as the singularity is approached ($a \to 0$), the repulsive potential grows faster ($\sim a^{-6}$) than the density of the fluid ($\sim a^{-3(1+w)}$) and so at some point the two terms become equal and the contraction is brought to a halt.

The form of the repulsive potential does not depend on the state of fluid filling the universe: the singularity is avoided thanks to the quantum geometry.

4. Integral quantization of constraints

4.1. Two approaches

First approach. On a general level, suppose that (X, ν) is a smooth n-dimensional manifold on which is defined the space $\mathcal{D}'(X)$ of distributions as the topological dual of compactly supported n-forms on X [12]. Some of these distributions, e.g. $\delta(u(x)) = \delta \circ u(x)$, express geometrical constraints, e.g. $u(x) = 0$. Given a POVM integral quantization determined by a family of density operators $x \mapsto \rho(x)$ resolving the identity in some Hilbert space $\mathcal{H}$, one extends the map $f \mapsto A_f$ to these objects. We thus obtain a quantum version, $A_{\delta\circ u}$ of these constraints.

Second approach. Another viewpoint in the quantization of constraints, such as the above $u(x) = 0$, is that of Dirac's [13]. It consists of determining the kernel of the operator A_u, which in our case is obtained from the integral quantization map $u \mapsto A_u$.

Lower symbols. One completes both approaches with the semi-classical analysis provided by their respective lower symbols $\widecheck{\delta \circ u}$ and $\check{u} = 0$. In the sequel, illustrative examples are provided by the complex plane with Lebesgue measure $(X, \nu) = (\mathbb{C}, \mathrm{d}z/\pi)$.

4.2. Example of constraints in the plane

In this example, $(X, \nu) = (\mathbb{C}, \mathrm{d}^2z/\pi)$, and we adopt the integral quantization based on the Weyl-Heisenberg group as it was described in Subsection 3.3. For instance, we can choose as weight functions the Cahill-Glauber function $\varpi_s(z) = e^{s|z|^2/2}$ for $s \leq -1$. This choice yields the family of density operators which are Weyl-Heisenberg displaced thermal states resolving the identity,

$$\rho_t(z) := D(z)\rho_t D(z)^\dagger\,,\ D(z) = e^{za^\dagger - \bar{z}a}\,, \quad \rho_t := (1-t)\sum_{n=0}^{\infty} t^n\,|n\rangle\langle n|\,,\ 0 \leq t := \frac{s+1}{s-1} < 1\,. \tag{52}$$

$$\int_{\mathbb{C}} \rho_t(z)\,\frac{\mathrm{d}^2 z}{\pi} = I\,. \tag{53}$$

The density operator ρ_t is viewed as a thermal state when $t = e^{-\hbar\omega/kT}$. Zero temperature, or equivalently $t = 0$, corresponds to the projector $\rho_0 = |0\rangle\langle 0|$ and the corresponding $\rho_0(z)$ reduces to the rank-one CS projector $|z\rangle\langle z|$. Note that the Weyl-Wigner integral quantization , and the related Wigner transform, correspond to the value $t = -1$ (i.e., $s = 0$ or $T = i\hbar\omega/\pi$ is purely imaginary). For this particular case in which ρ_{-1} loses its probabilistic meaning, $\rho_{-1} = 2\mathsf{P}$, where $\mathsf{P} = \sum_{n=0}^{\infty}(-1)^n|n\rangle\langle n|$ is the parity operator.

Let $f(z) = 0$ be an equation defining some subset of the complex plane, like points, curves, etc. The first approach towards quantization yields the operator

$$\delta \circ f \mapsto A_{\delta\circ f} = \int_{\mathbb{C}} \delta\left(f\left(z\right)\right)\,\rho_t(z)\,\frac{\mathrm{d}^2 z}{\pi}\,, \tag{54}$$

whose lower symbol is $\widecheck{\delta \circ f}(z) = \int_{\mathbb{C}} \delta\left(f\left(z'\right)\right)\langle z|\,\rho_t(z')\,|z\rangle\ \frac{\mathrm{d}^2 z'}{\pi}\,.$

The constraint can then be understood in terms of the kernel of the operator. In the **Dirac quantization** procedure we directly quantize the function $f(z)$

$$f \mapsto A_f = \int_{\mathbb{C}} f(z)\ \rho_t(z)\, \frac{\mathrm{d}^2 z}{\pi}\,, \tag{55}$$

We then analyze $\ker A_f$ through the zeros of the lower symbol $\check{f}(z) = \int_{\mathbb{C}} f(z')\,\mathrm{tr}(\rho_t(z)\rho_t(z'))\,\frac{\mathrm{d}^2 z'}{\pi}$.

Both methods are obviously not mathematically equivalent, except for a few cases. They are possibly *physically* equivalent, i.e. indistinguishable in terms of measurement.

Here are two simple examples. they are handled with CS, $\rho_0(z) = |z\rangle\langle z|$, for the sake of simplicity. A more thorough investigation can be found in [11].

4.2.1. The point: $f(z) = z - z_0 = 0$ Here we study the constraint $f(z) = z - z_0 = 0$. In the first approach, we have

$$A_{\pi\delta\circ f} = \int \pi\delta\,(z - z_0)\ |z\rangle\,\langle z|\,\frac{\mathrm{d}^2 z}{\pi} = |z_0\rangle\,\langle z_0|\ . \tag{56}$$

The lower symbol is purely Gaussian and concentrates about the point z_0.

$$\delta\,\check{\circ}\,f(z) = |\langle z|\ z_0\rangle|^2 = e^{-|z-z_0|^2}\ . \tag{57}$$

In the second approach one has

$$A_f = \int_{\mathbb{C}} (z - z_0)\,|z\rangle\,\langle z|\,\frac{\mathrm{d}^2 z}{\pi} = a - z_0 I\ . \tag{58}$$

where a is the lowering operator

$$a\,|z\rangle = z\,|z\rangle\ . \tag{59}$$

Therefore, $|v\rangle \in \ker A_f$ is such that $|v\rangle = \lambda\,|z_0\rangle\ ,\ \lambda \in \mathbb{C}$. The lower symbol of A_f is identical to the original f

$$\check{f}(z) = z - z_0 = f(z)\,. \tag{60}$$

which means that classical and semi-classical constraints are the same. Therefore the two methods already differ on this elementary example since the first one gives a smooth Gaussian portrait of the point z_0.

Note that the lower symbol of the projector $P_{z_0} = |z_0\rangle\,\langle z_0|$ corresponding to the element of the kernel is the same as (57)

$$\check{P}_{z_0} = e^{-|z-z_0|^2}\ . \tag{61}$$

It is only at this level that the two methods yield similar results.

4.2.2. The half-line $f(z) = P(\gamma, J) = \gamma - \gamma_0 = 0$ The constraint is given by $f(z) = P(J,\gamma) = \gamma - \gamma_0 = 0$. The first approach gives

$$\begin{aligned} A_{\pi\delta\circ f} &= \int \pi\delta\,(\gamma - \gamma_0)\,|z\rangle\,\langle z|\,\frac{\mathrm{d}^2 z}{\pi} \\ &= \sum_{n,n'\geq 0} |e_n\rangle\,\langle e_{n'}|\,\frac{\Gamma\left(\frac{n+n'}{2}+1\right)}{\sqrt{n!n'!}}\,\frac{1}{2}e^{i(n-n')\gamma_0}\,. \end{aligned}$$

Journal of Physics: Conference Series **597** (2015) 012037 doi:10.1088/1742-6596/597/1/012037

Its lower symbol is given by

$$\begin{aligned}\pi\delta\,\check{\circ}\,f\,(z) &= \int_0^\infty \exp\left[-\left|r\,e^{i\gamma} - r'\,e^{i\gamma_0}\right|^2\right]\,r'\,\mathrm{d}r' \\ &= \frac{\sqrt{\pi}}{2}\,e^{-r^2\sin^2(\gamma-\gamma_0)}\,\mathrm{erfc}(r\,\cos(\gamma-\gamma_0))\,. \end{aligned} \tag{62}$$

The second approach yields

$$\begin{aligned} A_f &= \int (\gamma-\gamma_0)\,|z\rangle\,\langle z|\,\frac{\mathrm{d}z}{\pi} = \pi I + i \sum_{\substack{n,n'\geq 0\\ n\neq n'}} \frac{\Gamma\left(\frac{n+n'}{2}+1\right)}{\sqrt{n!n'!}}\,\frac{1}{n-n'}\,|e_n\rangle\,\langle e_{n'}| - \gamma_0 I \\ &= A_{\daleth} - \gamma_0 I \end{aligned}$$

where $A_{\daleth}$ is the angle operator. In this case the spectrum of operator is continuous with support on $[0,2\pi]$, so the kernel is not properly defined in the Hilbert space, except in a distributional sense. The lower symbol of A_f is the Fourier series

$$\begin{aligned} \check{A}_f &= \langle J,\gamma|\,A_{\daleth}\,|J,\gamma\rangle - \gamma_0 = \pi + i\,e^{-J}\sum_{n\neq n'} \frac{\Gamma\left(\frac{n+n'}{2}+1\right)}{n!n'!}\,\frac{z^{n'}\,\bar{z}^n}{n'-n} - \gamma_0 \\ &= \pi - 2\sum_{q=1}^{\infty} d_q(\sqrt{J})\,\frac{\sin q\gamma}{q} - \gamma_0\,, \end{aligned} \tag{63}$$

where the function $d_q(r)$ is given by:

$$d_q(r) = e^{-r^2} r^q\,\frac{\Gamma(\frac{q}{2}+1)}{\Gamma(q+1)}\,{}_1F_1\left(\frac{q}{2}+1; q+1; r^2\right)\,. \tag{64}$$

It can be shown that this positive function is bounded by 1.

For large J this lower symbol tends to the Fourier series of the 2π-periodic angle function $\daleth(\gamma) = \gamma$ for $\gamma \in [0,2\pi)$:

$$\langle J,\gamma|\,A_{\daleth}\,|J,\gamma\rangle \approx \pi - 2\sum_{q=1}^{\infty}\frac{1}{q}\,\sin q\gamma\,. \tag{65}$$

Such a behavior is understood in terms of the classical limit of these quantum objects. Indeed, by restoring physical dimensions to our formulas, we know that the quantity $|z|^2 = J$ should appear divided by the Planck constant $\hbar$. Hence, the limit $J \to \infty$ in our previous expressions can also be considered as the classical limit $\hbar \to 0$.

5. Conclusion

The main idea put forward is a map between classical observables, functions on a set, and quantum observables, operators on a Hilbert space. In general this is not well-defined, because of the appearance of unbounded operators, domain issues, etc. It is possible to bypass such subtleties by working with the lower symbols of operators, which live in the classical setting, but carry some information from the noncommutative and probabilistic quantum setting.

Besides the freedom allowed by integral quantization, the advantages of the method with regard to other quantization procedures are manifold:

(i) Only a minimal amount of restrictions have to be imposed on the classical objects being quantized, which allows us to consider the quantization of more general objects, such as distributions.

(ii) Once a choice of a (positive) operator-valued measure has been made, which must be consistent with experiment, there is no more ambiguity (no ordering problem). To each classical object corresponds one and only one quantum object.

(iii) The method produces in essence a regularizing effect.

(iv) The method, through POVM choices, offers the possibility of keeping a full probabilistic content. In particular, the Weyl-Wigner integral quantization is not defined by a POVM.

Acknowledgments

J.P. Gazeau thanks the CBPF, the CNPq, and the UFABC for financial support. R. Fresneda thanks CNPq, FAPESP and CAPES for financial support.

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