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Fundamental solution of k-hyperbolic harmonic functions in odd spaces

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Abstract. We study k-hyperbolic harmonic functions

$$\Delta u - \frac{k}{x_n}\frac{\partial u}{\partial x_n} = 0,$$

in the upper half space $\mathbb{R}^{n+1}_+ = \{(x_0, x_1, ..., x_n) \mid x_i \in \mathbb{R}, x_n > 0 \text{ for } i = 0, ..., n\}$. The operator $x_n^{\frac{2k}{n-1}}\left(\Delta u - \frac{k}{x_n}\frac{\partial u}{\partial x_n}\right)$ is the Laplace-Beltrami operator with respect to the Riemannian metric $ds_k^2 = x_n^{-\frac{2k}{n-1}}(dx_0^2 + dx_1^2 + ... + dx_n^2)$. In case $k = n-1$ the Riemannian metric is the hyperbolic distance of Poincaré upper half space. The proposed functions are connected to the axially symmetric potentials studied notably by Weinstein, Huber and Leutwiler. We present the fundamental solution in case n is even using the hyperbolic metric. The main tool is the transformation of k-hyperbolic harmonic functions to eigenfunctions of the hyperbolic Laplace operator.

1. Introduction

We are studying function theory connected to the hyperbolic Riemannian metric

$$ds^2 = \frac{dx_0^2 + dx_1^2 + ... + dx_n^2}{x_n^2}$$

in the Poincaré upper half space

$$\mathbb{R}^{n+1}_+ = \{(x_0, x_1, ..., x_n) \mid x_i \in \mathbb{R}, x_n > 0 \text{ for } i = 0, ..., n\}.$$

This metric is interesting, since it is invariant under Möbius transformation mapping upper half space onto itself (see [7]). Moreover, Leutwiler noticed in [9] and [10] that the power function x^m $(m \in \mathbb{N}_0)$, calculated using Clifford algebras, is a conjugate gradient of a hyperbolic harmonic function f, that is

$$\Delta_h f = x_n^2 \Delta f - (n-1)\, x_n \frac{\partial f}{\partial x_n} = 0$$

where as usual

$$\Delta h = \frac{\partial^2 h}{\partial x_0^2} + ... + \frac{\partial^2 h}{\partial x_n^2}.$$

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and Δ_h is the hyperbolic Laplace-Beltrami operator with respect to the preceding metric. He started to study these type of functions.

In this paper, we study k-hyperbolic harmonic functions. Their function theory is investigated by the first author in [3] and for $k = n-1$ for example in [5]. The Riemannian metric connected to k-hyperbolic harmonic functions is

$$ds_k^2 = \frac{dx_0^2 + dx_1^2 + ... + dx_n^2}{x_n^{\frac{2k}{n-1}}}$$

and they satisfy the equation

$$x_n \Delta h - k \frac{\partial h}{\partial x_n} = 0$$

(see [13]). Weinstein introduced axially symmetric potential theory in [14] and his motivation was that if $p < 0$ is an integer then an axially symmetric harmonic function in $2-p$-dimensional space satisfies the preceding equation in the meridian plane (see for example [8]). It is known that k-hyperbolic harmonic functions are forming also a harmonic space in the sense of Contantinescu and Cornea [2] but this result is not giving any kind of concrete presentations for k-harmonic functions.

We also need to consider the generalized Weinstein equation

$$x_n^2 \Delta h - k x_n \frac{\partial h}{\partial x_n} + lh = 0 \tag{1}$$

in an open domain whose closure is contained in the upper half space. Earlier this equation has been researched for example by Leutwiler and Akim in [1] and in [11]. Our general technical assumption is that the constants $l, k \in \mathbb{R}$ satisfy $4l \leq (k+1)^2$. Our idea is that we transfer solutions of this equation to the solutions of Laplace-Beltrami equation of the hyperbolic metric in the Poincaré upper half space. In the main result, we present the fundamental solution of the equation (1) in terms of the hyperbolic distance function in case n is even. Earlier these results have been presented in $\mathbb{R}^3$ (see [6]).

2. Preliminaries

We recall the properties of the hyperbolic distance that we need later on (see the proof for example in [12]).

Theorem 2.1 *The hyperbolic distance $d_h(x,a)$ between the points $x = (x_0, ..., x_n)$ and $a = (a_0, ..., a_n)$ in $\mathbb{R}^{n+1}_+$ may be computed as*

$$d_h(x,a) = \text{arcosh}\ \lambda(x,a),$$

where

$$\begin{aligned}\lambda(x,a) &= \frac{(x_0 - a_0)^2 + ... + (x_{n-1} - a_{n-1})^2 + x_n^2 + a_n^2}{2x_n a_n} \\ &= \frac{|x-a|^2 + |x-\hat{a}|^2}{4x_n a_n}.\end{aligned}$$

In above $\hat{a} = (a_0, ..., a_{n-1}, -a_n)$ and $|x-a|$ is the usual Euclidean distance.

We also apply the simple calculation rules of the hyperbolic distance stated next.

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Lemma 2.2 *If $x = (x_0, x_1, ..., x_n)$ and $a = (a_0, a_1, ..., a_n)$ are points in $\mathbb{R}^{n+1}_+$ then*

$$|x - a|^2 = 2x_n a_n (\lambda(x, a) - 1), \tag{2}$$

$$|x - \hat{a}|^2 = 2x_n a_n (\lambda(x, a) + 1), \tag{3}$$

$$\frac{|x - a|^2}{|x - \hat{a}|^2} = \frac{\lambda(x, a) - 1}{\lambda(x, a) + 1} = \tanh^2(\frac{d_h(x, a)}{2}), \tag{4}$$

where $\widehat{a} = (a_0, .., a_{n-1}, -a_n)$.

We also note the relation between the Euclidean and hyperbolic balls.

Proposition 2.3 *The hyperbolic ball $B_h(a, r_h)$ with the hyperbolic center $a = (a_0, ..., a_n)$ and the radius r_h is the same as the Euclidean ball with the Euclidean center*

$$c_a(r_h) = a = (a_0, ..., a_n \cosh r_h)$$

and the Euclidean radius $r_e = a_n \sinh r_h$.

Definition 2.4 *Let $\Omega \subset \mathbb{R}^{n+1}_+$ be open. A twice continuously differentiable function $f : \Omega \to \mathbb{R}$ is $k-$hyperbolic harmonic if*

$$\Delta f - \frac{k}{x_n}\frac{\partial f}{\partial x_n} = 0$$

for any $x \in \Omega$.

An important tool for handling k-hyperbolic harmonic functions is their transformation to eigenfunctions of the hyperbolic Laplace operator, stated next.

Lemma 2.5 *Let Ω be an open set contained in $\mathbb{R}^{n+1}_+$. A function $f : \Omega \to \mathbb{R}$ satisfies the Weinstein equation (1) if and only if the function $g(x) = x_n^{\frac{n-k-1}{2}} f(x)$ satisfies the following equation*

$$\Delta_h g + \frac{1}{4}\left(n^2 - (k+1)^2 + 4l\right) g = 0. \tag{5}$$

Especially, a function $f : \Omega \to \mathbb{R}$ is k-hyperbolic harmonic if and only if the function $g(x) = x_n^{\frac{n-k-1}{2}} f(x)$ satisfies the following equation

$$\Delta_h g + \frac{1}{4}\left(n^2 - (k+1)^2\right) g = 0. \tag{6}$$

3. The fundamental solution in $\mathbb{R}^{n+1}_+$ when n is even

We are looking for a fundamental solutions of the equation (6) connected to the hyperbolic Laplace operator. The hyperbolic Laplace of functions depending on λ is computed in [4] as follows.

Lemma 3.1 *If f is twice continuously differentiable depending only on $\lambda = \lambda(x, e_n)$ then*

$$\Delta_h f(x) = \left(\lambda^2 - 1\right)\frac{\partial^2 f}{\partial \lambda^2} + (n+1)\lambda\frac{\partial f}{\partial \lambda}.$$

It is easier to compute the solutions in terms of the hyperbolic distance.

Theorem 3.2 *If f is twice continuously differentiable depending only on $r_h = d_h(x, e_n) = \operatorname{arcosh} \lambda(x, e_n)$ then the hyperbolic Laplace in $\mathbb{R}^{n+1}_+$ is given by*

$$\Delta_h f(r_h) = \frac{\partial^2 f}{\partial r_h^2} + n \coth r_h \frac{\partial f}{\partial r_h}.$$

Proof. Using $\lambda(x, e_n) = \cosh r_h$, we compute

$$\frac{\partial r_h}{\partial \lambda} = \frac{1}{\sinh r_h}$$

and

$$\frac{\partial^2 r_h}{\partial \lambda^2} = -\frac{\cosh r_h}{\sinh^3 r_h}.$$

Hence applying the chain rule we obtain

$$\begin{aligned} \frac{\partial f(r_h)}{\partial \lambda} &= \frac{\partial f}{\partial r_h}\frac{\partial r_h}{\partial \lambda} = \frac{\partial f}{\partial r_h}\frac{1}{\sinh r_h}, \\ \frac{\partial^2 f(\lambda)}{\partial \lambda^2} &= \frac{\partial^2 f}{\partial r_h^2}\left(\frac{\partial r_h}{\partial \lambda}\right)^2 + \frac{\partial f}{\partial r_h}\frac{\partial^2 r_h}{\partial \lambda^2} \\ &= \frac{\partial^2 f}{\partial r_h^2}\frac{1}{\sinh^2 r_h} - \frac{\partial f}{\partial r_h}\frac{\cosh r_h}{\sinh^3 r_h}, \end{aligned}$$

completing the proof by the preceding Lemma. ■

In case n even, the preceding equation may be solved inductively. The crucial step is the following result.

Lemma 3.3 *Let $s \in \mathbb{N}_0$. If f is a solution of the equation*

$$\frac{\partial^2 f}{\partial r_h^2} + 2s \coth r_h \frac{\partial f}{\partial r_h} = \gamma f \tag{7}$$

depending on $r_h = d_h(x, e_n)$ then the function $g(r_h) = \frac{1}{\sinh r_h}\frac{\partial f}{\partial r_h}(r_h)$ satisfies the equation

$$\frac{\partial^2 g}{\partial r_h^2} + 2(s+1)\coth r_h \frac{\partial g}{\partial r_h} = (\gamma - 1 - 2s)\, g.$$

Proof. We just compute

$$\begin{aligned} \frac{\partial^2 g}{\partial r_h^2} &= \left(-\frac{1}{\sinh r_h} + 2\frac{\cosh^2 r_h}{\sinh^3 r_h}\right)\frac{\partial f}{\partial r_h} - 2\frac{\cosh r_h}{\sinh^2 r_h}\frac{\partial^2 f}{\partial r_h^2} + \frac{1}{\sinh r_h}\frac{\partial^3 f}{\partial r_h^3}, \\ 2(s+1)\frac{\partial g}{\partial r_h}\frac{\cosh r_h}{\sinh r_h} &= -\frac{2(s+1)\cosh^2 r_h}{\sinh^3 r_h}\frac{\partial f}{\partial r_h} + \frac{2(s+1)\cosh r_h}{\sinh^2 r_h}\frac{\partial^2 f}{\partial^2 r_h}. \end{aligned}$$

Since $\frac{\partial^2 f}{\partial r_h^2} = \gamma f - 2s \coth r_h \frac{\partial f}{\partial r_h}$ we conclude

$$\begin{aligned} \frac{1}{\sinh r_h}\frac{\partial^3 f}{\partial r_h^3} &= \gamma g - 2s\frac{\cosh r_h}{\sinh^2 r_h}\frac{\partial^2 f}{\partial r_h^2} + 2s\frac{1}{\sinh^3 r_h}\frac{\partial f}{\partial r_h} \\ &= \gamma g - 2s\frac{\cosh r_h}{\sinh^2 r_h}\frac{\partial^2 f}{\partial r_h^2} + 2s\frac{\cosh^2 r_h}{\sinh^3 r_h}\frac{\partial f}{\partial r_h} - 2s\frac{1}{\sinh r_h}\frac{\partial f}{\partial r_h} \end{aligned}$$

and therefore
$$\frac{\partial^2 g}{\partial r_h^2}+2\left(s+1\right)\frac{\partial g}{\partial r_h}\frac{\cosh r_h}{\sinh r_h}=\left(\gamma-1-2s\right)g.$$
■

Applying the previous result inductively, we are able to solve our main equation.

Proposition 3.4 *Let A be the operator defined by $A=\frac{1}{\sinh(r_h)}\frac{\partial}{\partial r_h}$. If f is a solution of the equation*
$$\frac{\partial^2 f}{\partial r_h^2}=\gamma f$$
then $g=A^m f$ satisfies the equation
$$\frac{\partial^2 g}{\partial r_h^2}+2m\frac{\partial g}{\partial r_h}\frac{\cosh r_h}{\sinh r_h}=\left(\gamma-m^2\right)g.$$

Proof. Applying inductively the previous lemma, we may deduce the result as follows
$$\begin{aligned}\frac{\partial^2 g}{\partial r_h^2}+2m\frac{\partial g}{\partial r_h}\frac{\cosh r_h}{\sinh r_h}&=\left(\gamma-\sum_{j=0}^{m-1}\left(1+2j\right)\right)g\\&=\left(\gamma-\left(m+\left(m-1\right)m\right)\right)g=\left(\gamma-m^2\right)g.\end{aligned}$$
■

Corollary 3.5 *If f is a solution of the equation*
$$\frac{\partial^2 f}{\partial r_h^2}=(\gamma+m^2)f$$
then $g=A^m f$ satisfies the equation
$$\frac{\partial^2 g}{\partial r_h^2}+2m\frac{\partial g}{\partial r_h}\frac{\cosh r_h}{\sinh r_h}=\gamma g.$$

We are ready to state our main result.

Theorem 3.6 *Let $\gamma=\frac{(k+1)^2-4l}{4}-m^2$, $\alpha=\sqrt{(k+1)^2-4l}$ and $n=2m$. The solution of the equation*
$$\frac{\partial^2 g}{\partial r_h^2}+2m\frac{\partial g}{\partial r_h}\frac{\cosh r_h}{\sinh r_h}=\gamma g$$
depending on the hyperbolic distance $r_h=d_h\left(x,a\right)$ with a pole at $x=a$ is
$$F_{n,k,l}\left(r_h\right)=\frac{s_{n,k,l}\left(r_h\right)}{\sinh^{n-1}r_h}$$
where $s_{n,k,l}\left(r_h\right)=\sum_{j=-m+1}^{j=m-1}a_{j,n,l}\cosh\left(\frac{r_n(\alpha+2j)}{2}\right)$ for some constant $a_{j,n,l}$ defined inductively by
$$\begin{aligned}a_{j,n+2,l}&=\frac{a_{(j-1),n}\left(\alpha+2j-2n\right)-a_{(j+1),n,l}\left(\alpha+2\left(j+n\right)\right)}{4}\\a_{j,n,0}&=0\qquad \text{if } j<1-m \text{ or } j>m-1,\\a_{0,2,0}&=1\end{aligned}$$

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and

$$s_{n,k,l}(0) = \frac{(-1)^{m-1}(n-1)!!}{2}.$$

Moreover, the function

$$H_{n,k}(x,a) = (-1)^{m-1} x_n^{\frac{k+1-n}{2}} a_n^{\frac{k+1-n}{2}} F_{n,k,0}(d_h(a,x))$$

is k-hyperbolic with respect to the both variables a and x.

Proof. Using the preceding corollary we obtain that the function $G_{n,k,l}(r_h) = CA^m f$ is the eigenfunction corresponding to the value γ when we choose

$$f(r_h) = \begin{cases} \sinh\left(\frac{\alpha r_h}{2}\right), & k \neq -1, \\ r_h, & k = -1. \end{cases}$$

Setting $C = \frac{2}{\alpha}$, we deduce that

$$F_{2,k,l}(r_h) = \frac{\cosh\left(\frac{\alpha r_h}{2}\right)}{\sinh r_h}$$

is the eigenfunction corresponding to the value γ and the result holds when $m = 1$ and $a_{0,2,0} = 1$. Assume that the result holds for some $n = 2m$. Then

$$\begin{aligned} F_{n+2,k,l}(r_h) = & \frac{\sum_{j=-m+1}^{j=m-1} a_{j,n,l}(\alpha+2j)\sinh\left(\frac{r_n(\alpha+2j)}{2}\right)\sinh r_h}{2\sinh^{n+1} r_h} \\ & - \frac{(n-1)\sum_{j=-m+1}^{j=m-1} a_{j,n,l}\cosh\left(\frac{r_n(\alpha+2j)}{2}\right)\cosh r_n}{\sinh^{n+1} r_h}. \end{aligned} \tag{8}$$

Applying the hyperbolic identities, we obtain

$$\begin{aligned} F_{n+2,k,l}(r_h) = & \frac{\sum_{j=1-m}^{j=m-1} a_{j,n,l}(\alpha+2(j-n+1))\cosh\left(\frac{r_n(|k+1|+2(j+1))}{2}\right)}{4\sinh^{n+1} r_h} \\ & - \frac{\sum_{j=1-m}^{j=m-1} a_{j,n,l}(\alpha+2(j+n-1))\cosh\left(\frac{r_n(\alpha+2(j-1))}{2}\right)}{4\sinh^{n+1} r_h}. \end{aligned}$$

Setting $a_{j,n,l} = 0$ if $j < 1-m$ or $j > m-1$, we obtain

$$a_{j,n+2,l} = \frac{a_{(j-1),n}(\alpha+2j-2n) - a_{(j+1),n}(\alpha+2(j+n))}{4}$$

completing the proof of the first assertion. The value of $s_{n,k,l}(0)$ follows inductively from (8). ■

The preceding function leads to the fundamental solution. For the proof we need the Green's identity (see for example [1]).

Lemma 3.7 *Let $R \subset \mathbb{R}_+^{n+1}$ be an open set with the smooth boundary contained $\mathbb{R}_+^{n+1}$ and denote the volume element corresponding to the Riemannian metric*

$$ds^2 = \frac{dx_0^2 + dx_1^2 + ... + dx_n^2}{x_n^{\frac{2k}{n-1}}}$$

by $dx_{(k)} = x_n^{-\frac{k(n+1)}{n-1}} dx$, *the surface elements by* $d\sigma_{(k)} = x_n^{-\frac{kn}{n-1}} d\sigma$ *and the outer normal* $\frac{\partial u}{\partial m_k} = x_n^{\frac{k}{n-1}} \frac{\partial u}{\partial m}$ *where* m *is the outer normal to the the surface* ∂R. *Then the Laplace-Beltrami operator is*

$$\Delta_k = x_n^{\frac{2k}{n-1}} \left(\Delta - \frac{k}{x_n} \frac{\partial}{\partial x_n} \right)$$

and

$$\int_R \left(u \Delta_k v dx_{(k)} - v \Delta_k u dx_{(k)} \right) = \int_{\partial R} \left(u \frac{\partial v}{\partial m_k} - v \frac{\partial u}{\partial m_k} \right) d\sigma_{(k)}$$

for any functions u *and* v *that are twice continuously differentiable functions in an open neighborhood containing the closure* $\overline{R}$ *of* R.

As usual, we obtain setting $v = 1$ the result

Proposition 3.8 *If* $R \subset \mathbb{R}_+^{n+1}$ *be an open set and* $f : R \to \mathbb{R}$ *is* k*-hyperbolic harmonic then*

$$\int_{\partial B_h(y,\rho)} \frac{\partial u}{\partial m_k} d\sigma_{(k)} = 0$$

for any $B_h(y, \rho)$ *with* $\overline{B_h(y, \rho)} \subset R$.

We need to prove that the preceding function $H_{n,k}(y, a)$ is Lebesgue integrable and that its normal derivative has some nice limiting property.

Lemma 3.9 *The function* $H_{n,k}(y, a)$ *is Lebesgue integrable in the hyperbolic ball* $B_h(a, R_h)$ *and*

$$\int_{B_h(a,R_h)} \left| F_{n,k,0}\left(d_h(a, y)\right) \right| \frac{dy}{y_n^{\frac{2kn}{n-1}}} \leq M \sinh^2 r_h$$

for some positive $M > 0$.

Proof. It is enough to prove the statement for $a = (0, ..., 1)$. We recall that

$$\frac{s_{n,k}\left(d_h(y, a)\right)}{\sinh^{n-1} d_h(y, a)} = \frac{s_{n,k}\left(d_h(y, a)\right)}{|y - \cosh R_h a|^{n-1}}$$

in $B_h(a, R_h)$. Since $s_{n,k}\left(d_h(y, a)\right) y_n^{-\frac{2kn}{n-1}}$ is a continuous function, we only need to consider the integral

$$\int_{B_h(a,R_h)} \sinh^{1-n} d_h(y, a)\, dy = \int_{B_e(\cosh R_h a, \sinh R_h)} \frac{dy}{|y - \cosh R_h a|^{n-1}},$$

since $\sinh d_h(y, a) = |y - \cosh R_h a|$ by Proposition 2.3. Hence we have

$$\int_{B_e(\cosh R_h a, R_h)} \sinh^{1-n} d_h(y, a)\, dy = \int_0^{\sinh r_h} r \int_{\partial B_e(\cosh R_h a, 1)} dy dr = \frac{\omega_n \sinh^2 r_h}{2},$$

where ω_n is the surface measure of the unit ball in $\mathbb{R}^{n+1}$. ■

Lemma 3.10 *Let $\Omega \subset \mathbb{R}_+^{n+1}$ be open. Let u be continuous in Ω. Denote $H_{n,k}(y,x) = (-1)^{\frac{n}{2}-1} y_n^{\frac{k+1-n}{2}} x_n^{\frac{k+1-n}{2}} F_{n,k,0}(y,x)$. Then*

$$\lim_{R_h \to 0} \frac{\int_{\partial B_h(x,R_h)} u \frac{\partial H_{n,k}}{\partial m_k} d\sigma_{(k)}}{2^{\frac{n+2}{2}} \pi^{\frac{n}{2}}} = -u(x).$$

for any hyperbolic balls $B_h(x,r_h)$ satisfying $\overline{B_h(x,r_h)} \subset \Omega$.

Proof. We abbreviate $F_{n,k,0} = F_{n,k}$. Using Proposition 2.3 we infer that in $\partial B_h(x,R_h)$ the outside pointing normal at y is

$$m = (\nu_0, \nu_1, ..., \nu_n) = \frac{(y_0 - x_0, y_1 - x_1, ..., y_n - x_n \cosh R_h)}{x_n \sinh R_h}.$$

Denote $r_h = d_h(y,x)$. We first compute

$$\begin{aligned}(-1)^{\frac{n}{2}-1} \frac{\partial H_{n,k}}{\partial m_k} &= (-1)^{\frac{n}{2}-1} y_n^{\frac{k}{n-1}} \frac{\partial H_{n,k}}{\partial m} = (-1)^{\frac{n}{2}-1} y_n^{\frac{k}{n-1}} (m, \operatorname{grad} H_{n,k}) \\ &= y_n^{\frac{(n-1)(k+1-n)+2k}{2(n-1)}} x_n^{\frac{k+1-n}{2}} \frac{\partial F_{n,k}}{\partial r_h} \sum_{i=0}^{n} \nu_i \frac{\partial r_h}{\partial y_i} \\ &+ \frac{k+1-n}{2} y_n^{\frac{(n-1)(k-1-n)+2k}{2(n-1)}} x_n^{\frac{k+1-n}{2}} \nu_n F_{n,k}.\end{aligned}$$

Applying Lemma 2.1 we infer

$$\frac{\partial r_h}{\partial y_i} = \frac{\partial r_h}{\partial y_i} = \frac{\partial \arccos \lambda(y,x)}{\partial y_i} = \frac{y_i - x_i - x_n (\cosh r_h - 1) \delta_{in}}{y_n x_n \sinh r_h}$$

and therefore we conclude

$$\sum_{i=0}^{n} \nu_i \frac{\partial r_h}{\partial y_i} = \frac{1}{y_n}.$$

Hence we have

$$\begin{aligned}(-1)^{\frac{n}{2}-1} \frac{\partial H_{n,k}}{\partial m_k}(y) = &\, y_n^{\frac{(n-1)(k-1-n)+2k}{2(n-1)}} x_n^{\frac{k+1-n}{2}} \frac{\partial F_{n,k}}{\partial r_h} + \\ &\frac{k+1-n}{2} y_n^{\frac{(n-1)(k-1-n)+2k}{2(n-1)}} x_n^{\frac{k+1-n}{2}} \nu_n F_{n,k} \\ = &\, y_n^{\frac{(n-1)(k-1-n)+2k}{2(n-1)}} x_n^{\frac{k+1-n}{2}} \frac{s'_{k,n}(r_h)}{\sinh^{n-1} r_h} - \\ &(n-1)\, y_n^{\frac{(n-1)(k-1-n)+2k}{2(n-1)}} x_n^{\frac{k+1-n}{2}} \frac{s_{n,k}(r_h) \cosh r_h}{\sinh^n r_h} + \\ &\frac{k+1-n}{2} y_n^{\frac{(n-1)(k-1-n)+2k}{2(n-1)}} x_n^{\frac{k+1-n}{2}} \nu_n F_{n,k}.\end{aligned}$$

Since $B_h(x,R_h) = B(x_e, x_n \sin R_h)$ for $x_e = (x_0, x_1, ..., x_n \cosh R_h)$ we obtain

$$\lim_{R_h \to 0} \frac{1}{\omega_n x_n^n \sinh^n R_h} \int_{\partial B_h(x,R_h)} y_n^{\frac{(n-1)(k-1-n)+2k}{2(n-1)}} x_n^{\frac{k+1+n}{2}} \sinh R_h s'_{k,n}(R_h)\, d\sigma_{(k)} = 0.$$

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Similarly, we compute that

$$\lim_{R_h\to 0}\frac{1}{\omega_n x_n^n \sinh^n R_h}\int_{\partial B_h(x,R_h)}\frac{u(y)(y_n - x_n\cosh R_h)\, s_{n,k}(R_h)}{x_n}d\sigma_{(k)} = 0.$$

Since $\omega_n = (-1)^{\frac{n}{2}-1}\frac{2^{\frac{n+2}{2}}\pi^{\frac{n}{2}}}{s_{n,k}(0)}$ when n is even, manipulating the last integral we obtain

$$\begin{aligned}
&\lim_{R_h\to 0}\frac{-(n-1)(-1)^{\frac{n}{2}-1}}{\omega_n x_n^n\sinh^n R_h}\int_{\partial B_h(x,R_h)}u(y)\, y_n^{\frac{(n-1)(k-1-n)+2k}{2(n-1)}}x_n^{\frac{k+1+n}{2}}s_{n,k}(r_h)\cosh R_h d\sigma_{(k)}\\
&=\lim_{r_h\to 0}\frac{-(n-1)\,x_n^{\frac{k+1+n}{2}}s_{n,k}(0)\cosh R_h}{2^{\frac{n+2}{2}}\pi^{\frac{n}{2}}x_n^n\sinh^n R_h}\int_{\partial B_h(x,R_h)}u(y)\frac{(-1)^{\frac{n}{2}-1}s_{n,k}(r_h)}{y_n^{\frac{k+1+n}{2}}s_{n,k}(0)}d\sigma\\
&=\lim_{r_h\to 0}\frac{-(n-1)\,x_n^{\frac{k+1+n}{2}}\cosh R_h}{\omega_n x_n^n\sinh^n R_h}\int_{\partial B_h(x,R_h)}u(y)\frac{s_{n,k}(r_h)}{y_n^{\frac{k+1+n}{2}}s_{n,k}(0)}d\sigma\\
&=-(n-1)\,u(x),
\end{aligned}$$

completing the proof. ■

Our most important result is the following formula.

Theorem 3.11 *Let $\Omega\subset\mathbb{R}_+^{n+1}$ be open and $B_h(y,\rho)$ a hyperbolic ball with the hyperbolic center y and the hyperbolic radius ρ satisfying $\overline{B_h(y,\rho)}\subset\Omega$. If u is a twice continuously differentiable function in Ω and $x\in B_h(y,\rho)$ then*

$$\begin{aligned}
u(x) = &\frac{1}{2^{\frac{n+2}{2}}\pi^{\frac{n}{2}}(n-1)}\int_{\partial B_h(y,\rho)}\left(H_{n,k}(y,x)\frac{\partial u(y)}{\partial m_k} - u(y)\frac{\partial H_{n,k}(y,x)}{\partial m_k}\right)d\sigma_{(k)}(y)\\
&-\frac{1}{2^{\frac{n+2}{2}}\pi^{\frac{n}{2}}(n-1)}\int_{B_h(y,\rho)}\Delta_k u(y)\, H_{n,k}(y,x)\, dx_{(k)}
\end{aligned}$$

where $d\sigma_{(k)}$, $dx_{(k)}$ and $\frac{\partial}{\partial m_k}$ are the same as in Lemma 3.7.

Proof. Denote $B_h(y,\rho)=B$ and pick a hyperbolic ball such that $\overline{B_h(x,R_h)}\subset B$. Since $H_{h,k}$ is k-hyperbolic harmonic we obtain

$$\begin{aligned}
\int_{B\backslash B_h(x,R_h)}H_{n,k}\Delta_k u dx_{(k)} = &\int_{\partial R}\left(H_{n,k}\frac{\partial u}{\partial m_k} - u\frac{\partial H_{n,k}}{\partial m_k}\right)d\sigma_{(k)}\\
&-\int_{\partial B_h(x,R_h)}\left(u\frac{\partial H_{n,k}}{\partial m_k} + H_{n,k}\frac{\partial u}{\partial m_k}\right)d\sigma_{(k)}.
\end{aligned}$$

Since $\frac{\partial u}{\partial m_k}$ and $y_n^{\frac{k+1-n}{2}-\frac{2kn}{n-1}}x_n^{\frac{k+1-n}{2}}s_{n,k}(d_h(x,y))$ are bounded we obtain

$$\int_{\partial B_h(x,R_h)}\left|H_{n,k}(x,y)\frac{\partial u}{\partial m_k}\right|d\sigma_{(k)}(y)\le\frac{M}{\sinh^{n-1}R}\int_{\partial B_h(x,R_h)}d\sigma = M\omega_n\sinh R_h$$

and therefore

$$\lim_{R_h\to 0}\int_{\partial B_h(x,R_h)}\left|H_{n,k}(x,y)\frac{\partial u}{\partial m_k}\right|d\sigma_{(k)}(y)=0.$$

Applying Lemma 3.10 we conclude the result. ■

Corollary 3.12 *If $\phi\in\mathcal{C}_0^\infty\left(\mathbb{R}_+^{n+1}\right)$ and $\rho>0$ the radius such that supp$(\phi)\subset B_h(y,\rho)$ then*

$$\phi(x) = -\frac{1}{2^{\frac{n+2}{2}}\pi^{\frac{n}{2}}(n-1)}\int_{B_h(y,\rho)}\Delta_k\phi(y)\, H_{n,k}(y,x)\, dx_{(k)}.$$

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Conclusion

We have found the fundamental k-hyperbolic solution in odd dimensional spaces. Earlier these results have only been proved in $\mathbb{R}^3_+$ (see [6]). Using this result in consecutive papers, we will obtain kernels in hyperbolic function theory based on k-hyperbolic functions and Cauchy type integral formulas.

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