Geometry of surfaces associated to Grassmannian sigma models

L Delisle[1], **V Hussin**[2,3] **and W J Zakrzewski**[4]

[1] Institut de mathématiques de Jussieu–Paris Rive Gauche, UP7D–Campus des Grands Moulins, Bâtiment Sophie Germain, Cases 7012, 75205 Paris Cedex 13.
[2] Département de mathématiques et de statistique, Université de Montréal, C. P. 6128, Succ. Centre-ville, Montréal (Québec) H3C 3J7, Canada.
[3] Centre de recherches mathématiquxes, Université de Montréal, C. P. 6128, Succ. Centre-ville, Montréal (Québec) H3C 3J7, Canada.
[4] Department of mathematical sciences, University of Durham, Durham DH1 3LE, United Kingdom.

E-mail: laurent.delisle@imj-prg.fr, hussin@dms.umontreal.ca, w.j.zakrzewski@durham.ac.uk

Abstract. We investigate the geometric characteristics of constant Gaussian curvature surfaces obtained from solutions of the $G(m, n)$ sigma model. Most of these solutions are related to the Veronese sequence. We show that we can distinguish surfaces with the same Gaussian curvature using additional quantities like the topological charge and the mean curvature. The cases of $G(1, n) = \mathbb{C}P^{n-1}$ and $G(2, n)$ are used to illustrate these characteristics.

1. Introduction

In recent papers [1,2], we have classified some relevant solutions of the Grassmannian $G(m, n)$ sigma model that are associated to constant Gaussian curvature surfaces in $su(n)$. In our construction, we have found, among others, some non-equivalent solutions with the same constant Gaussian curvature. In the non-holomorphic case [2], we have considered a particular set of solutions starting from the knowledge of the corresponding solutions of $G(1, n) = \mathbb{C}P^{n-1}$, the so-called Veronese solutions. In the holomorphic case [1], we presented some conjectures and constructed some solutions which are not related to the Veronese ones. These results clarified and extended results obtained elsewhere [3-9].

In this contribution, we aim to show that some of the surfaces that have the same constant Gaussian curvature and correspond to non-equivalent solutions of $G(m, n)$, up to a gauge transformation, may be distinguished by other geometric characteristics such as the topological charge and the mean curvature. The case of $G(2, n)$ will be discussed in detail to show how this works out.

In Section 2, we discuss the $G(m, n)$ sigma model and we define the geometric quantities for the surfaces associated to $G(m, n)$. In particular, we recall a class of solutions of the model that lead to surfaces with constant Gaussian curvature and explain the relation with the Veronese sequence. We compute explicitly the additional geometric characteristics of these surfaces. In Section 3, we show how these quantities could explain the differences between surfaces of same constant Gaussian curvature. We re-visit the case of $G(1, n) = \mathbb{C}P^{n-1}$ and give some general results for $G(m,n)$. We show, in the case of $G(2,n)$, how different solutions with the same Gaussian curvature can be distinguished by calculating their topological charge and/or their mean curvature. We conclude this section with the case of non-Veronese holomorphic solutions. Section 4 presents our conclusions and future outlook.

2. Surfaces associated to solutions of $G(m,n)$

2.1. The model

The two-dimensional $G(m,n)$ sigma model is a field theory [10] defined on the complex plane $\mathbb{C}$ which has the Grassmannian manifold $G(m,n)$ as its target space:

$$G(m,n) \cong \frac{U(n)}{U(m)\times U(n-m)}, \quad n>m, \tag{1}$$

where $U(k)$ is the set of $k\times k$ unitary matrices. The field $Z(x_+,x_-)$ defined on an open and simply connected subset Ω of $\mathbb{C}$ thus takes values in $G(m,n)$. The elements Z of $G(m,n)$ are parametrized by $n\times m$ matrices and satisfy $Z^\dagger Z=\mathbb{I}_m$. Moreover, they correspond to critical points of the energy functional defined via the Lagrangian density

$$\mathcal{L}(Z)=\frac{1}{2}\mathrm{Tr}\left[(D_+Z)^\dagger D_+Z+(D_-Z)^\dagger D_-Z\right], \tag{2}$$

where $D_\pm\Lambda=\partial_\pm\Lambda-\Lambda(Z^\dagger\partial_\pm Z)$ are the covariant derivatives, $\partial_\pm=\partial_{x_\pm}$ and (x_+,x_-) are complex local coordinates on Ω. We consider the case of $\Omega=\mathbb{C}$ and require the energy of these fields to be finite. To achieve this we have to impose the boundary conditions $D_\pm Z\longrightarrow 0$ as $|x_\pm|\longrightarrow\infty$. With such boundary conditions, the complex plane $\mathbb{C}$ is compactified into the two-sphere S^2 via the stereographic projection and, as a consequence, the fields Z are harmonic maps [11] from S^2 into the Grassmann manifold $G(m,n)$.

Using the variation of the energy, we deduce the Euler-Lagrange equations of the model given as $(\pm\longleftrightarrow\mp)$

$$D_+D_-Z+Z(D_-Z)^\dagger D_-Z=0, \quad Z^\dagger Z=\mathbb{I}_m. \tag{3}$$

The finite energy solutions of these equations are fully known in the $G(1,n)\cong\mathbb{C}P^{n-1}$ case [11]. They are given by

$$Z_i=\frac{P_+^i f}{|P_+^i f|}, \quad i=0,1,\cdots,n-1, \tag{4}$$

where $f=f(x_+)\in\mathbb{C}^n$ is holomorphic and P_+ is an orthogonalizing operator defined recursively as

$$P_+^0 f=f, \quad P_+f=\partial_+f-\frac{f^\dagger\partial_+f}{|f|^2}f, \quad P_+^i f=P_+(P_+^{i-1}f), \quad P_+^n f=0. \tag{5}$$

For the $G(m,n)$ model with $m\geq 2$, the complete set of solutions is not known, but we can use the solutions (4) of the $G(1,n)$ model to construct particular classes of them:

$$Z_{(i_1,i_2,\cdots,i_m)}=\left(\frac{P_+^{i_1}f}{|P_+^{i_1}f|},\frac{P_+^{i_2}f}{|P_+^{i_2}f|},\cdots,\frac{P_+^{i_m}f}{|P_+^{i_m}f|}\right), \quad 0\leq i_1<i_2<\cdots<i_m\leq n-1. \tag{6}$$

A convenient way to reformulate the model in a gauge-invariant way involves using orthogonal projectors [10]. Indeed, for $G(m,n)$, we define a rank m hermitian orthogonal projector $\mathbb{P}$ as

$$\mathbb{P}=ZZ^\dagger. \tag{7}$$

This projector satisfies

$$\mathbb{P}^2 = \mathbb{P}^\dagger = \mathbb{P}, \quad \mathrm{Tr}(\mathbb{P}) = m. \tag{8}$$

The Lagrangian density (2) and the Euler-Lagrange equations (3) can be rewritten in an equivalent way as

$$\mathcal{L}(\mathbb{P}) = \frac{1}{2}\mathrm{Tr}(\partial_+\mathbb{P}\partial_-\mathbb{P}), \quad [\partial_+\partial_-\mathbb{P}, \mathbb{P}] = 0, \quad \mathbb{P}^2 = \mathbb{P}. \tag{9}$$

A solution of the type (6) leads to a projector of the form:

$$\mathbb{P}_\beta = \sum_{j=0}^{n-1} \beta_j \, \mathbb{P}_j, \quad \mathbb{P}_j = Z_j Z_j^\dagger = \frac{P_+^j f (P_+^j f)^\dagger}{|P_+^j f|^2}, \tag{10}$$

where β is a $n-$column vector such that $\beta_j = 0$ or 1 for all j and $\sum_{j=0}^{n-1} \beta_j = m$.

The key, in constructing surfaces from the solutions of the $G(m, n)$ model, is to observe that the Euler-Lagrange equations (9) may be rewritten as a conservation law:

$$\partial_+\mathbf{L} - \partial_-\mathbf{L}^\dagger = 0, \quad \mathbf{L} = [\partial_-\mathbb{P}, \mathbb{P}]. \tag{11}$$

Then, using the Poincaré lemma and the fact that Ω is simply connected [12], we may define a surface $\mathbf{X} \in su(n)$ (the set of $n \times n$ hermitian and traceless matrices) via its tangent space as

$$d\mathbf{X} = \mathbf{L}^\dagger dx_+ + \mathbf{L} dx_- \tag{12}$$

or explicitly

$$\partial_+\mathbf{X} = [\partial_+\mathbb{P}, \mathbb{P}], \quad \partial_-\mathbf{X} = -[\partial_-\mathbb{P}, \mathbb{P}]. \tag{13}$$

2.2. Lagrangian and topological densities

Let us recall that the topological density is defined by [10]

$$\mathcal{Q}(\mathbb{P}) = \frac{1}{2}\mathrm{Tr}\left[(D_+Z)^\dagger D_+Z - (D_-Z)^\dagger D_-Z\right] = \frac{1}{2}\mathrm{Tr}\left[\mathbb{P}[\partial_-\mathbb{P}, \partial_+\mathbb{P}]\right], \tag{14}$$

showing, in particular, that for a holomorphic (or anti-holomorphic) solution, which satisfies $D_-Z = 0$ (or $D_+Z = 0$), it coincides with the Lagrangian density (up to a sign).

Hence, for the solutions of type (10), we get the following explicit expressions for the Lagrangian density (9) and the topological density (14):

$$\mathcal{L}(\mathbb{P}_\beta) = \frac{1}{2}\sum_{j=1}^{n-1}(\beta_{j-1} - \beta_j)^2 \frac{|P_+^j f|^2}{|P_+^{j-1} f|^2}, \quad \mathcal{Q}(\mathbb{P}_\beta) = \frac{1}{2}\sum_{j-1}^{n-1}(\beta_{j-1} - \beta_j) \frac{|P_+^j f|^2}{|P_+^{j-1} f|^2}. \tag{15}$$

Let us exhibit some properties of the topological charge. First we rewrite the topological density as

$$\mathcal{Q}(\mathbb{P}_\beta) = \mathcal{Q}\left(\sum_{j=0}^{n-1} \beta_j \mathbb{P}_j\right) = \frac{1}{2}\sum_{j=0}^{n-1} \beta_j \left(\frac{|P_+^{j+1} f|^2}{|P_+^j f|^2} - \frac{|P_+^j f|^2}{|P_+^{j-1} f|^2}\right) = \sum_{j=0}^{n-1} \beta_j \mathcal{Q}(\mathbb{P}_j), \tag{16}$$

showing that $\mathcal{Q}$ is a purely additive quantity. Furthermore, using the topological property

$$\mathcal{Q}(\mathbb{P}_j) = \frac{|P_+^{j+1} f|^2}{|P_+^j f|^2} - \frac{|P_+^j f|^2}{|P_+^{j-1} f|^2} = \partial_+\partial_- \ln(|P_+^j f|^2), \tag{17}$$

the topological density takes the compact form

$$\mathcal{Q}(\mathbb{P}_\beta) = \partial_+\partial_- \ln \prod_{j=0}^{n-1} |P_+^j f|^{\beta_j}. \tag{18}$$

Journal of Physics: Conference Series **597** (2015) 012029 doi:10.1088/1742-6596/597/1/012029

2.3. Mean and Gaussian curvatures

In order to extract some geometric properties of the surface $\mathbf{X}$ defined as in (12), we introduce a scalar product on the Lie algebra $su(n)$:

$$\langle A, B\rangle = \frac{1}{2}\mathrm{Tr}(AB), \quad A, B \in su(n). \tag{19}$$

The first fundamental form [12] of the surface $\mathbf{X}$ is given by:

$$\mathbf{I} = \langle d\mathbf{X}, d\mathbf{X}\rangle = g_{++}dx_+^2 + 2g_{+-}dx_+dx_- + g_{--}dx_-^2, \tag{20}$$

where $g_{\mu\nu}$ are the components of the metric tensor and are given by

$$g_{\pm\pm} = \langle \partial_\pm \mathbf{X}, \partial_\pm \mathbf{X}\rangle = -\langle \partial_\pm \mathbb{P}, \partial_\pm \mathbb{P}\rangle, \quad g_{\pm\mp} = \langle \partial_\pm \mathbf{X}, \partial_\mp \mathbf{X}\rangle = \langle \partial_\pm \mathbb{P}, \partial_\mp \mathbb{P}\rangle. \tag{21}$$

Since we are interested in solutions $\mathbb{P}$ given in (10), we can show that our surfaces are conformal maps and that the metric components take the form:

$$g_{\pm\pm} = 0, \quad g_{+-} = g_{-+} = \frac{1}{2}\mathrm{Tr}(\partial_+\mathbb{P}\partial_-\mathbb{P}). \tag{22}$$

Note that the expression for g_{+-} is identical to the expression for the Lagrangian density (9). Using the Brioschi formula [12], we see that the Gaussian curvature $\mathcal{K}$ of the surface $\mathbf{X}$ associated to the solution $\mathbb{P}$:

$$\mathcal{K} = -\frac{1}{g_{+-}}\partial_+\partial_- \ln g_{+-}. \tag{23}$$

Let us now calculate the expression for the mean curvature $\mathcal{H}$ [12] associated to solutions of the model. As we have shown the considered surfaces are conformal maps and we know that the expression for the mean curvature is given as

$$\mathcal{H} = \frac{1}{2}\mathrm{Tr}(\mathbf{II}(\mathbf{I}^{-1})), \tag{24}$$

where $\mathbf{II}$ is the second fundamental form defined as

$$\mathbf{II} = \langle \partial_+^2 \mathbf{X}, N\rangle dx_+^2 + 2\langle \partial_+\partial_- \mathbf{X}, N\rangle dx_+dx_- + \langle \partial_-^2 \mathbf{X}, N\rangle dx_-^2 = -\langle d\mathbf{X}, dN\rangle. \tag{25}$$

In the above expression, N is a normal unit vector to the surface $\mathbf{X}$ and thus satisfies $\langle d\mathbf{X}, N\rangle = 0$. Using the conformal property of the surfaces the mean curvature is given by

$$\mathcal{H} = \frac{\langle \partial_+\partial_- \mathbf{X}, N\rangle}{g_{+-}}. \tag{26}$$

Due to the expression (13) and from the Euler-Lagrange equations (9), we easily get :

$$\partial_+\partial_- \mathbf{X} = [\partial_+\mathbb{P}, \partial_-\mathbb{P}] = [\partial_+\mathbf{X}, \partial_-\mathbf{X}]. \tag{27}$$

We may thus define a unit normal vector N to the surface $\mathbf{X}$ as

$$N = \frac{[\partial_+\mathbf{X}, \partial_-\mathbf{X}]}{\|[\partial_+\mathbf{X}, \partial_-\mathbf{X}]\|} = \frac{\partial_+\partial_-\mathbf{X}}{\|\partial_+\partial_-\mathbf{X}\|} \in su(n) \tag{28}$$

remembering that $\langle d\mathbf{X}, N\rangle = 0$. Hence, we see that the mean curvature $\mathcal{H}$ is given by

$$\mathcal{H} = \frac{\|\partial_+\partial_-\mathbf{X}\|}{g_{+-}} = 2\frac{\|[\partial_+\mathbb{P}, \partial_-\mathbb{P}]\|}{\mathrm{Tr}(\partial_+\mathbb{P}\partial_-\mathbb{P})}. \tag{29}$$

3. Solutions of $G(m,n)$, their associated surfaces and geometric characteristics

3.1. Special case of $\mathbb{C}P^{n-1}$ and the Veronese sequence

In the late 80's, Bolton and *al.*[11] fully classified constant Gaussian curvature surfaces $\mathbf{X}$ associated to the solutions (4) of the $\mathbb{C}P^{n-1}$ model. Indeed, the set of all solutions is obtained from the Veronese holomorphic curve f defined as

$$f(x_+) = \left(1, \sqrt{\binom{n-1}{1}}x_+, \cdots, \sqrt{\binom{n-1}{r}}x_+^r, \cdots, x_+^{n-1}\right)^T. \tag{30}$$

This holomorphic curve satisfies the following identity, which will prove to be useful in the rest of the paper:

$$\frac{|P_+^i f|^2}{|P_+^{i-1} f|^2} = \frac{\alpha_{i,n}}{(1+|x|^2)^2} \tag{31}$$

where

$$\alpha_{i,n} = i(n-i), \quad i = 1, 2, \cdots, n-1. \tag{32}$$

In this case, the Gaussian curvature $\mathcal{K}$ takes the form

$$\mathcal{K}(Z_i) = \frac{4}{r_i(1,n)}, \quad r_i(1,n) = r_i = n-1+2i(n-1-i). \tag{33}$$

We see that the quantity r_i admits an obvious symmetry given by

$$r_i = r_{n-1-i}, \tag{34}$$

which shows that some surfaces associated to non-equivalent solutions Z_i and Z_{n-1-i} have the same value of constant Gaussian curvature. Note that when n is odd, we omit $i = \frac{n-1}{2}$ since $Z_i = Z_{n-1-i}$.

Thus we need other geometric quantities to differentiate them. In this case, the topological density is sufficient since we have

$$\mathcal{Q}(Z_i) = \frac{q_i(1,n)}{2(1+|x|^2)^2}, \quad q_i(1,n) = q_i = n-1-2i, \tag{35}$$

where q_i satisfy the relation:

$$q_{n-1-i} = -q_i. \tag{36}$$

To go further, we express the quantity $\alpha_{i,n}$ in terms of the r's and q's given, respectively, in (33) and (35). We get

$$\alpha_{m+j,n} = \frac{1}{2}\left[r_m + (2j-1)q_m\right] - j(j-1), \tag{37}$$

which will help us to express the geometric expressions in terms of the r's and the q's. For example, we have

$$\alpha_{m,n} = \frac{1}{2}\left[r_m - q_m\right], \quad \alpha_{m+1,n} = \frac{1}{2}\left[r_m + q_m\right]. \tag{38}$$

The mean curvature $\mathcal{H}_i$, associated to the solution Z_i, is constant and is given by

$$\mathcal{H}_i = 2\frac{\sqrt{\alpha_{i,n}^2 - \alpha_{i,n}\alpha_{i+1,n} + \alpha_{i+1,n}^2}}{\alpha_{i,n} + \alpha_{i+1,n}} = \frac{\sqrt{r_i^2 + 3q_i^2}}{r_i}. \tag{39}$$

This geometric quantity is not necessary to differentiate surfaces of $\mathbb{C}P^{n-1}$ and of equal Gaussian curvature, but it will become relevant for higher dimensional Grassmannians.

Journal of Physics: Conference Series **597** (2015) 012029 doi:10.1088/1742-6596/597/1/012029

3.2. Geometric characteristics of the Veronese curves of $G(m,n)$

For the general Grassmannian $G(m,n)$, we take the solution $\mathbb{P}_\beta$ given in (10) with f chosen as the Veronese holomorphic curve (30). The Lagrangian and topological densities are then given by

$$\mathcal{L}(\mathbb{P}_\beta) = \frac{r_\beta(m,n)}{2(1+|x|^2)^2}, \quad \mathcal{Q}(\mathbb{P}_\beta) = \frac{q_\beta(m,n)}{2(1+|x|^2)^2}, \tag{40}$$

where

$$r_\beta(m,n) = \sum_{j=1}^{n-1}(\beta_{j-1}-\beta_j)^2\alpha_{j,n}, \quad q_\beta(m,n) = \sum_{j=1}^{n-1}(\beta_{j-1}-\beta_j)\alpha_{j,n}. \tag{41}$$

Due to the fact that β_j are 0 or 1, we easily deduce the following expressions:

$$r_\beta(m,n) - q_\beta(m,n) = 2\sum_{j=1}^{n-1}\beta_j\alpha_{j,n} - 2\sum_{j=1}^{n-1}\beta_j\beta_{j-1}\alpha_{j,n}, \tag{42}$$

$$r_\beta(m,n) + q_\beta(m,n) = 2\sum_{j=1}^{n-1}\beta_{j-1}\alpha_{j,n} - 2\sum_{j=1}^{n-1}\beta_j\beta_{j-1}\alpha_{j,n}. \tag{43}$$

These expressions are slightly different from the ones obtained in the $\mathbb{C}P^{n-1}$ case. Indeed, they exhibit interactions between consecutive projectors $\mathbb{P}_{j-1}$ and $\mathbb{P}_j$ in the general expression of $\mathbb{P}_\beta$, which is absent in the $\mathbb{C}P^{n-1}$ case.

Let us recall that the Gaussian curvature of the surface associated to the solution $\mathbb{P}_\beta$ is given by

$$\mathcal{K} = \frac{4}{r_\beta(m,n)}. \tag{44}$$

Moreover, we can also calculate the numerator of the mean curvature expression given in (29) and we get

$$\begin{aligned} \|[\partial_+\mathbb{P}_\beta, \partial_-\mathbb{P}_\beta]\|^2 \ \propto\ & \sum_{j=1}^{n-1}(\beta_{j-1}-\beta_j)^2\alpha_{j,n}((\beta_{j-1}-\beta_j)^2\alpha_{j,n} - \frac{1}{2}(\beta_j-\beta_{j+1})^2\alpha_{j+1,n} \\ & - \frac{1}{2}(\beta_{j-2}-\beta_{j-1})^2\alpha_{j-1,n}). \end{aligned} \tag{45}$$

For example, the holomorphic solutions, in the $G(m,n)$ case, which are described by $\beta_i = 1$ for $i = 0,1,\cdots,m-1$ and $\beta_i = 0$ for $i = m, m+1, \cdots, n-1$ lead to

$$r_\beta^{(hol)}(m,n) = q_\beta^{(hol)}(m,n) = \alpha_{m,n} = m(n-m), \quad \mathcal{H}_\beta^{(hol)}(m,n) = 2. \tag{46}$$

Let us illustrate, in the following subsection, the case of $G(2,n)$ and the need to use further geometric characteristics to distinguish surfaces with the same Gaussian curvature.

3.2.1. The case of $G(2,n)$: some examples In this case, we have already computed the Gaussian curvature for some surfaces associated to non-holomorphic solutions of $G(2,n)$. For example, we have found [2] that

$$r_{2,3}(2,7) = r_{0,5}(2,7) = 22. \tag{47}$$

If we now compute the topological charge we find that

$$q_{2,3}(2,7) = q_{0,5}(2,7) = 2. \tag{48}$$

This means that we need other quantities to differentiate the geometry of these two surfaces.

The explicit forms of the quantities $r_\beta(2,n)$ and $q_\beta(2,n)$ are obtained directly from (42) and (43). Indeed, we have to distinguish two cases: when $\beta_j = \beta_{j+1} = 1$ (**interaction**), we get

$$q_{j,j+1}(2,n) = 2(n-2-2j), \tag{49}$$
$$r_{j,j+1}(2,n) = 2(n-2+j(n-2-j)) = q_{j,j+1}(2,n) + 2\alpha_j; \tag{50}$$

and when $\beta_j = \beta_k = 1$ for $k > j+1$ (**absence of interaction**), we get

$$q_{j,k}(2,n) = 2(n-1-j-k), \tag{51}$$
$$r_{j,k}(2,n) = 2(n-1+j(n-1-j)+k(n-1-k)) = q_{j,k}(2,n) + 2\alpha_j + 2\alpha_k. \tag{52}$$

Finally, we have (note that, for simplicity, we have set $\alpha_i = \alpha_{i,n}$)

$$\mathcal{H}_{i,i+1} = 2\frac{\sqrt{\alpha_i^2 + \alpha_{i+2}^2}}{\alpha_i + \alpha_{i+2}}, \tag{53}$$
$$\mathcal{H}_{i,i+2} = 2\frac{\sqrt{\alpha_i^2 - \alpha_i\alpha_{i+1} + \alpha_{i+1}^2 - \alpha_{i+1}\alpha_{i+2} + \alpha_{i+2}^2 - \alpha_{i+2}\alpha_{i+3} + \alpha_{i+3}^2}}{\alpha_i + \alpha_{i+1} + \alpha_{i+2} + \alpha_{i+3}}, \tag{54}$$
$$\mathcal{H}_{i,j>i+2} = 2\frac{\sqrt{\alpha_i^2 - \alpha_i\alpha_{i+1} + \alpha_{i+1}^2 + \alpha_j^2 - \alpha_j\alpha_{j+1} + \alpha_{j+1}^2}}{\alpha_i + \alpha_{i+1} + \alpha_j + \alpha_{j+1}}. \tag{55}$$

The mean curvature is constant in each cases and it can be used to differentiate the geometry of the surfaces associated to $\mathbb{P}_{2,3}$ and $\mathbb{P}_{0,5}$ in $G(2,7)$. Indeed, we have

$$\mathcal{H}_{2,3} = \frac{2\sqrt{61}}{11}, \quad \mathcal{H}_{0,5} = \frac{4\sqrt{7}}{11}. \tag{56}$$

In Table 1, we give the examples of $G(2,n)$ with $n = 4,5,6$.

Table 1. The $G(2,4)$, $G(2,5)$ and $G(2,6)$ models

(i,j)	$r_{i,j}$	$q_{i,j}$	$\mathcal{H}_{i,j}$
$(0,1)$	4	4	2
$(1,2)$	6	0	$\sqrt{2}$
$(0,2)$	10	2	$\sqrt{\frac{2}{5}}$

(i,j)	$r_{i,j}$	$q_{i,j}$	$\mathcal{H}_{i,j}$
$(0,1)$	6	6	2
$(1,2)$	10	2	$\frac{2\sqrt{13}}{5}$
$(0,2)$	16	4	$\frac{\sqrt{7}}{4}$
$(0,3)$	14	2	$\frac{2\sqrt{11}}{7}$
$(0,4)$	8	0	$\sqrt{2}$
$(1,3)$	20	0	$\frac{1}{\sqrt{5}}$

(i,j)	$r_{i,j}$	$q_{i,j}$	$\mathcal{H}_{i,j}$
$(0,1)$	8	8	2
$(1,2)$	14	4	$\frac{\sqrt{106}}{7}$
$(2,3)$	16	0	$\sqrt{2}$
$(0,2)$	22	6	$\frac{\sqrt{58}}{11}$
$(0,3)$	22	4	$\frac{7\sqrt{2}}{11}$
$(0,4)$	18	2	$\frac{\sqrt{74}}{9}$
$(0,5)$	10	0	$\sqrt{2}$
$(1,3)$	30	2	$\frac{\sqrt{2}}{3}$
$(1,4)$	26	0	$\frac{7\sqrt{2}}{13}$

3.2.2. The case of $G(2,n)$: more general results In this section, we give a partial answer to the following question: If $q_{i,j} = q_{k,l}$ and $r_{i,j} = r_{k,l}$, is the mean curvature sufficient to differentiate the surfaces associated to $\mathbb{P}_{i,j}$ and $\mathbb{P}_{k,l}$?

To answer this question, we have to consider three cases.

The first case arises when $j = i+1$ and $l = k+1$. In this case, $q_{i,i+1} = q_{k,k+1}$ and so $k = i$ and we are, thus, dealing with the same solution. We may exclude this case.

The second one is corresponds to $j > i+1$ and $l > k+1$. In this case, the condition $q_{i,j} = q_{k,l}$ leads to $i+j = k+l$ which is equivalent to $l = i+j-k$. Then one wants $r_{ij} = r_{kl}$ which implies that

$$r_{i,j} - r_{k,l} = r_{i,j} - r_{k,i+j-k} = 4(i-k)(j-k) = 0 \quad \Longleftrightarrow \quad i = k \quad \text{or} \quad j = k. \tag{57}$$

But if $i = k$, then $j = l$ and so we are dealing with the same solution. Moreover, when $j = k$, we have $i = l$, but this contradicts the original assumption that $j > i+1$. So we may ignore this case too.

The third case is the one we have encountered for different grids: $j = i+1$ and $l > k+1$. The constraint $q_{i,i+1} = q_{k,l}$ leads to $l_{k,i} = 2i-k+1$ and together with $l > k+1$ implies $i > k$. Solving the constraint $r_{i,i+1} = r_{k,l}$, we get an expression for the dimension n which is given by

$$n_{k,i} = 3i + 1 - 4k + \frac{2k(1+k)}{1+i}, \quad i > k. \tag{58}$$

So let us now look for a couple (k,i) such that n_{ki} is an integer. Once we have found such a couple, we must check that it satisfies the constraints $l_{k,i} = 2i-k+1 < n$ and $i < n-1$.

We have

- $n_{0,i} = 3i+1 \in \mathbb{N}, \quad i > 0$: This shows that the projectors $\mathbb{P}_{i,i+1}$ and $\mathbb{P}_{0,2i+1}$ have the same Gaussian curvature and topological charge. But can their corresponding mean curvatures be different? If they are the same, we have

$$\left(\frac{\mathcal{H}_{0,2i+1}}{\mathcal{H}_{i,i+1}}\right)^2_{n=3i+1} = \frac{2+i+3i^2+i^3+2i^4}{2-6i+i^2+8i^3+4i^4} = 1 \quad \Longleftrightarrow \quad i = 1. \tag{59}$$

  However, the case $i = 1$ can be easily understood since, in this case, $n_{0,1} = 4$ and we are thus comparing the projectors $\mathbb{P}_{1,2}$ and $\mathbb{P}_{0,3}$ which correspond to the same solution of the $G(2,4)$ model. So we see that, in this case, the mean curvatures are different.
- $n_{1,i} = -3 + 3i + \frac{4}{1+i} \in \mathbb{N}, \quad i > 1$: This shows that the only admissible i is $i = 3$. In this case, we have $n_{1,3} = 7$ and we are comparing the projectors $\mathbb{P}_{3,4}$ and $\mathbb{P}_{1,6}$. Using the completeness relation, this is the same as comparing the projectors $\mathbb{P}_{2,3}$ and $\mathbb{P}_{0,5}$ which is consistent with the discussion of subsection 3.2.1. As we already know, the mean curvatures are different.
- $n_{2,i} = -7 + 3i + \frac{12}{1+i} \in \mathbb{N}, \quad i > 2$: Putting all the constraints together, this shows that the only admissible i are $i = 11$ and $i = 5$. In the first case, we get $n_{2,11} = 27$ and we are comparing the projectors $\mathbb{P}_{11,12}$ and $\mathbb{P}_{2,21}$. The second possibility leads to $n_{2,5} = 10$ and we are, then, comparing the projectors $\mathbb{P}_{5,6}$ and $\mathbb{P}_{2,9}$. As before, we can show that the mean curvatures are different.

Let us make some further comments. Indeed, if we set $l = n-1$, then we have that $n = 2i+2-k$ which can then be compared with the formula for $n_{k,i}$. Doing so, we obtain $i = k-1$ or $i = 1+2k$. The case $i = k-1$ must be rejected due to the constraint that $i > k$. This means that we are left with the unique choice $i = 1+2k$. Thus we have $n_{k,1+2k} = 4+3k$

and we are comparing surfaces associated to the projectors $\mathbb{P}_{1+2k,2+2k}$ and $\mathbb{P}_{k,3+3k}$. This general result is consistent with the above examples. Furthermore, we can show that

$$\left(\frac{\mathcal{H}^{(4+3k)}_{k,3+3k}}{\mathcal{H}^{(4+3k)}_{1+2k,2+2k}}\right)^2 = \frac{9+18k+18k^2+9k^3+2k^4}{9+36k+49k^2+24k^3+4k^4} = 1 \quad \Longleftrightarrow \quad k = -\frac{9}{2}, -2, -1, 0. \tag{60}$$

Since $k > 0$, the mean curvatures are different. Note that the case $k = 0$ is to be rejected as previously discussed.

In Table 2, we give some possible values of i. This table contains all the examples mentioned

Table 2. Some possible values of i in the expression for $n_{k,i}$

i	$n_{k,i}$	$l_{k,i}$	Comments
$2k(1+k)-1$	$6k^2+2k-1$	$4k^2+3k-1$	$k>0$
$k(1+k)-1$	$k(3k-1)$	$2k^2+k-1$	$k>1$
$2k+1$	$3k+4$	$3(k+1)$	$k \geq 0$
$2k-1$	$3k-1$	$3k-1$	N/A

above. The above discussion works for $k = 3, 4$, but for $k = 5$ new cases arise. Indeed, there are two of them: $n_{5,19} = 41$ with $l_{5,19} = 34$ and $n_{5,14} = 27$ with $l_{5,14} = 24$. The later one can be explained using the fact that $2k(1+k)$ in the expression of $n_{k,i}$ is always divisible by 4. We have summarized all this information in Table 3. We conjecture that, that as k increases, the

Table 3. Values of i for which $2k(1+k)$ is divisible by 4

i	$n_{k,i}$	$l_{k,i}$	Comments
$m(2m+3)$	$6m^2+m+1$	$4m(m+1)$	$k=2m+1 \quad m \geq 2$
$m(2m+1)-1$	$6m^2-5m+2$	$4m^2-1$	$k=2m \quad m \geq 2$

number of cases increases too and it is totally dependent on the prime decomposition of the factor $2k(1+k)$ in the expression for $n_{k,i}$.

3.3. Non-Veronese holomorphic solutions

We have conjectured [1] that we can construct a holomorphic solution in $G(m,n)$ of constant Gaussian curvature $\mathcal{K} = \frac{4}{r}$ for all integer values of $r = 1, 2, \cdots, \alpha_{m,n}$. The maximal value $r = \alpha_{m,n} = m(n-m)$ is obtained from the Veronese holomorphic curve (30) and its $m-1$ consecutive derivatives. The values of $r = 1, 2, \cdots, m(n-1-m)$ are obtained from the natural immersion of $G(m, n-1)$ into $G(m,n)$. The other values are not obtained from such immersions nor from the Veronese curve. Furthermore, for the same value of the missing r, we may find non-equivalent solutions $Z_i = \hat{Z}_i\hat{L}_i$, $i = 1, 2$. As an example, in the $G(2,5)$ case, we have obtained [1] two non-equivalent holomorphic solutions corresponding to $r = 5$ parametrized by

$$\hat{Z}_1^T = \begin{pmatrix} 1 & 0 & \sqrt{5}x_+ & \sqrt{5}x_+^2 & 0 \\ 0 & 1 & \sqrt{5}x_+^2 & \frac{7}{\sqrt{5}}x_+^3 & \frac{1}{\sqrt{5}}x_+^3 \end{pmatrix}, \quad \hat{Z}_2^T = \begin{pmatrix} 1 & 0 & x_+ & \frac{1}{\sqrt{5}}x_+^2 & 0 \\ 0 & 1 & 2x_+ & \frac{7}{\sqrt{5}}x_+^2 & \sqrt{5}x_+^3 \end{pmatrix}. \tag{61}$$

Of course, we know that, in the holomorphic case, the Lagrangian density corresponds to the topological one. Since the Lagrangian density of these two solutions is the same this is also the case for the topological density. Hence, the mean curvature will distinguish them. Indeed, using $Z = \hat{Z}\hat{L}$ that satisfies $\hat{Z}^\dagger \hat{Z} = (\hat{L}\hat{L}^\dagger)^{-1}$, we get $\mathbb{P} = \hat{Z}(\hat{Z}^\dagger \hat{Z})^{-1}\hat{Z}^\dagger$. In this case, the mean curvature is not a constant, *i.e.* $\mathcal{H}_i = \mathcal{H}_i(|x|^2)$. Moreover, we can show that $\mathcal{H}_1 \neq \mathcal{H}_2$. Indeed from (29), we get

$$\left(\frac{\mathcal{H}_1}{\mathcal{H}_2}\right)^2 = \frac{\mathcal{P}_1(|x|^2)}{\mathcal{P}_2(|x|^2)}, \quad \mathcal{P}_2(y) = y^6 \mathcal{P}_1\left(\frac{1}{y}\right) = 25 + 110y + 285y^2 + 428y^3 + 355y^4 + 150y^5 + 25y^6.$$

4. Conclusions and Outlook

In this paper, we have studied various geometric properties of surfaces constructed from the solutions of the two-dimensional $G(m,n)$ Sigma model. The aim of our work was to see whether we can differentiate surfaces of equal constant Gaussian curvature by involving also the topological density and the mean curvature. This problem originated from the complete classification of constant Gaussian curvature surfaces [11] associated to Veronese holomorphic curves for $\mathbb{C}P^{n-1}$. In this classification, some non-equivalent solutions had, due to a symmetry property, identical Gaussian curvatures. In these cases, the topological charges had different values.

We have not fully solved the problem which has turned out to be more complicated than originally envisaged. So here we report where we are at present. We have generalized our previous results to more general Grassmannians. We have obtained explicit expressions for the Gaussian curvature, the topological density and the mean curvature for these solutions. We have shown, in the case of $G(2,n)$, that some non-equivalent solutions may have identical Gaussian curvature and also identical topological densities. This has led us to consider the second fundamental form of these surfaces and we have computed their mean curvature. Some partial results for the $G(2,n)$ case show that the mean curvature is sufficient to distinguish surfaces with identical Gaussian curvatures and topological charges. Furthermore, we have shown that the projectors having this property are all of the form $\mathbb{P}_{i,i+1}$ and $\mathbb{P}_{k,l}$ with $l > k+1$.

The case of holomorphic solutions which are not of the Veronese type and that cannot be obtained from immersions of lower dimensional Grassmannians is really challenging since we have no general formula for these cases. Our results are complete for the $G(2,5)$ model where we have shown that all solutions of equal Gaussian curvature and topological density have distinct mean curvature.

Acknowledgements

This work has been supported in part by research grants from Natural sciences and engineering research council of Canada (NSERC). Laurent Delisle acknowledges a NSERC Postdoctoral fellowship.

References

[1] Delisle L, Hussin V and Zakrzewski W J 2013 *Journal of Geometry and Physics* **66**, 24
[2] Delisle L, Hussin V and Zakrzewski W J 2013 *Journal of Geometry and Physics* **71** 1
[3] Li Z-Q and Yu Z-H 1999 *Manuscripta Math.* **100** 305
[4] Jiao X and Peng J 2003 *Transaction of the American Mathematical Society* **355** 3715
[5] Jiao X and Peng J 2004 *Differential geometry and its applications* **20** 267
[6] Jiao X and Peng J 2010 *Front. Math. China* **5** 297
[7] Fei J, Jiao X and Xu X 2011 *Proc. Indian Acad. Sci.* **121** 181
[8] Peng C K and Xu X W 2014 *Israel Journal of Mathematics* **202** 1
[9] Peng C K and Xu X W 2014 *Journal de Mathématiques Pures et Appliquées*, to be published
[10] Zakrzewski W J 1989 *Low Dimensional Sigma Models* (Bristol: Adam Hilger)
[11] Bolton J, Jensen G R, Rigoli M and Woodward L M 1988 *Math. Ann.* **279** 599
[12] Hélein F 2001 *Constant mean curvature surfaces, harmonic maps and integrable systems* (ETH Zürich: Birkhäuser)