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Warped product of Hamiltonians and extensions of Hamiltonian systems

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Abstract. The aim of this article is to set the theory of extensions of Hamiltonian systems, developed in a series of previous papers, in the framework of warped products of Hamiltonian systems. Some illustrative examples and plots are given.

1. Introduction

In Sec. 2 the definition of warped product of Riemannian manifolds is recalled. This geometric concept suggests the dynamic idea of warped product of Hamiltonian systems. Some properties are introduced as well as two examples of application: the geodesic Hamiltonian on the torus $\mathbb{T}^2$, the Jacobi-Calogero and Wolfes systems. In Sec. 3 we motivate and introduce the extension procedure (developed in a series of papers [4]-[9]) which transforms natural Hamiltonians with N degrees of freedom into $(N+1)$-dimensional ones possessing a non-trivial first integral of higher degree in the momenta. The interesting fact is that the new integral can be computed in a simple recursive way. The extended Hamiltonians perfectly fit in the framework of warped products. Moreover, the geometric necessary conditions for the extension procedure are deeply related with the geometry of warped products of Riemannian manifolds. The fundamental examples of Hamiltonian systems which illustrate the extension procedure are the well-known Tremblay-Turbiner-Winternitz (TTW) system [11] , [19], (that is a generalization of the Jacobi -Calogero and Wolfes systems) and the anisotropic caged oscillator [11], [20]. The dynamics of the TTW system and of the geodesic flow on $\mathbb{T}^2$ as warped product of Hamiltonians is sketched in three figures. The superintegrability of the TTW Hamiltonian is then proved in a new way thanks to its identification as an extended Hamiltonian. Moreover, it is straightforward to see that the extension of an N-dimensional superintegrable systems is an $N+1$-dimensional superintegrable system. Then, the extension procedure allows the construction of new superintegrable systems from known ones [7]. In Sec. 4 the extension procedure is summarized in a simple algorithm.

2. Warped products

Given n Riemannian manifolds (Q_k, ds_k^2), $(k = 1, \dots, n)$, their warped product is the Riemannian manifold $Q = \times_{k=1}^n Q_k$, with metric $ds^2 = \alpha^k ds_k^2$, where $\alpha^k : Q \to \mathbb{R}$ or $\mathbb{C}$. A simple example of a non-trivial warped product of Riemannian manifolds is the Euclidean plane with metric $ds^2 = dr^2 + r^2 d\theta^2$ in polar coordinates (r, θ) which can be interpreted as warped product of $(\mathbb{R}^1)^+ \times \mathbb{S}^1$ with $\alpha^1 = 1$, $\alpha^2 = r^2$.

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Given a Riemannian manifold, the possibility of writing it as warped product of lower dimensional manifolds is characterized by several theorems involving its geodesic completeness, curvature, etc. (see for example [18]).

The concept of warped product can be straightforwardly generalized to Hamiltonian systems. Let H_k be n non-degenerate Hamiltonians on T^*Q_k with canonical coordinates (q^{i_k}, p_{j_k}). Let α^k be n functions defined on $Q = \times_{k=1}^n Q_k \to \mathbb{R}$ (we do not consider here the more general case of α_k defined on the whole T^*Q). Let

$$H = \alpha^k H_k \tag{1}$$

be a Hamiltonian in $\times_{k=1}^n T^*Q_k = T^*Q$ with the obvious canonical coordinates.

The Hamiltonian vector field (on T^*Q_k) generated by H_k is

$$X_k = \frac{\partial H_k}{\partial p_{i_h}} \frac{\partial}{\partial q^{i_h}} - \frac{\partial H_k}{\partial q^{i_h}} \frac{\partial}{\partial p_{i_h}},$$

hence, the Hamiltonian vector field (on Q) generated by H is given by

$$\begin{aligned} X_H &= \alpha^k \frac{\partial H_k}{\partial p_{i_h}} \frac{\partial}{\partial q^{i_h}} - \alpha^k \frac{\partial H_k}{\partial q^{i_h}} \frac{\partial}{\partial p_{i_h}} - H_k \frac{\partial \alpha^k}{\partial q^{i_h}} \frac{\partial}{\partial p_{i_h}} \\ &= \alpha^k X_k - H_k \frac{\partial \alpha^k}{\partial q^{i_h}} \frac{\partial}{\partial p_{i_h}} \end{aligned}$$

Therefore, the functions

$$\alpha^k = \frac{dt_k}{d\tau}, \tag{2}$$

determine a position-dependent time-scaling and we can write

$$\frac{dq^{i_k}}{d\tau} = \frac{d\hat{q}^{i_k}}{d\tau}, \tag{3}$$

meaning that the Hamilton equations for the configuration coordinates on each Q_k coincide with the Hamilton equations for the configuration coordinates on Q rescaled for the Hamiltonian parameter τ.

An example of such a rescaling (for $k = 1$) is provided by the classical Jacobi transform, mapping a natural Hamiltonian $H_1 = G + V - E = \frac{1}{2} g^{ij} p_i p_j + V(q^h) - E$ ($E \in \mathbb{R}$) in the geodesic Hamiltonian $H = \frac{G}{E-V}$ whose metric is conformal to the original one. In this case $\alpha^1 = (E - V)^{-1}$ and the two Hamiltonian vector fields are parallel on the submanifold $H_1 = 0$.

Moreover, the Hamilton equations for the momenta on Q

$$\frac{dp_{i_k}}{d\tau} = \frac{d\hat{p}_{i_k}}{d\tau} - \frac{\partial \alpha^h}{\partial q^{i_k}} H_h. \tag{4}$$

mean that the momentum components of the Hamiltonian vector field of H on T^*Q coincide with the Hamiltonian fields of each H_k rescaled according to the parameter τ and modified by a term in each point equal to the linear combination of the differentials of the α^k with coefficients given by the energies in those points of the non-interacting systems H_k. We may say that in H the interaction among the H_k is determined in each point by the energies H_k, assumed free, oriented according to the differentials of the α^k. The α^k themselves determine the time-scaling among the Hamiltonian parameters of each H_k and that of H.

For any function f we have

$$\frac{df}{d\tau} = \{H, f\} = \{\alpha^k, f\}H_k + \alpha^k\{H_k, f\},$$

where $\{,\}$ are the canonical Poisson brackets on T^*Q. Hence, if $f = H_m$ then

$$\frac{dH_m}{d\tau} = -H_k\{H_m, \alpha^k\}, \tag{5}$$

where $\{H_m, \alpha^k\}$ represents the "time-variation" of the function α^k with respect to the Hamiltonian parameter of the system H_m assumed non-interacting with the other systems. We may therefore say that each system of Hamiltonian H_m is interacting with the whole system H with an exchange of energy determined by (5).

If the Hamiltonians $H_k = \frac{1}{2}g_k^{i_k j_k}p_{i_k}p_{j_k} + V(q^{i_k})$ are natural, then also their warped product H, as defined here, is a natural Hamiltonian, with the warped product of the (Q_k, g_k) as configuration manifold.

2.1. Example: the geodesic Hamiltonian on $\mathbb{T}^2$

The Torus $\mathbb{T}^2$ with the metric inherited from $\mathbb{E}^3$ may be represented by

$$\begin{aligned} x &= (R + r\cos u)\cos v, \\ y &= (R + r\cos u)\sin v, \\ z &= r\sin u, \end{aligned}$$

$(r, R \in \mathbb{R})$ with metric tensor

$$g_{uu} = r^2, \quad g_{uv} = 0, \quad g_{vv} = (R + r\cos u)^2.$$

The torus is then $\mathbb{S}^1 \times \mathbb{S}^1$ with metric tensor given by the warped product $ds^2 = r^2du^2 + (R + r\cos u)^2dv^2$. The geodesic flow has Hamiltonian

$$H = \frac{1}{2r^2}p_u^2 + \frac{1}{2(R + r\cos u)^2}p_v^2,$$

that is the warped product of $H_1 = \frac{1}{2}p_u^2$, $H_2 = \frac{1}{2}p_v^2$ with $\alpha^1 = \frac{1}{r^2}$, $\alpha^2 = \frac{1}{(R+r\cos u)^2}$. To both H_1 and H_2 can be added scalar potentials depending on u and v respectively. In Figure 1 a segment of geodesic is shown, together with its projection on the spaces $u = p_u = 0$, $v = p_v = 0$, respectively. The graphs of the percentage time variations of H_1 and H_2 are also drawn. The function H_2 is a constant of the motion for H.

2.2. Example: the Jacobi-Calogero and Wolfes Hamiltonians

Let three points (x^1, x^2, x^3) with unitary masses on a line be interacting with scalar potential

$$V_{JC} = \frac{k}{(x^1 - x^2)^2} + \frac{k}{(x^2 - x^3)^2} + \frac{k}{(x^3 - x^1)^2},$$

or

$$V_W = \frac{k}{(x^1 + x^2 - 2x^3)^2} + \frac{k}{(x^2 + x^3 - 2x^1)^2} + \frac{k}{(x^3 + x^1 - 2x^2)^2}.$$

These functions are respectively known as Jacobi-Calogero's (V_{JC}) and Wolfes' (V_W) potentials, and they are more generally considered with additional harmonic terms or added together (see

[3] and references therein). The Jacobi-Calogero potential is a two-body interaction, while the Wolfes is a three-body interaction. Both potentials are integrable and superintegrable in both their classical and quantum formulations. The corresponding Hamiltonians share the same inertial term

$$\frac{1}{2}\left(p_1^2+p_2^2+p_3^2\right),$$

and the coordinates (x^i) can be considered as Cartesian coordinates in $\mathbb{R}^3$ so that the Hamiltonians of the systems are functions on $T^*\mathbb{R}^3$. When written in cylindrical coordinates (r,ϕ,z) with axis z passing trough the points $(0,0,0)$ and $(1,1,1)$, the Hamiltonian of the two systems become

$$H=\frac{1}{2}(p_r^2+p_z^2)+\frac{1}{r^2}\left(\frac{1}{2}p_\phi^2+V(\phi)\right),$$

where $V(\phi)=k_I\sin^{-2}(3\phi)$ for the Jacobi-Calogero and $V(\phi)=k_{II}\cos^{-2}(3\phi)$ for the Wolfes potential, and k_I, k_{II} are suitable constants. After the remarkable observation that the two potentials coincide under a $\frac{\pi}{6}$ rotation around the axis, we see that H, for every $V(\phi)$, is the warped product of the two Hamiltonians

$$H_1=\frac{1}{2}(p_r^2+p_z^2),\quad H_2=\frac{1}{2}p_\phi^2+V(\phi),$$

with $\alpha^1=1$, $\alpha^2=r^{-2}$. It is easy to check that H_2 is always a first integral of H.

3. Extensions of Hamiltonian systems

Other interesting examples fitting into this framework are two families of superintegrable Hamiltonian systems with two degrees of freedom, intensively studied in the last few years, whose potential depends on a rational parameter k :

- the TTW-system [12, 13, 19]

$$H=\frac{1}{2}p_r^2+\omega^2r^2+\frac{1}{r^2}\left(\frac{1}{2}p_\psi^2+\frac{a}{\sin^2 k\psi}+\frac{b}{\cos^2 k\psi}\right)$$

- the anisotropic caged oscillator, [20, 11]

$$H=\frac{1}{2}p_x^2+\frac{1}{2}p_y^2+\omega^2x^2+k\omega^2y^2+\frac{a}{x^2}+\frac{b}{y^2}.$$

The interest for these systems arises from the fact that they are maximally superintegrable for any rational value of the parameter k, in their classical as well as quantum realizations. Their quantum version has been intensively studied. Moreover, the reduction of the three-dimensional Jacobi-Calogero and Wolfes systems on the (r,ϕ) plane coincide with the TTW-system for $a=k_I$, $b=\omega=0$ and $a=\omega=0$, $b=k_{II}$, respectively.

The previous systems have Hamiltonians of type

$$H=\frac{1}{2}p_u^2+f(u)+\alpha^2(u)\left(\frac{1}{2}g^{ij}p_ip_j+V(q^i)\right)\tag{6}$$

i.e., they are constructed by merging an N-dimensional natural Hamiltonian $H_2=L=\frac{1}{2}g^{ij}p_ip_j+V(q^i)$ with a 1-dimensional natural Hamiltonian $H_1=\frac{1}{2}p_u^2+f(u)$ through the functions $\alpha^1=1$, $\alpha^2=\alpha^2(u)$.

Since these two examples are both superintegrable systems (i.e., they admit a third independent first integral K other than H and H_2, associated with the separation of variables), it is worth to investigate if it is their structure of warped products the responsible of the existence of K. Namely, we search for conditions on $\alpha^2(u)$, g^{ij}, $V(q^i)$, $f(u)$ such that H admits a polynomial in the momenta first-integral K, of degree determined by the rational parameter k. Therefore, in the following we will not assume separation of variables, that is essential in the previous proofs of the superintegrability of these systems, and focus on the existence of a single constant of motion. The results, proved in several papers, are summarized below.

Roughly speaking, given a natural Hamiltonian $L = \frac{1}{2}g^{ij}p_i p_j + V(q^i)$ with N degrees of freedom, we call *extensions of* L the Hamiltonians with $N+1$ degrees of freedom of the form

$$H=\frac{1}{2}p_u^2 + f(u) + \alpha^2(u)L,$$

such that, starting from a first order differential operator of the form

$$p_u + w\gamma(u)X_L,$$

where X_L is the Hamiltonian vector field of L and w a suitable constant, a first integral K can be constructed by applying iteratively the operator to a non-trivial function $G(q^i, p_i)$. A series of results of increasing generality exposed in [3]-[9], ended in the following Theorem, proved in [9]:

	$c=0$	$c\neq 0$
$\gamma =$	$-Au$	$\dfrac{1}{T_\kappa(cu)}$
$\alpha^2 = -\dfrac{m^2}{n^2}\gamma' =$	$\dfrac{m^2}{n^2}A$	$\dfrac{m^2 c}{n^2 S_\kappa^2(cu)}$
$f =$	$\dfrac{m^2}{n^2}L_0 A^2 u^2 + \dfrac{\omega}{A^2u^2}$	$\omega T_\kappa^2(cu)$

Table 1. Functions involved in the modified (m,n)-extension of L (A, ω constants)

Theorem 1. *Let L be a natural Hamiltonian such that there exist two constants c, L_0, not both zero, such that equation*

$$X_L^2(G) = -2(cL + L_0)G \tag{7}$$

admits a non trivial solution G (i.e., $X_L(G) \neq 0$) and let γ, α^2 and f be defined by Table 1 where the functions

$$S_\kappa(x) = \begin{cases} \frac{\sin\sqrt{\kappa}x}{\sqrt{\kappa}} & \kappa > 0 \\ x & \kappa = 0 \\ \frac{\sinh\sqrt{|\kappa|}x}{\sqrt{|\kappa|}} & \kappa < 0 \end{cases} \qquad T_\kappa(x) = \begin{cases} \frac{\tan\sqrt{\kappa}x}{\sqrt{\kappa}} & \kappa > 0 \\ x & \kappa = 0 \\ \frac{\tanh\sqrt{|\kappa|}x}{\sqrt{|\kappa|}} & \kappa < 0 \end{cases}$$

$$C_\kappa(x) = \begin{cases} \cos\sqrt{\kappa}x & \kappa > 0 \\ 1 & \kappa = 0 \\ \cosh\sqrt{|\kappa|}x & \kappa < 0 \end{cases}$$

are known as Trigonometric Tagged Functions *and resume in a single function the standard trigonometric and hyperbolic functions [15].*

Then, the Hamiltonian H depending on the rational parameter m/n, called modified (m,n)-extension of L,

$$H = \frac{1}{2}p_u^2 + f(u) + \alpha^2(u)L = \frac{1}{2}p_u^2 + \frac{m^2}{n^2}L_0\gamma^2 + \frac{\omega}{\gamma^2} - \frac{m^2}{n^2}\gamma' L$$

admits a first integral of the form

$$K_{2s,n} = \left(\left(p_u + \frac{2s}{n^2}\gamma X_L\right)^2 + 2\omega\gamma\right)^s G_n$$

for $m = 2s$ and

$$K_{4s+2,2n} = \left(\left(p_u + \frac{4s+2}{4n^2}\gamma X_L\right)^2 + 2\omega\gamma\right)^{2s+1} G_{2n}$$

for $m = 2s+1$, where G_n and G_{2n} are the n-th and $2n$-th terms of the recursion

$$G_1 = G, \qquad G_{n+1} = X_L(G)\,G_n + \frac{1}{n}G\,X_L(G_n).$$

The extended Hamiltonian H is therefore based on different Riemannian manifolds according to the values of κ (which can be arbitrarily chosen), and of c, (which is determined by the curvature of the metric tensor g [6]). When L is a one-dimensional Hamiltonian, the metric tensor g is not defined and the possible extensions H and their configuration manifolds can be summarized by Table 2 in the case $c \neq 0$, while for $c = 0$ the extension is

$$H = \frac{1}{2}p_u^2 + \frac{m^2}{n^2}L_0A^2u^2 + \frac{\omega}{A^2u^2} + \frac{m^2}{n^2}AL,$$

and is based on either the Euclidean or the Minkowski plane, according to the sign of A.

When L is not one-dimensional, the curvature of its base manifold appears into the integrability conditions of (8) [6]. For example, when L is based on $\mathbb{S}^2$, and G is assumed to be independent from the momenta, then c must be equal to the curvature of the sphere (see Theorem 2 below). For $\kappa = 1$, $\alpha^2 = m^2n^{-2}c\sin^{-2}u$ and H is based on a Riemannian manifold, which is S^3 only for the trivial case $m = n$. For $\kappa = 0$, $\alpha^2 = m^2n^{-2}cu^{-2}$ and the base of H is $\mathbb{E}^3$ for $m = n$. For κ negative we have $\alpha^2 = m^2n^{-2}c\sin h^{-2}u$ and the base of H is again a Riemannian manifold, conformally flat only for $m = n$. When L is based on the pseudosphere $\mathbb{H}^2$, whose curvature is negative, c must be negative and the extensions are based on a Lorentzian manifold, whose geometry is again determined by the sign of κ.

An intrinsic characterization of extended Hamiltonians systems is given in [6] and requires the existence on the base manifold of H of a particular conformal Killing vector.

The general TTW system and the caged anisotropic oscillator can be written in form of extensions on any two-dimensional manifold of constant curvature (for superintegrable systems with higher-order symmetries on flat and curved manifolds see also [1, 2, 10, 14, 16, 17]). This has been done for the Euclidean plane in [9] and reported (for the TTW system) as an example below. Hence, by Theorem 1, this allows to compute directly the additional first integral and prove superintegrability in a constructive way.

	$c=1$	Manifold
$\kappa=1$	$H=\frac{1}{2}p_u^2+\omega\tan^2 u+\frac{m^2}{n^2\sin^2 u}L$	Sphere
$\kappa=0$	$H=\frac{1}{2}p_u^2+\omega u^2+\frac{m^2}{n^2u^2}L$	Euclidean plane
$\kappa=-1$	$H=\frac{1}{2}p_u^2+\omega\tanh^2 u+\frac{m^2}{n^2\sinh^2 u}L$	Pseudo-sphere
	$c=-1$	
$\kappa=1$	$H=\frac{1}{2}p_u^2+\omega\tan^2 u-\frac{m^2}{n^2\sin^2 u}L$	de Sitter
$\kappa=0$	$H=\frac{1}{2}p_u^2+\omega u^2-\frac{m^2}{n^2u^2}L$	Minkowski plane
$\kappa=-1$	$H=\frac{1}{2}p_u^2+\omega\tanh^2 u-\frac{m^2}{n^2\sinh^2 u}L$	anti-de Sitter

Table 2. The possible (m,n)-extensions of $L=\frac{1}{2}p_v^2+V(v)$ and their configuration manifolds

3.1. Example: The Tremblay-Turbiner-Winternitz system

By setting $k=\frac{m}{n}$, $u=r$, $v=k\psi$, the complete TTW system in the Euclidean plane becomes

$$H=\frac{1}{2}p_u^2+\frac{1}{u^2}\left(\frac{m^2}{2n^2}p_v^2+\frac{a}{\sin^2 v}+\frac{b}{\cos^2 v}\right)+\omega u^2.$$

From [9] we know that H can be written as

$$H=\frac{1}{2}p_u^2+\frac{(2m)^2}{n^2u^2}\left(\frac{1}{2}p_q^2+\frac{c_1+c_2\cos q}{\sin^2 q}\right)+\omega u^2,$$

where $q=2v$ and $c_1=\frac{a+b}{2k^2}$ $c_2=\frac{a-b}{2k^2}$. The factorized Hamiltonian $L(q,p_q)$ fulfills the hypotheses of Theorem 1 for $c=1$, $L_0=0$ and $G=p_q\sin q$. The coefficient of L is obtained for $\kappa=0$, that is $\gamma=1/u$. The corresponding first integral is of the form

$$\bar{K}_{2m,n}=\left(\left(p_u+\frac{2m}{n^2u}X_L\right)^2+2\frac{\omega}{u}\right)^m G_n.$$

We show in Figures 2 and 3 the trajectories of the particular case $k_2=0$ of the TTW system with harmonic term. In each figure the first two graphs show the relative change (w.r.t. the initial condition) of H_1 and H_2. The two following plots represent the projection of the integral curves in the subspaces $p_u=u=0$ and $p_v=v=0$, respectively. The last one displays the trajectory of the system in the plane (u,v) with axes $x=u\cos v$, $y=y\sin v$. These curves are drawn in the non-Euclidean space with metric $g^{uu}=1$, $g^{vv}=\frac{m^2}{u^2}$, $g^{uv}=0$, where m is now the rational parameter. If the v coordinate is reparametrized into mv, the system can be represented in the Euclidean space, but the graph become self-overlapping for $m<\frac{1}{2}$. Since m is rational, the trajectories always close.

The two-dimensional reductions of the Jacobi-Calogero and Wolfes systems are clearly members of the TTW family, then, they are extended Hamiltonian systems. On the contrary, the Hamiltonian of the geodesic flow of $\mathbb{T}^2$ of Example 2.1 is not an extended Hamiltonian.

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The key point of the method is the existence of solutions of (7), this implies global geometric properties of the Riemannian configuration manifold of L.

As instance, in the simplest case when we assume $G = G(q^i)$, independent of the momenta, (7) splits into

$$\nabla^i\nabla^j G + cg^{ij}G = 0,$$
$$\nabla_i V\nabla^i G - 2(cV + L_0)G = 0.$$

The solution $G(q^i)$ may depend on up to $N + 1$ parameters a_i.

Theorem 2. *The function $G(q^i)$ solution of (7) satisfies*

$$\nabla^i\nabla^j G + cg^{ij}G = 0, \tag{8}$$

and depends on $N + 1$ parameters, and we say that G is complete, iff g is of constant curvature c.

These are the only non trivial solutions on a two-dimensional manifold.

Families of solutions depending on less than $N + 1$ parameters can be found on manifolds of non-constant curvature of dimension ≥ 3.

Remark 1. Equation (8) is fundamental in theory of warped manifolds [18]. For example, the existence of non-trivial solutions of (8) on a complete Riemannian manifold of metric g with $c \neq 0$, implies that g has constant non zero curvature. Moreover, in the neighbourhood of non-stationary points of the geodesic flow of any Riemannian manifold, the existence of solutions implies that g itself has the structure of warped product, and that there exist local coordinates $(u^1, \dots, u^n)$ such that

$$ds^2 = (G'(u^n))^2 f_{ij}(u^h)du^i du^j + (du^n)^2,$$

where the f_{ij} depend on $(u^1, \dots, u^{n-1})$ and G' is the ordinary derivative of G with respect to u^n ([18], Lemma 1.2). This form of the metric tensor is a necessary condition for writing a Hamiltonian based on the manifold as extended Hamiltonian.

4. Conclusions

The extension procedure, when it occurs, maps separable systems into separable systems and (maximally) superintegrable systems into (maximally) superintegrable systems. Indeed, if L admits separation of variables, then also its extension H does, but separation of L it is not a requirement of the method. Moreover, if the Hamiltonian L is superintegrable with $2n - 1$ first integrals, then its extension H is superintegrable with $(2n - 1) + 2 = 2(n + 1) - 1$ first integrals (the new integrals being the Hamiltonian H itself and the first integral K). Therefore, the extension of superintegrable systems is a powerful method for building new superintegrable systems from old ones [7].

We can resume the extension procedure, here for the case of $G = G(q^i)$, in a simple algorithm

1 - Consider L on a (pseudo-)Riemannian manifold of metric g and constant curvature c

2 - Solve equation $\nabla^i\nabla^j G + cg^{ij}G = 0$ for the functions $G(q^i; a_1, \dots, a_{n+1})$

3 - Solve equation $\nabla_i V\nabla^i G - 2(cV + L_0)G = 0$ for the potential V

4 - Determine the extension H in the two cases $c = 0$ or $c \neq 0$.

5 - Compute the additional first integral as described in Theorem 1.

The dynamics of extended Hamiltonian systems can be understood within the framework of warped products of Hamiltonian systems, since any extension is a warped product of Hamiltonians.

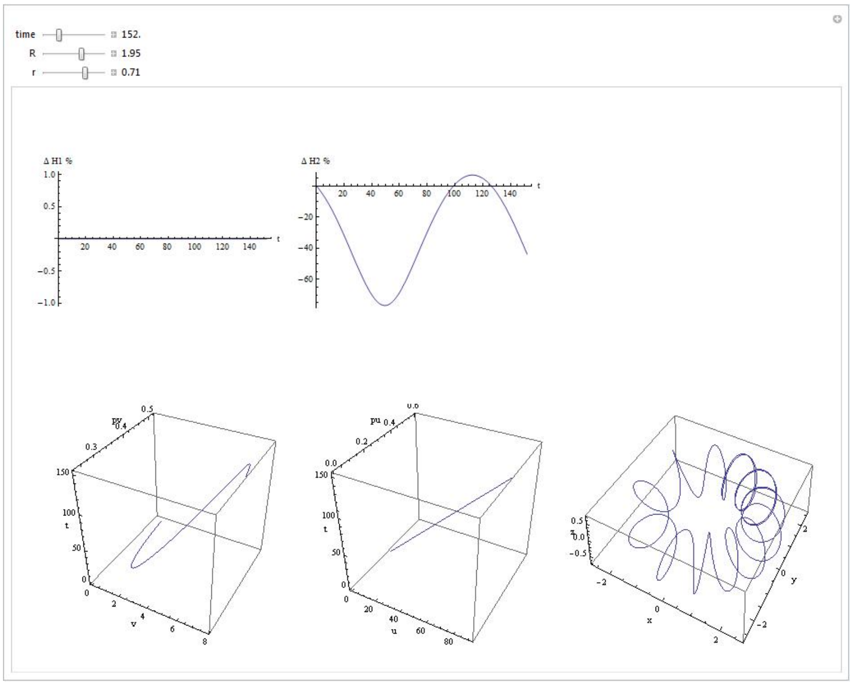

Figure 1. Geodesic on the Torus

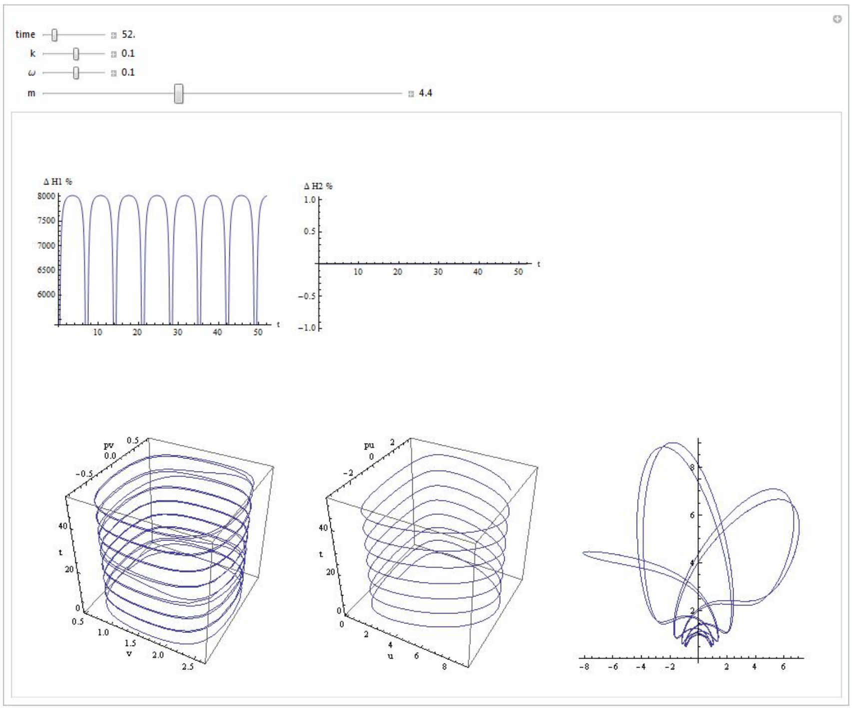

Figure 2. Tremblay-Turbiner-Winternitz system A

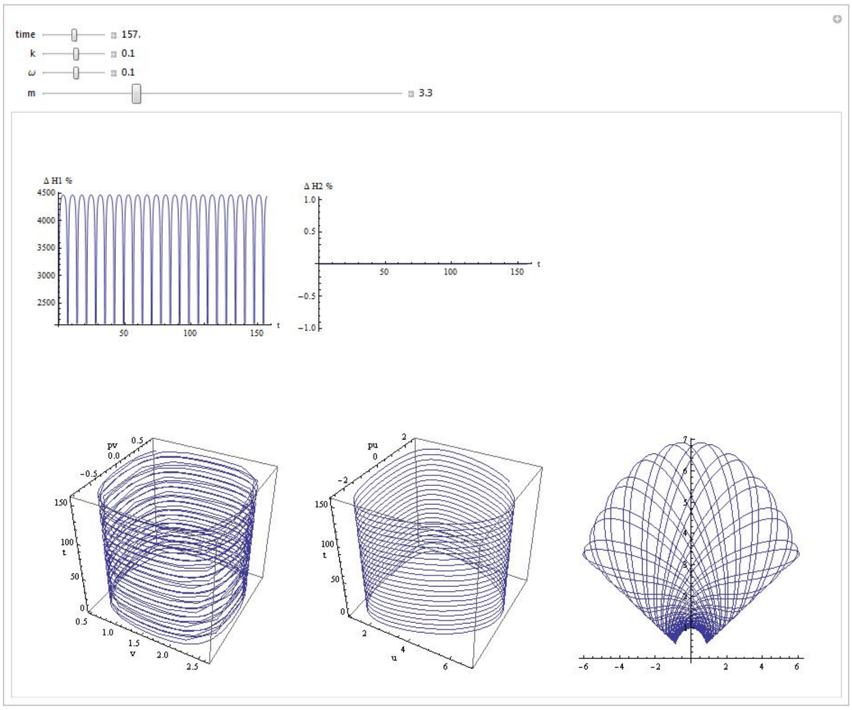

Figure 3. Tremblay-Turbiner-Winternitz system B

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