Auxiliary functions for rational ODEs with 2 and 3 singular values

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Abstract. We examine auxiliary functions which are related to certain second order nonlinear differential equations with 2 and 3 singular values. The auxiliary functions satisfy first order linear differential equations. It is shown that after imposing suitable conditions on the coefficients of the linear equations and on the form of the auxiliary functions one can reduce the nonlinear differential equations to the (modified) third and fifth Painlevé equations and the fourth Painlevé equation.

1. Introduction

The singularity structure of a given nonlinear differential equation is in general very complicated. Fixed singularities occur at points where the equation itself is singular in some sense. The location of the singularities may also depend on the initial conditions and such singularities are called movable. The coefficients of the equation are not necessarily singular at such points. It is known from the theory of ordinary differential equations [9] that linear equations do not have movable singularities. Nonlinear equations in general may have movable poles, algebraic branch points, essential singularities and others (see [9] for numerous examples of possible singularities).

Classes of second-order rational ordinary differential equations, admitting certain families of formal algebraic series solutions, is considered in [1, 2, 3, 10]. For all solutions of these equations, it is shown that any movable singularity that can be reached by analytic continuation along a finite-length curve is an algebraic branch point (or, in particular cases, a pole). As it is shown in [1, 3], this yields in particular the known fact that the only movable singularities of solutions of the Painlevé equations (P_I)—(P_{VI}) are poles for generic values of parameters. This is known as the Painlevé property. In the technical part of the proof certain auxiliary functions with certain properties appear and play an important role.

Solutions of the Painlevé equations are often referred to as the Painlevé transcendents or nonlinear special functions and have numerous and significant applications in mathematics and mathematical physics. The Painlevé equations are irreducible and satisfy second order nonlinear differential equations. The third, fourth and fifth Painlevé equations are given respectively by equations (10), (13), (15). There exist also modified versions of the third and fifth Painlevé equations (e.g., (6)) which are related to the original ones by a change of variables. For more information on the Painlevé equations the reader is referred to the book [5].

There are several proofs of the meromophic nature of the Painleveé equations (see, for instance, [8, 7, 6, 5, 10] and [1, 3] with further references therein). In [1, 2, 3, 10] the proof of the fact that any movable singularity that can be reached by analytic continuation along a rectifiable curve is an algebraic branch point relies on the Cauchy theorem for certain systems of differential equations in the complex domain and the existence of certain auxiliary functions (which are not unique) which allow constructing such systems and which remain bounded as the singularities are approached. These functions satisfy first-order linear ODEs with some potentially singular terms that can be removed. Each singular value of the equation should be treated separately and, hence, one needs to consider several bounded functions. Note that the study of similar auxiliary functions for the Painlevé equations was also done in [4] (and in the references therein).

The construction of auxiliary functions in [4, 3, 10] may look a bit artificial and therefore, in this paper we are interested to understand the construction of auxiliary functions for the Painlevé type equations in general. We consider several second-order rational ordinary differential equations of the form (3), (7), (11) with 2 singular values $w = 0, \infty$ and equations (14), (16) with 3 singular values $w = 0, 1, \infty$. In each case we analyse the auxiliary functions

$$V(z) = E(z,w)w'^2 + F(z,w)w' + G(z,w) \tag{1}$$

which satisfy the first order linear equations of the form

$$V' = P(z,w)V + Q(z,w)w' + R(z,w) \tag{2}$$

and which are related to the nonlinear differential equations for $w(z)$. We show that under suitable conditions on functions E, F, G, P, Q, R the nonlinear equations are reduced to the Painlevé equations. The most important conditions on the functions are that they are rational and P, Q and R are holomorphic at singular values of the nonlinear equation. We get the (modified) third and fifth Painlevé equations and the fourth Painlevé equation. By a similar argument as in Lemma 4.3 [10] one immediately gets that under those conditions the function $V(z)$ remains bounded as $w(z)$ stays away from singular values and thus the results of the present paper can be used to improve the proof of the Painlevé property for the Painlevé equation [10] if combined with the methods in [1, 2, 3]. Moreover, we show that there is always a freedom in the choice of P for the Painlevé equations and discuss several examples.

2. Main results

2.1. The modified third Painlevé equation

Let $w(z)$ be defined by the equation

$$w'' = \frac{w'^2}{w} + \sum_{n=-1}^{3} a_n(z)w^n, \tag{3}$$

where $' = d/dz$ and $a_n(z)$ are analytic functions of z. Singular values of this equation are $w = 0$ and $w = \infty$. Take

$$E(z,w) = \frac{1}{w^2} \tag{4}$$

in (1). The choice of E is unique if we assume that the function V satisfies equation (2).

Assume that the function G satisfying

$$\frac{\partial G}{\partial w} = -\frac{2}{w^2}\sum_{n=-1}^{3} a_n(z)w^n$$

is rational in w. This yields

$$G(z,w) = g(z) + \frac{a_{-1}(z)}{w^2} + \frac{2a_0(z)}{w} - 2a_2(z)w - a_3(z)w^2 \tag{5}$$

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and $a_1(z) = 0$ (to avoid the logarithmic term). Here $g(z)$ is an arbitrary function.

By a simple calculation the function P satisfies

$$P = w\left(F + w\frac{\partial F}{\partial w}\right).$$

Since we want this function to be analytic at singular values $w(z) = 0$ and $w(z) = \infty$ and also be non-constant, there is a freedom of the choice. The function F should be rational in w, so one can assume that

$$P = \frac{p_1(z)}{w+1} + \frac{p_2(z)}{(w+1)^2}.$$

The functions $p_1(z) \neq 0$ and $p_2(z) \neq 0$ are to be determined from other conditions. By solving a differential equation

$$w\left(F + w\frac{\partial F}{\partial w}\right) = \frac{p_1(z)}{w+1} + \frac{p_2(z)}{(w+1)^2}$$

for $F(z, w)$ we get

$$F = \frac{f(z) - p_1(z)/(w+1) + \log(w/(w+1))(p_1 + p_2)}{w}.$$

The function F is rational if the condition $p_2(z) = -p_1(z)$ is imposed and it is given by

$$F(z, w) = \frac{p_2(z)}{w(w+1)} + \frac{f(z)}{w},$$

where $f(z)$ is an arbitrary function. The coefficient Q in the differential equation (2) is given by

$$Q = wF^2 - w^2 F F_w + F_z,$$

where $F_w = \partial F/\partial w$ and $F_z = \partial F/\partial z$. From the proof in [3] we observe that the functions P, Q and R in (2) should be holomorphic at $w = 0$ and $w = \infty$ and the function Q has a second order zero at $w = \infty$. Observe that with our choice of P as above, the function P is automatically holomorphic at $w = 0$ and $w = \infty$. Other conditions give the following system of differential equations:

$$\begin{aligned}
&p_2'(z) + f'(z) = 0, \;\; f'(z) = 0,\\
&p_2(z)a_{-1}(z) + a_{-1}(z)f(z) + a_{-1}'(z) = 0,\\
&p_2(z)a_0(z) + a_0(z)f(z) + 2a_0'(z) = 0,\\
&a_3(z)f(z) - a_3'(z) = 0, \;\; a_2(z)f(z) - 2a_2'(z) = 0,
\end{aligned}$$

which is solved by

$$\begin{gathered}
f(z) = C_1, \;\; p_2(z) = C_2, \;\; a_{-1}(z) = \delta e^{(-C_2 - C_1)z},\\
a_0(z) = \beta e^{(-C_2 - C_1)z/2}, \;\; a_3(z) = \gamma e^{C_1 z}, \;\; a_2 = \alpha e^{C_1 z/2},
\end{gathered}$$

where C_1 and C_2 are arbitrary constants. Finally, taking $C_1 = 2a$ and $C_2 - C_1 = 2b$, we get that the equation (3) has the following form:

$$w'' = \frac{w'^2}{w} + \gamma e^{2az} w^3 + \alpha e^{az} w^2 + \beta e^{bz} + \frac{\delta e^{2bz}}{w},$$

where a, b, α, β, γ, δ are arbitrary constants. In particular, if $C_1 = 0,\ C_2 = 2$ $(a = 0,\ b = 1)$ we get the modified third Painlevé equation given by

$$w'' = \frac{w'^2}{w} + \alpha w^2 + \beta + e^{2z}\left(\gamma w^3 + \frac{\delta}{w}\right). \tag{6}$$

For the modified third Painlevé equation we get the auxiliary function given by

$$V(z) = \frac{w'^2}{w} - \frac{2w'}{w(w+1)} - \gamma w^2 - 2\alpha w + \frac{2\beta e^z}{w} + \frac{\delta e^{2z}}{w^2} + g(z).$$

We can summarize the results in this section as follows.

Theorem 2.1 *Assume the following conditions for the coefficients of (1) and (2):*

(i) the function E satisfies (4);

(ii) the function G satisfying

$$\frac{\partial G}{\partial w} = -\frac{2}{w^2}\sum_{n=-1}^{3} a_n(z)w^n$$

is rational in w and is given by (5);

(iii) the function F satisfying

$$w\left(F + w\frac{\partial F}{\partial w}\right) = P := \frac{g_1(z)}{w+1} + \frac{g_2(z)}{(w+1)^2}$$

is rational in w, where $g_1(z)$ and $g_2(z)$ are functions to be determined from other conditions;

(iv) the functions P, Q and R in (2) are holomorphic at $w = 0$ and $w = \infty$;

(v) the function Q has a second order zero at $w = \infty$.

Then equation (3) has the following form:

$$w'' = \frac{w'^2}{w} + \gamma e^{2az}w^3 + \alpha e^{az}w^2 + \beta e^{bz} + \frac{\delta e^{2bz}}{w},$$

where a, b, α, β, γ, δ are arbitrary constants.

We remark that the function P should have at least a second order pole at $w = -1$ and by taking more terms in the expression of P we can get a similar statement. If the function P is either constant or has the first order pole, then the function F is not rational. We also note that we can choose another function P with a pole of second order at $w = 1$ and get the same modified Painlevé equation. Moreover, this function is equal (up to the arbitrary function $g(z)$) to the function U in [7].

2.2. The third Painlevé equation

Let the function $w(z)$ satisfy the equation

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \sum_{n=-1}^{3} a_n(z)w^n, \tag{7}$$

where $' = d/dz$ and $a_n(z)$ are analytic functions of z as before. Singular values of this equation are $w = 0$ and $w = \infty$. We take

$$E(z, w) = \frac{z^2}{w^2} \tag{8}$$

in (1).

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Theorem 2.2 *Assume condition (8) and the following conditions for the coefficients of (1) and (2):*

(i) the function G satisfying

$$\frac{\partial G}{\partial w} = -\frac{2z^2}{w^2}\sum_{n=-1}^{3} a_n(z)w^n$$

is rational in w;

(ii) the function F satisfying

$$\frac{w}{z^2}\left(F + w\frac{\partial F}{\partial w}\right) = P := \frac{p_1(z)}{w-1} + \frac{p_2(z)}{(w-1)^2}$$

with $p_1(z) \neq 0$, $p_2(z) \neq 0$ is rational in w;

(iii) the functions P, Q and R in (2) are holomorphic at $w = 0$ and $w = \infty$;

(iv) the function Q has a second order zero at $w = \infty$.

Then equation (7) has the following form:

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \gamma z^{2a}w^3 + \alpha z^{a-1}w^2 + \beta z^b + \frac{\delta z^{2b+2}}{w}, \tag{9}$$

where a, b, α, β, γ, δ are arbitrary constants.

Proof 1 *Condition 1 implies $a_1(z) = 0$ since*

$$G(z,w) = g(z) + \frac{a_{-1}(z)z^2}{w^2} + \frac{2z^2a_0}{w} - 2z^2a_1(z)\log w - 2z^2a_2(z)w - z^2a_3(z)w^2.$$

In this case the function G does not have logarithmic terms. As in the previous section, we could choose $P \neq 0$ equal to any function which is holomorphic at $w = 0$ and $w = \infty$ such that the differential equation for F has a rational solution. If $P(z,w) = p_1(z)/(w-1)$, then F has a logarithmic term if $p_1(z) \neq 0$. Hence, if we choose P as in the theorem, then

$$F(z,w) = \frac{f(z) + z^2((p_1(z) - p_2(z))\log((w-1)/w) - p_2(z)/(w-1))}{w}.$$

Condition 2 implies $p_1(z) = p_2(z)$ and the function

$$F(z,w) = \frac{f(z)}{w} - \frac{z^2p_1(z)}{w(w-1)},$$

where $f(z)$, $p_1(z)$ are the functions to be determined from other conditions. We have already chosen the function P such that it is automatically holomorphic at $w = 0$ and $w = \infty$. Substituting into

$$Q = \frac{z^2w^2F_z - w^4FF_w - w^3F^2 - zw^2F}{z^2w^2}$$

and to the expression for the function R, which is cumbersome so we omit it, we analyse other conditions in the theorem on Q and R. Their holomorphicity at $w = 0$ and $w = \infty$ gives the following system of differential equations:

$$\begin{aligned}
&z^2p_1(z) - f(z) + z^3p_1'(z) + zf'(z) = 0,\\
&zf'(z) - f(z) = 0, \;\; 2z^2a_2'(z) - a_2(z)f(z) + 4za_2(z) = 0,\\
&2za_{-1}(z) + z^2p_1(z)a_{-1}(z) + a_{-1}(z)f(z) + z^2a_{-1}'(z),\\
&4za_0(z) + z^2p_1(z)a_0(z) + a_0(z)f(z) + 2z^2a_0'(z) = 0,\\
&z^2a_3'(z) - a_3(z)f(z) + 2za_3(z) = 0,
\end{aligned}$$

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which is solved by

$$f(z) = az, \;\; p_1(z) = b/z, \;\; a_{-1}(z) = \delta z^{-2-a-b},$$
$$a_0(z) = \beta z^{-2-a/2-b/2}, \;\; a_3(z) = \gamma z^{a-2}, \;\; a_2 = \alpha z^{a/2-2}.$$

This yields the statement of the theorem after renaming the constants $a \to 2(1+a), b \to -2(3+a+b)$. We remark that one can take more terms in the expression of P and get a similar statement.

In particular, if $b = -1$, $a = 0$ in (9), we get the third Painlevé equation given by

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{1}{z}\left(\alpha w^2 + \beta\right) + \gamma w^3 + \frac{\delta}{w}, \tag{10}$$

where α, β, γ, δ are arbitrary complex parameters. In this case the auxiliary function $V(z)$ is given by

$$V(z) = \frac{z^2 w'^2}{w^2} + \frac{2z(w+1)w'}{w(w-1)} - \gamma z^2 w^2 - 2\alpha z w + \frac{2\beta z}{w} + \frac{\delta z^2}{w^2} + g(z).$$

One can immediately see that the function coincides with the function U in [10] up to the arbitrary function $g(z)$.

2.3. The fourth Painlevé equation

Assume that the function $w(z)$ satisfies an equation of the form

$$w'' = \frac{w'^2}{2w} + \sum_{n=-1}^{3} a_n(z) w^n \tag{11}$$

with singular values $w = 0$ and $w = \infty$.

Theorem 2.3 *Let*

$$E(z, w) = \frac{1}{w}$$

in (1). We impose the following conditions on the coefficients of (1) and (2):

(i) the function G satisfying

$$\frac{\partial G}{\partial w} = -\frac{2}{w} \sum_{n=-1}^{3} a_n(z) w^n$$

is rational in w;

(ii) the function F satisfying

$$\frac{1}{2}F + w\frac{\partial F}{\partial w} = P := \frac{p_1(z)}{w+1} + \frac{p_2(z)}{(w+1)^2}$$

is rational in w with $p_1(z) \neq 0$ and $p_2(z) \neq 0$;

(iii) the functions P, Q and R are holomorphic at $w = 0$ and $w = \infty$;

(iv) the function Q has a second order zero at $w = \infty$.

Then equation (11) has the following form:

$$w'' = \frac{w'^2}{2w} + a_3 w^3 + \left(a_2 - \frac{4a_3 cz}{3}\right) w^2 + \frac{1}{6}\left(cz(2a_3 cz - 3a_2) - 12\alpha\right) w + \frac{\beta}{w}, \tag{12}$$

where a_2, a_3 and c are arbitrary constants.

Proof 2 *Condition 1 implies* $a_0(z) = 0$ *since* $G(z,w) = g(z) + 2a_{-1}(z)/w - 2a_0(z)\log w - 2a_1(z)w - a_2(z)w^2 - 2a_3(z)w^3/3$. *We further get* $p_2(z) = -2p_1(z)$ *since*

$$F(z,w) = \frac{f(z) + \mathrm{arctg}\sqrt{w}(2p_1(z) + p_2(z)) + p_2(z)\sqrt{w}/(w+1)}{\sqrt{w}}.$$

Thus, the function

$$F(z,w) = -\frac{2p_1(z)}{w+1},$$

where $p_1(z)$ *is to be determined from other conditions given by*

$$p_1'(z) = 0, \;\; a_{-1}'(z) = 0, \;\; a_3'(z) = 0$$
$$4p_1(z)a_3(z) + 3a_2'(z) = 0, \;\; 2a_1'(z) + p_1(z)a_2(z) = 0.$$

Solving for coefficients $a_i(z)$ *and* $p_1(z)$ *we get the statement of the theorem.*

In particular, if $a_3 = 3/2$, $a_2 = 0$, $c = -2$ in (12) we get the fourth Painlevé equation given by

$$w'' = \frac{w'^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \tag{13}$$

where α and β are arbitrary complex parameters. The auxiliary function is given by

$$V(z) = \frac{w'^2}{w} + \frac{4w'}{w+1} + \frac{2\beta}{w} - 4(z^2 - \alpha)w - 4zw^2 - w^3 + g(z).$$

One can analise the auxiliary functions in [10] and [5] to see that up to a choice of P and a constant functions our construction gives essentially the same auxiliary functions.

2.4. The fifth Painlevé equation
Assume that the function $w(z)$ satisfies

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)w'^2 - \frac{w'}{z} + \sum_{n=-1}^{3} a_n(z)w^n + \frac{b(z)}{w-1} \tag{14}$$

with singular values $w = 0$, 1 and $w = \infty$. We choose the function

$$V(z) = \frac{z^2 w'^2}{w(w-1)^2} + F(z,w)w' + G(z,w)$$

which satisfies (2). Similarly to the previous subsections we state the following theorem.

Theorem 2.4 *Let the function* G *satisfying*

$$\frac{\partial G}{\partial w} = -\frac{2z^2}{w(w-1)^2}\left(\sum_{n=-1}^{3} a_n(z)w^n + \frac{b(z)}{w-1}\right)$$

be rational in w. *Take the function* F *in the form*

$$F(z,w) = -\frac{2z^2 p(z)}{3(w-1)(w+1)^2}$$

which satisfies

$$\frac{(w-1)(F(3w-1)+2w(w-1)F_w)}{2z^2} = P := \frac{p(z)}{(w+1)} - \frac{10p(z)}{3(w+1)^2} + \frac{8p(z)}{3(w+1)^3}.$$

If the functions P, Q and R are holomorphic at $w=0$, 1 and $w=\infty$ and the function Q has a second order zero at $w=\infty$, then the coefficients in (14) are given by

$$b(z) = 2a_{-1}(z) + a_0(z) - a_2(z) - 2a_3(z), \quad p(z) = \frac{c}{z}, \quad a_3(z) = -\frac{a_2(z)}{2},$$

$$a_2(z) = -\frac{2\alpha}{z^2},$$

$$a_1(z) = \frac{\delta z^{c/6} + \gamma z^{c/12} + \beta + \alpha}{z^2},$$

$$a_0(z) = \frac{2\delta z^{c/6} - 2\beta}{z^2}, \quad a_{-1}(z) = \frac{\beta}{z^2},$$

where c, α, β, γ, δ are arbitrary. In particular, if $c=12$, we get the fifth Painlevé equation given by

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right) w'^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \gamma\frac{w}{z} + \delta\frac{w(w+1)}{w-1}. \tag{15}$$

The proof is similar to the previous sections.

Next we remark about the dependence of the resulting equation on the choice of P. Indeed, we make certain general assumptions (like the holomorphicity at certain points) on the coefficients in the proof of the theorem. Therefore, we expect that certain changes in the choice of P might lead to a different result. For example, the choice in [10] was that $F = -4z/(w^2-1)$. In this case one should take P in the form $P = 2(w-1)^2/(w+1)^2$. However, this function is holomorphic at $w = 0, 1, \infty$ and one can show that equation (14) is still (15). We could have chosen $P(z,w) = p(z) + p_1(z)/(w+1) + p_2(z)(w+1)^2$ and get (15) as well.

2.5. The modified fifth Painlevé equation

If the function $w(z)$ satisfies

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right) w'^2 + \sum_{n=-1}^{3} a_n(z)w^n + \frac{b(z)}{w-1} \tag{16}$$

with singular values $w=0$, 1 and $w=\infty$, and the function $V(z)$ is given by

$$V(z) = \frac{w'^2}{w(w-1)^2} + F(z,w)w' + G(z,w),$$

we have

Theorem 2.5 *Let the function G satisfying*

$$\frac{\partial G}{\partial w} = -\frac{2}{w(w-1)^2}\left(\sum_{n=-1}^{3} a_n(z)w^n + \frac{b(z)}{w-1}\right)$$

be rational in w. Take the function F in the form

$$F(z,w) = -\frac{2p(z)}{3(w-1)(w+1)^2}$$

which satisfies

$$\frac{1}{2}(w-1)\left(F(3w-1)+2w(w-1)\frac{\partial F}{\partial w}\right) = P := \frac{p(z)}{(w+1)} - \frac{10p(z)}{3(w+1)^2} + \frac{8p(z)}{3(w+1)^3}.$$

If the functions P, Q and R are holomorphic at $w = 0$, 1 and $w = \infty$ and the function Q has a second order zero at $w = \infty$, then equation (16) takes the following form:

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)w'^2 + (w-1)^2\left(\alpha w + \frac{\beta}{w}\right) + \gamma e^{cz/12}w + \delta e^{cz/6}\frac{w(w+1)}{w-1},$$

where c, α, β, γ, δ are arbitrary. In particular, if $c = 12$, we get the modified fifth Painlevé equation.

Proof 3 *The proof is computational. The function F is rational provided $b(z) = 2a_{-1}(z) + a_0(z) - a_2(z) - 2a_3(z)$ and $a_2(z) = -2a_3$. Moreover, one gets*

$$p(z) = c,\ \ a_{-1}(z) = \beta,\ \ a_3(z) = \alpha,\ \ a_0(z) = -2\beta + 2\delta e^{cz/6},$$
$$a_1(z) = \alpha + \beta + \delta e^{cz/6} + \gamma e^{cz/12}.$$

3. Discussion

In this paper we analised the principles of the construction of the auxiliary bounded functions related to the Painlevé equations. Auxiliary functions satisfy the property that they are analytic around every zero and pole for the (modified) third and fourth Painlevé equations and additionally at every 1-point for the (modified) fifth Painlevé equation. We showed that in particular, our construction yields the functions considered in [5, 7, 10] in the proofs of the Painlevé property of the Painlevé equations. In principle, similar computations can be done for the sixth Painlevé equation with 4 singular values $w = 0, 1, z, \infty$, but the calculations become cumbersome and more involved.

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