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Factorization method for the truncated harmonic oscillator

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Abstract. Factorization procedures of first and second order are used to generate Hamiltonians with known spectra departing from the harmonic oscillator with an infinite potential barrier. Certain systems obtained in a straightforward way through said method possess differential ladder operators of both types, third and fourth order. Since systems with this kind of operators are linked with the Painlevé IV and V equations respectively, several solutions of these non-linear second-order differential equations will be simply found.

1. Introduction

The idea of factorizing a differential operator in Quantum Mechanics yields a mighty technique for solving Schrödinger equation [1, 2, 3, 4]. This method can be presented in several ways, supplying us with interesting new results concerning the solvability of quantum problems [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] and the spectral design of new potentials [11, 13, 15].

When applied to the harmonic oscillator, this procedure deforms the potential as well as the underlying algebra of the system in a specific way [18, 17, 10, 19]. These deformations of the harmonic oscillator algebra, which are called Polynomial Heisenberg Algebras (PHA), turn out to be characterized by differential ladder operators of order $m+1$, with the commutator between them being polynomials of degree m in the Hamiltonian. While systems realizing the second order PHA are connected to the Painlevé IV (PIV) equation, those realizing the third order algebras are linked to the Painlevé V (PV) equation [21, 20, 24, 23, 22, 17, 25].

These results have been used recently to produce plenty of non-singular solutions to the PIV and PV equations [14, 26, 27]. With the aim of studying some singular solutions supplied by said connections, we will continue the analysis now for the harmonic oscillator with an infinite potential barrier at $x = 0$ [7, 28, 29], which we will call truncated harmonic oscillator. Caution must be employed to avoid that singularities appear inside the domain of the potential.

The organization of this work is the following: in Section 2 we will review the factorization method as needed in the rest of the paper as well as the way in which the Hamiltonians built from the harmonic oscillator yields realizations of the second and third order PHA's. We will also present the procedure which connects these systems with the PIV and PV equations to produce solutions of these non-linear second-order differential equations. Section 3 is devoted to analyze the harmonic oscillator with an infinite potential barrier at the origin, its first and second order transformations, and the systems thus obtained. In Section 4 we will obtain several solutions to the PIV equation by using the extremal states of the supersymmetric partners of the truncated harmonic oscillator. Thus far, Sections 2-4 comprise a review of a previous work which can be found in [28, 29]. However, Section 5 contains original results on how to obtain several solutions to the PV equation respectively by using said extremal states. Finally, in Section 6 we will summarize the results attained.

2. Factorization method

Let us suppose the existence of a one-dimensional Schrödinger Hamiltonian $H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x)$, together with a k-th order differential operator A^+. Then, for another one-dimensional Schrödinger Hamiltonian $\tilde{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \tilde{V}(x)$ which is "intertwined" with H through the relation

$$\tilde{H}A^+ = A^+H,$$

it is true that solutions of the stationary Schrödinger equation $\tilde{H}\phi_n(x) = E_n\phi_n(x)$ are connected to those of $H\psi_n(x) = E_n\psi_n(x)$ by

$$\phi_n(x) \equiv C_n A^+\psi_n(x),$$

where C_n is a normalization constant. We will call $\tilde{H}$ and H supersymmetric partners, and we will assume that both are Hermitian so that the potentials V and $\tilde{V}$ are real.

If the new k-th order differential operator $A \equiv (A^+)^\dagger$ is introduced, then there is also an intertwining relation involving A, which can be written as $HA = A\tilde{H}$. Even more, the products of A and A^+ become factorized as follows

$$A^+A = \prod_{i=1}^{k}(\tilde{H} - \epsilon_i), \quad AA^+ = \prod_{i=1}^{k}(H - \epsilon_i).$$

For $k = 1$ the intertwining operator A and the new potential $\tilde{V}$ are determined by a solution to the stationary Schrödinger equation $Hu = \epsilon u$ as follows:

$$\begin{aligned} A^+ &= \frac{1}{\sqrt{2}}\left[-\frac{d}{dx} + [\ln(u)]'\right], \\ \tilde{V} &= V - [\ln(u)]'', \end{aligned}$$

where ϵ is called *factorization energy* and u *transformation function* or *seed solution.*

For a second order intertwining operator ($k = 2$) the procedure is fixed by a pair of transformation functions u_i, which are also solutions of $Hu_i = \epsilon_i u_i$, $i = 1, 2$. In a similar fashion as for $k = 1$ now we obtain

$$\begin{aligned} A^+ &= \frac{1}{2}\left\{\frac{d^2}{dx^2} - [\ln W(u_1, u_2)]'\frac{d}{dx} + \frac{1}{2}\left([\ln W(u_1, u_2)]'' + [\ln W(u_1, u_2)]'^2\right) - 2V + \epsilon_1 + \epsilon_2\right\}, \\ \tilde{V} &= V - [\ln W(u_1, u_2)]'', \end{aligned}$$

where $W(u_1, u_2)$ is the Wronskian of the two seed solutions u_1, u_2.

The general case follows straightforwardly by induction from the previous cases [31, 32, 33, 34, 35, 36, 37]. As an example of this general procedure next we apply the method to the harmonic oscillator.

2.1. Harmonic oscillator

For the specific harmonic oscillator Hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2},$$

a transformation of order k leads to a Hamiltonian $\tilde{H}$ having at most k new levels created at the positions defined by the factorization energies ϵ_j, $j = 1, .., k$.

In such a case, the eigenfunctions $\phi_n(x)$ of $\tilde{H}$ associated to the eigenvalues $E_n = n + \frac{1}{2}$ of H are

$$\phi_n(x) = \frac{A^+\psi_n(x)}{\sqrt{(E_n - \epsilon_1)...(E_n - \epsilon_k)}},$$

while the ones $\phi_{\epsilon_j}(x)$ associated to the new eigenvalues ϵ_j, are

$$\phi_{\epsilon_j} \propto \frac{W(u_1, .., u_{j-1}, u_{j+1}, ..., u_k)}{W(u_1, .., u_k)}.$$

The k transformation functions u_j, $j = 1, ..., k$ are chosen from the general solutions of the stationary Schrödinger equation $-u_j'' + V u_j = \epsilon_j u_j$ with $V = \frac{x^2}{2}$:

$$u_j(x) = e^{-x^2/2}\left[{}_1F_1\left(\frac{1-2\epsilon_j}{4}, \frac{1}{2}, x^2\right) + 2\nu_j \frac{\Gamma(\frac{3-2\epsilon_j}{4})}{\Gamma(\frac{1-2\epsilon_j}{4})}\, x\, {}_1F_1\left(\frac{3-2\epsilon_j}{4}, \frac{3}{2}, x^2\right)\right]. \quad (1)$$

In order to have a non-singular transformation, the Wronskian $W(u_1, .., u_k)$ must have no zeros on the real axis. This is achieved by demanding $\epsilon_k < \epsilon_{k-1} < ... < \epsilon_1 < E_0 = 1/2$ and $|\nu_1| < 1$, $|\nu_2| > 1$, $|\nu_3| < 1$,... so that the new potential

$$\tilde{V}(x) = \frac{x^2}{2} - [\ln W(u_1, ..., u_k)]''$$

won't have singularities in its domain.

Hamiltonians $\tilde{H}$ obtained by this procedure possess natural ladder operators, $L^\pm = A^+ a^\pm A$. Since $a^\pm$ are the usual first order ladder operators of the harmonic oscillator, then $L^\pm$ are differential operators of order $2k + 1$. The set of operators $\{\tilde{H}, L^+, L^-\}$ realizes the so-called Polynomial Heisenberg Algebras (PHA), which in general are described by the commutators

$$[\tilde{H}, L^\pm] = \pm L^\pm,$$
$$[L^-, L^+] = P_{2k}(\tilde{H}),$$

where $P_{2k}(\tilde{H})$ is a polynomial of degree $2k$ on the Hamiltonian $\tilde{H}$, which defines the system as a deformation of the harmonic oscillator.

In particular, systems with third order ladder operators $L^\pm$ realize a second order PHA. By factorizing them as $L^+ = L_1^+ L_2^+$ and $L^- = L_2^- L_1^-$ where $L_{1,2}^- = (L_{1,2}^+)^\dagger$ and $L_1^+ = \frac{1}{\sqrt{2}}\left[-\frac{d}{dx} + f(x)\right]$, $L_2^+ = \frac{1}{2}\left[\frac{d^2}{dx^2} + g(x)\frac{d}{dx} + h(x)\right]$, we find at the end the following set of equations:

$$f(x) = x + g(x),$$
$$h = \frac{g'}{2} - \frac{g^2}{2} - 2xg - x^2 + \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1 - 1,$$
$$V = \frac{x^2}{2} - \frac{g'}{2} + \frac{g^2}{2} + xg + \varepsilon_1 - \frac{1}{2},$$
$$\frac{d^2 g}{dx^2} = \frac{1}{2g}\left(\frac{dg}{dx}\right)^2 + \frac{3}{2}g^3 + 4xg^2 + 2(x^2 - a)g + \frac{b}{g}.$$

The last equation can be recognized as the Painlevé IV (PIV) differential equation with constant parameters $a = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1 - 1$ and $b = -2(\varepsilon_2 - \varepsilon_3)^2$. Thus, we get a connection between second order PHA's and the PIV equation. We can either solve such a non-linear differential equation in order to realize a second order PHA, or take a known realization of a second order PHA and obtain then specific solutions to the PIV equation.

The last is done by using the factorization

$$N \equiv L^+L^- = \left(\tilde{H} - \varepsilon_1\right)\left(\tilde{H} - \varepsilon_2\right)\left(\tilde{H} - \varepsilon_3\right), \tag{2}$$

which indicates that there are three extremal states ϕ_λ, with $\lambda = \varepsilon_1, \varepsilon_2, \varepsilon_3$, corresponding to the eigenvalues ε_1, ε_2 and ε_3 respectively such that $L^-\phi_\lambda = L_2^- L_1^- \phi_\lambda = 0$ and thus $N\phi_\lambda = 0$. In particular,

$$L_1^-\phi_\lambda = \frac{1}{\sqrt{2}}\left[\frac{d}{dx} + f(x)\right]\phi_\lambda = 0$$

satisfies the extremal state condition. Without lost of generality, we denote by ϕ_{ε_1} such a state, but let us keep in mind that there are three possible identifications. Finally, let us recall that $f(x) = x + g(x)$ to obtain

$$g(x) = -x - [\ln \phi_{\varepsilon_1}]', \tag{3}$$

which is a solution to the PIV equation with fixed parameters a, b.

On the other hand, systems with fourth order ladder operators $L^+ = L_1^+L_2^+$ and $L^- = L_2^-L_1^-$ realize a third order PHA [10, 25]. Choosing $L_1^+ = \frac{1}{2}\left[\frac{d^2}{dx^2} + g_1(x)\frac{d}{dx} + h_1(x)\right]$, $L_2^+ = \frac{1}{2}\left[\frac{d^2}{dx^2} + g(x)\frac{d}{dx} + h(x)\right]$ and $L_{1,2}^- = (L_{1,2}^+)^\dagger$ we find this other set of equations:

$$\begin{gathered} g_1(x) = -g(x) - x, \\ g(x) = \tfrac{x}{w-1}, \\ \frac{d^2w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{(w-1)^2}{z^2}\left(aw + \frac{b}{w}\right) + c\frac{w}{z} + d\frac{w(w+1)}{w-1}. \end{gathered} \tag{4}$$

Equation (4) is recognized as the Painlevé V (PV) differential equation, with $z = x^2$ and $a = \frac{(\varepsilon_1-\varepsilon_2)^2}{2}$, $b = -\frac{(\varepsilon_3-\varepsilon_4)^2}{2}$, $c = \frac{\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4-1}{2}$, $d = -\frac{1}{8}$ being fixed parameters.

This time we are concerned with the factorization

$$N \equiv L^+L^- = \left(\tilde{H} - \varepsilon_1\right)\left(\tilde{H} - \varepsilon_2\right)\left(\tilde{H} - \varepsilon_3\right)\left(\tilde{H} - \varepsilon_4\right) \tag{5}$$

which shows that there are four extremal states such that $L^-\phi_\lambda = L_2^-L_1^-\phi_\lambda = 0$ and thus $N\phi_\lambda = 0$, $\lambda = \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ corresponding to the eigenvalues ε_1, ε_2, ε_3 and ε_4 respectively.

In a similar fashion as the previous case, but with lengthier calculations, without lost of generality we can write $g_1 = (\ln[W(\phi_{\varepsilon_3}, \phi_{\varepsilon_4})])'$ thus obtaining $g(x) = -x - (\ln[W(\phi_{\varepsilon_3}, \phi_{\varepsilon_4})])'$. The use of two extremal states in this construction enables to fix now six different identifications for ϕ_{ε_3} and ϕ_{ε_4} leading to g.

Finally, through the change of variable $x = \sqrt{z}$, the function

$$w(z) = 1 + \frac{\sqrt{z}}{g(\sqrt{z})}.$$

produces the solution to the PV equation with fixed parameters a, b, c and d [21, 37].

3. Truncated harmonic oscillator

Adding an infinite potential barrier to the previous potential leads to modify the boundary condition at $x = 0$. The Hamiltonian of this system is

$$H_0 = -\frac{1}{2}\frac{d^2}{dx^2} + V_0, \qquad V_0 = \begin{cases} \frac{x^2}{2} & \text{if } x > 0, \\ \infty & \text{if } x \leq 0. \end{cases}$$

While both functions (and energies)

$$\psi_n(x) \propto x\, e^{-x^2/2}\, {}_1F_1\Big(-n, \frac{3}{2}, x^2\Big), \qquad E_n = 2n + \frac{3}{2},$$

$$\chi_n(x) \propto e^{-x^2/2}\, {}_1F_1\Big(-n, \frac{1}{2}, x^2\Big), \qquad \mathcal{E}_n = 2n + \frac{1}{2},$$

satisfy $H_0\psi_n = E_n\psi_n$ and $H_0\chi_n = \mathcal{E}_n\chi_n$ respectively, only the ψ_n satisfy the boundary condition at the origin making each E_n an eigenvalue of H_0, which is not the case for $\mathcal{E}_n$.

In order to implement the factorization method, with a major control on the singularities of the new potentials, we will use transformation functions as in equation (1) but with a definite parity, i.e., we will proceed by cases taking $u_j(x) = xe^{-x^2/2}\, {}_1F_1\left(\frac{3-2\epsilon_j}{4}, \frac{3}{2}, x^2\right)$ as the odd transformation function and $u_j(x) = e^{-x^2/2}\, {}_1F_1\left(\frac{1-2\epsilon_j}{4}, \frac{1}{2}, x^2\right)$ as the even one.

3.1. Intertwining operator with $k = 1$

Implementing a first order intertwining $H_1A^+ = A^+H_0$ using an odd transformation function u induces new Hamiltonians H_1 with potentials of the form

$$V_1 = V_0 + 1 + \frac{1}{x^2} - \left\{ \ln\left[{}_1F_1\Big(\frac{3-2\epsilon}{4}, \frac{3}{2}, x^2\Big)\right] \right\}'', \tag{6}$$

with eigenfunctions and eigenvalues given by $\phi_n \propto A^+\psi_n$ and $E_n = 2n + \frac{3}{2}$. For $\epsilon < \frac{3}{2}$ there is no new level of H_1 at ϵ, while in the limit $\epsilon = \frac{3}{2}$ one can erase the level $E_0 = \frac{3}{2}$.

On the other hand, the use of an even transformation function u produces new Hamiltonians H_1 with potentials of the form

$$V_1 = V_0 + 1 - \left\{ \ln\left[{}_1F_1\left(\frac{1-2\epsilon}{4}, \frac{1}{2}, x^2\right)\right] \right\}''. \tag{7}$$

In this case the eigenfunctions and eigenvalues of H_1 become $\phi_n \propto A^+\chi_n$ and $\mathcal{E}_n = 2n + \frac{1}{2}$. Again, for $\epsilon < \frac{1}{2}$ there is no new level of H_1 at ϵ, and with the choice $\epsilon = \frac{1}{2}$ one can erase the level $E_0 = \frac{1}{2}$.

In any case, for both of these first order intertwinings the new Hamiltonians are isospectral to H_0, up to a shift in the ground state energy. This will not hold anymore for second order intertwinings.

3.2. Intertwining operator with $k = 2$

A second order intertwining $H_2A^+ = A^+H_0$, where $H_2 = -\frac{1}{2}\frac{d^2}{dx^2} + V_2$, requires to use two transformation functions, u_1 and u_2 associated to ϵ_1 and ϵ_2 respectively, as described in Section 2. Without lost of generality, let us suppose that $\epsilon_2 < \epsilon_1$. We have found four non-equivalent combinations for these seed solutions [28, 29]. The results obtained are to be described in what follows.

If both u_1 and u_2 are odd then the new potential takes the form

$$V_2 = V_0 + \frac{3}{x^2} + 2 - [\ln \omega_1(x)]'' ,$$

where the function $\omega_1(x)$ is free of zeros in the interval $(0,\infty)$ as long as $\epsilon_1, \epsilon_2 < \frac{3}{2}$ or $\frac{3+4n}{2} < \epsilon_1, \epsilon_2 < \frac{3+4(n+1)}{2}$. The set of eigenvalues of H_2 is given by $\{E_n = 2n + \frac{3}{2}, \quad n = 0, 1, ...\}$ with corresponding eigenfunctions $\phi_n(x) \propto A^+\psi_n(x)$. As a limiting case, when one of the factorization energies takes the value E_n, then this level is erased from the set of eigenvalues.

In the case that u_1 and u_2 are both even it is found that

$$V_2 = V_0 + \frac{1}{x^2} + 2 - [\ln \omega_2(x)]'' ,$$

where now the function $\omega_2(x)$ will not have zeros in $(0,\infty)$ whenever $\epsilon_1, \epsilon_2 < \frac{1}{2}$ or $\frac{1+4n}{2} < \epsilon_1, \epsilon_2 < \frac{1+4(n+1)}{2}$. In general, H_2 has eigenvalues $\mathcal{E}_n = 2n + \frac{1}{2}$, $n = 0, 1, ...$, with $\phi_n(x) \propto A^+\chi_n(x)$ being their corresponding eigenfunctions. However when one of the factorization energies becomes $\mathcal{E}_n$, this level is missing from the set of eigenvalues.

For u_1 being odd and u_2 even with $\frac{1+4n}{2} < \epsilon_2 < \epsilon_1 < \frac{3+4n}{2}$, the new potential becomes

$$V_2 = V_0 + 2 - [\ln \omega_3(x)]'' .$$

It does not have any singularity in the domain $(0,\infty)$ other than the original one at $x = 0$. The eigenfunctions transform as $\phi_n(x) \propto A^+\psi_n(x)$ with eigenvalues $E_n = 2n + \frac{3}{2}$. The limiting case of this choice occurs when $\epsilon_1 = E_n$, which erases the level E_n. In addition, for $\epsilon_2 \neq \mathcal{E}_n$ the level ϵ_2 is added with $\phi_{\epsilon_2} \propto \frac{u_1}{W(u_1,u_2)}$.

Finally, when u_1 is chosen even and u_2 odd, the new potential turns out to be

$$V_2 = V_0 + 2 - [\ln \omega_4(x)]'' ,$$

where the function $\omega_4(x)$ does not have zeros in $(0,\infty)$ whenever $\epsilon_2 < \epsilon_1 < \frac{1}{2}$ or $\frac{3+4n}{2} < \epsilon_2 < \epsilon_1 < \frac{5+4n}{2}$. The eigenfunctions of H_2 are $\phi_n(x) \propto A^+\psi_n$ with eigenvalues $E_n = 2n + \frac{3}{2}$. With the choice $\epsilon_2 = E_n$ we can erase precisely this level. For $\epsilon_1 \neq \mathcal{E}_n$ there is an extra level at ϵ_1 with eigenfunction $\phi_{\epsilon_1} \propto \frac{u_2}{W(u_1,u_2)}$.

4. Solutions to the Painlevé IV equation

When performing a first order transformation on the truncated harmonic oscillator the supersymmetric partners H_1 possess natural third-order ladder operators $L^\pm = A^+a^\pm A$, thus we require just to identify the three extremal states to connect with specific solutions to the PIV equation. From equation (2) we know that the extremal states of H_1 are related to the eigenvalues ϵ, $\epsilon + 1$, $\frac{1}{2}$, and they are given by

$$\phi_{\varepsilon_1} \propto \frac{1}{u}, \qquad \phi_{\varepsilon_2} \propto A^+a^+u, \qquad \phi_{\varepsilon_3} \propto A^+e^{\frac{-x^2}{2}}. \tag{8}$$

Once again, one chooses a definite parity for $u(x)$ in order to control better the position of the singularities introduced by the transformation. For an odd transformation function, equation (3) gives a solution to the PIV equation associated to the first extremal state of equation (8) of the form

$$g_1 = \frac{1}{x} - 2x + \left(1 - \frac{2}{3}\epsilon\right) x \, \frac{{}_1F_1(\frac{7-2\epsilon}{4}; \frac{5}{2}; x^2)}{{}_1F_1(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)}.$$

Meanwhile, for an even transformation function this procedure gives a solution to the PIV equation associated also to the first extremal state of equation (8) of the form

$$g_1 = -2x + (1-2\epsilon)\, x\, \frac{{}_1F_1(\frac{5-2\epsilon}{4}; \frac{3}{2}; x^2)}{{}_1F_1(\frac{1-2\epsilon}{4}; \frac{1}{2}; x^2)}.$$

Once g_1 is obtained, independently of the parity of $u(x)$ the other two solutions to the PIV equation are obtained through the expressions

$$g_2 = -g_1 - 2x - 2\left[\tfrac{x+(2\epsilon-x^2)(g_1+x)+(g_1+x)^3}{x^2-2\epsilon-1-(g_1+x)^2}\right],$$
$$g_3 = -\tfrac{g_1'+2}{g_1+2x}.$$

Now, a second order transformation generates Hamiltonians H_2 having natural fifth-order ladder operators $L^{\pm} = A^+a^{\pm}A$. Thus we need to employ the reduction theorem described in [14] which ensures that if $u_2(x) = a^- u_1(x)$ and $\epsilon_2 = \epsilon_1 - 1$, then the system has as well third order ladder operators $l^{\pm}$ such that

$$l^+l^- = (H_2 - \epsilon_1 + 1)(H_2 - \epsilon_1 - 1)(H_2 - 1/2).$$

Therefore, the sought extremal states corresponding to the eigenvalues $\epsilon_1 - 1$, $\epsilon_1 + 1$, $\frac{1}{2}$ are

$$\phi_{\varepsilon_1} \propto \frac{u_1}{W[u_1,u_2]}, \qquad \phi_{\varepsilon_2} \propto A^+a^+u_1, \qquad \phi_{\varepsilon_3} \propto A^+e^{\frac{-x^2}{2}}.$$

From the procedure described previously, based on equation (3), we get now the following solutions to the PIV equation

$$\begin{aligned} g_1 &= -x-\alpha+2\left[\frac{x+\alpha}{x^2+1-2\epsilon_1-\alpha^2}\right], \\ g_2 &= g_1 + \frac{2\alpha^2-2x^2+2(2\epsilon_1+1)}{\alpha-g_1-x}, \\ g_3 &= \frac{(x+\alpha)g_1^2+\left[2\epsilon_1-1+(x+\alpha)^2\right]g_1+(2\epsilon_1-3)(x+\alpha)}{(x+\alpha)^2+(x+\alpha)g_1+2\epsilon_1-1}, \end{aligned}$$

where $\alpha = \frac{u_1'}{u_1}$.

5. Solutions to the Painlevé V equation

As for the PV equation, let us recall first that the truncated harmonic oscillator Hamiltonian has a pair of second-order ladder operators $(a^+)^2$ and $(a^-)^2$ which connect in a natural way the eigenfunctions of H_0.

In this way, for a first-order transformation the Hamiltonian H_1 possesses also natural fourth order ladder operators $L^{\pm} = A^+(a^{\pm})^2A$, which satisfy the conditions described in the paragraphs after equation (5). Using now the six different identifications of the extremal states ϕ_{ε_3} and ϕ_{ε_4} supplies us with several explicit solutions to the PV equation:

$$w_1 = 1 + \frac{2\sqrt{2z}(1+2\epsilon-z+\sqrt{2z}\alpha)}{\sqrt{2z}(2+z)-4(1+z)\alpha+2\sqrt{2z}\alpha^2},$$
$$w_2 = \frac{\left(2\alpha+\sqrt{2z}\right)\left(4\alpha-4\epsilon\sqrt{2z}+\sqrt{2}z^{3/2}-2\sqrt{2}\alpha^2\sqrt{z}-4\sqrt{2z}\right)}{\left(2\alpha-\sqrt{2z}-\right)\left(4\alpha-4\epsilon\sqrt{2z}+\sqrt{2}z^{3/2}-2\sqrt{2}\alpha^2\sqrt{z}+4\sqrt{2z}\right)},$$

$$w_3 = 1+\frac{\sqrt{2z}\left(8\epsilon z-2z^2+4\alpha^2 z+4\sqrt{2z}\alpha-16\right)}{8\alpha-2\sqrt{2}z^{3/2}\left(\alpha^2+2\epsilon-3\right)+4\alpha z\left(\alpha^2+2\epsilon-1\right)-8\sqrt{2z}\left(\alpha^2+\epsilon-1\right)+\sqrt{2}z^{5/2}-2\alpha z^2},$$

$$w_4 = \frac{2\sqrt{2}-2\alpha\sqrt{z}-\sqrt{2}z}{2\sqrt{2}-2\alpha\sqrt{z}+\sqrt{2}z},$$

$$w_5 = \frac{2\alpha+\sqrt{2z}}{2\alpha-\sqrt{2z}},$$

$$w_6 = -\frac{-4\alpha+\sqrt{2}z^{3/2}+2\sqrt{2}\left(\alpha^2-1\right)\sqrt{z}+4\alpha z}{4\alpha-2\sqrt{2}\sqrt{z}\left(\alpha^2+2\epsilon-2\right)+\sqrt{2}z^{3/2}},$$

where again we have $\alpha = \frac{u'}{u}$.

Finally, for a second-order transformation the natural ladder operators $L^{\pm} = A^{+}(a^{\pm})^2 A$ of H_2 are now of sixth order. So, we find ourselves in the need of a slight generalization of the reduction theorem employed above. Such generalization requires now that $u_2 = (a^-)^2 u_1$, with $\epsilon_2 = \epsilon_1 - 2$, thus providing us with the fourth order ladder operators $l^{\pm}$ for H_2 such that

$$l^+ l^- = \left(H_2 - \frac{1}{2}\right)\left(H_2 - \frac{3}{2}\right)\left(H_2 - \epsilon_1 + 2\right)\left(H_2 - \epsilon_1 - 2\right).$$

Analytic expressions for solutions to the PV equation obtained through this method, with a second-order transformation and an arbitrary factorization energy ϵ_1, are too long to be depicted here. However, some simple examples for specific values of ϵ_1 can be shown.

If we set $\varepsilon_1 = \epsilon_1 - 2,\ \varepsilon_2 = \frac{3}{2},\ \varepsilon_3 = \frac{1}{2},\ \varepsilon_4 = \epsilon_1 + 2$, we get:

$$w(z) = \frac{3-z}{2} \quad \text{with} \quad \epsilon_1 = \frac{3}{2} \quad \text{and } u_1(x) \text{ odd}$$

For the identification $\varepsilon_1 = \frac{1}{2},\ \varepsilon_2 = \epsilon_1 - 2,\ \varepsilon_3 = \frac{3}{2},\ \varepsilon_4 = \epsilon_1 + 2$, it is obtained:

$$w(z) = \frac{1-z}{2} \quad \text{with} \quad \epsilon_1 = \frac{1}{2} \quad \text{and } u_1(x) \text{ even}$$

Finally, for the permutation $\varepsilon_1 = \epsilon_1 - 2,\ \varepsilon_2 = \epsilon_1 + 2,\ \varepsilon_3 = \frac{1}{2},\ \varepsilon_4 = \frac{3}{2}$, we arrive at:

$$w(z) = \frac{1+2z-z^2}{4+4z} \quad \text{with} \quad \epsilon_1 = \frac{3}{2} \quad \text{and } u_1(x) \text{ even}$$

6. Conclusions

In this work we have obtained sundry supersymmetric partners of the harmonic oscillator with an infinite potential barrier. In particular those generated from the first-order technique turned out to be isospectral to the truncated harmonic oscillator, while those obtained from second-order SUSY offered greater possibilities for spectral design, e.g., it is possible to erase one or two consecutive levels in the energy spectrum. It is also possible to add a new level to the original spectrum in almost any position on the energy axis.

For each parity choice of the transformation functions we have found the factorization energy domain allowing the non-singular first and second order supersymmetric transformations in $(0, \infty)$. It must be noted that, for the initial singular potential the first-order transformations with $u(x)$ even and the second-order ones with both $u_1(x)$ and $u_2(x)$ even behave in a peculiar way, since they transform the eigenfunctions of the harmonic oscillator which are not physical solutions of the original singular potential into those which are eigenfunctions of the new Hamiltonian and vice versa.

Even more, a simple and direct procedure to obtain several explicit solutions to the Painlevé IV and V equations was implemented using the extremal states associated to the supersymmetric partners of the harmonic oscillator with an infinite potential barrier at the origin.

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