

# Lie groups, algebraic special functions and Jacobi polynomials

Enrico Celeghini<sup>1,2</sup>, Mariano A del Olmo<sup>2</sup> and Miguel A Velasco<sup>3</sup>

<sup>1</sup>Dipartimento di Fisica, Università di Firenze and INFN-Sezione di Firenze  
I50019 Sesto Fiorentino, Firenze, Italy

<sup>2</sup>Departamento de Física Teórica and IMUVA, Universidad de Valladolid,  
E-47011, Valladolid, Spain

<sup>3</sup>CIEMAT, Madrid, Spain

E-mail: celeghini@fi.infn.it, olmo@fta.uva.es, miguelangel.velasco@ciemat.es

**Abstract.** We present an attempt of classification of special functions in terms of a set of formal properties mainly based on their “ladder symmetry” algebras. Thus, the “algebraic special functions” are defined. We discuss, here, the case related with the Jacobi polynomials. We construct the relevant objects associated with the corresponding symmetry group  $SU(2, 2)$ : generators, subgroups, invariants and representations. Harmonic analyses based on significant subgroups for the Jacobi functions are developed.

## 1. Introduction

In a recent series of papers [1, 2, 3] we have revisited the connection between special functions, in particular the classical orthogonal polynomials, with Lie groups, differential equations and Hilbert spaces. The revision started from the Hermite polynomials in the quantum harmonic oscillator to arrive to the study of the symmetries of quantum systems where, in many cases, orthogonal polynomials are involved in the construction of bases of  $L^2$  spaces [4, 5, 6, 7].

Our approach is based on ideas of Wigner (symmetry group) [8, 9] and of Truesdell (“functions with additional properties”) [10]. A lot of work has been done in this direction since the seminal book by Wigner, in particular we can mention Vilenkin [11, 12, 13] and Miller [14, 15]. Both have made an impressive work in the study of the symmetries of special functions.

Starting from Miller, Vilenkin and Truesdell we have defined the “algebraic special functions” (ASF) as those functions with the following set of formal properties: they admit a set of ladder operators that span a Lie algebra  $\mathcal{G} = \text{Lie}(G)$ , they support a unitary irreducible representation (UIR) of  $G$  and they are bases of Hilbert spaces.

The main facts associated to these ASF are: the defining second order differential equation of the corresponding family of ASF can be obtained by the factorization method [16, 17] from the Casimir of each possible subalgebra of  $\mathcal{G}$  or from the Casimir of the full algebra  $\mathcal{G}$  associated to a specific representation; the set of operators acting on the Hilbert space is homomorphic to the Universal Enveloping Algebra (UEA) built on  $\mathcal{G}$ ; and the exponential map defines all possible changes of bases in the corresponding Hilbert space. So, they are well adapted to the quantum mechanics framework.



We have got some interesting results that we can summarize as follows:

- (i) for Hermite, Laguerre and Legendre polynomials we have obtained symmetry Lie groups of rank one. Thus, the Heisenberg-Weyl group  $H(1)$  for algebraic Hermite functions

$$K_n(x) = \frac{e^{-x^2/2}}{\sqrt{2^n n!} \sqrt{\pi}} H_n(x), \quad \forall n \in \mathbb{N},$$

where  $H_n(x)$  are the Hermite polynomials. However the Lie group  $SU(1,1)$  describes the Legendre polynomials  $P_n(x)$  and also the algebraic Laguerre functions

$$M_n(x) = e^{-x/2} L_n(x), \quad \forall n \in \mathbb{N},$$

where  $L_n(x)$  are the Laguerre polynomials [2].

- (ii) Groups of rank two are obtained for the associated Legendre polynomials, spherical harmonics, Gegenbauer polynomials and Jacobi polynomials  $J_n^{(\alpha,\alpha)}(x)$ . For instance, for the associated Legendre polynomials  $P_l^m(x)$  we get the algebraic associated Legendre functions

$$T_l^m(x) = \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(x), \quad l \in \mathbb{N}, m \in \mathbb{Z}, 0 \leq |m| \leq l,$$

that as well as the spherical harmonics support a particular UIR of  $SO(3,2)$  [1].

We present in this paper a case whose symmetry group has rank three: the Jacobi polynomials  $J_n^{(\alpha,\beta)}(x)$ . As we have noted in the previous cases the rank of the group is related with the number of independent parameters labeling the orthogonal polynomials. The idea is to connect an operator to each label, such that the polynomials are eigenvectors of it. These operators will span the Cartan subalgebra of the corresponding symmetry Lie algebra.

Some of the results described here have been obtained by Miller many years ago [18, 19] in connection with the wave equation in four dimensions and the Gaussian hypergeometric functions  ${}_2F_1(\alpha, \beta, \gamma; x)$ , generalization of the Jacobi polynomials  $J_n^{(\alpha,\beta)}(x)$ . The conformal group  $SU(2,2)$  found by Miller is recovered here. Later, in Ref. [20] Floreanini and Vinet construct, following the approach of Miller and the results of Kalnins and Miller [21], a representation of the euclidean algebra in four dimensions  $e(4)$  supported by functions of three variables that include the Jacobi polynomials. In these cases the problem of the coexistence of variations of the parameters  $\alpha, \beta, \gamma, n$  and derivatives in the variable  $x$  was solved in a creative manner by means of the introduction of new variables and new functions that include the hypergeometric functions and the Jacobi polynomials, respectively. Thus, the ladder operators act on the space of these auxiliary functions instead on the space of the hypergeometric functions or of Jacobi polynomials where only the corresponding recurrence relations are defined.

In our approach we realize all recurrence relations as true operators in the vector space of Jacobi polynomials  $J_n^{(\alpha,\beta)}(x)$  associating the parameters  $n, \alpha, \beta$  to operators, that have the parameters as eigenvalues. We can thus define the operators in the space of the corresponding orthogonal polynomials and describe the algebraic structure in the vector space of these polynomials.

The realization of the ladder operators acting on the Jacobi polynomials allows us to construct objects associated to a Lie group: generators, invariants, subgroups, universal enveloping algebra, representations, etc.

Different subspaces of the Jacobi functions are found to support unitary irreducible representations of subgroups of  $SU(2,2)$ . In particular, it looks interesting for physical

applications that the  $SU(2,2)$ -UIR can be splitted in two subrepresentations dividing the Jacobi functions in bosonic ( $SO(3,2)$ ) and fermionic ( $Spin(3,2)$ ) ones, stressing the relation of the algebraic Jacobi functions (AJF) with the Wigner  $d_j$ -matrices [22, 23].

We also present harmonic analysis approaches based on the groups  $SU(1,1)$ ,  $SO(3,2)$  and  $SU(2,2)$ , respectively. In this way subsets of AJF are bases of Hilbert spaces. The operators acting on the associated spaces of square integrable functions belong to the corresponding universal enveloping algebra.

## 2. Algebraic Jacobi functions

The Jacobi polynomials,  $J_n^{(\alpha,\beta)}(x)$ , are polynomials of degree  $n \in \mathbb{N}$  with  $\alpha, \beta \in \mathbb{R}$  and  $\alpha > -1$ ,  $\beta > -1$  [24, 25]. Firstly we take three new (discrete) variables  $(j, m, q)$  instead of  $(n, \alpha, \beta)$  to label them. The relation between these two set of parameters is

$$j := n + \frac{\alpha + \beta}{2}, \quad m := \frac{\alpha + \beta}{2}, \quad q := \frac{\alpha - \beta}{2}.$$

Next we include a  $x$ -depending factor related with the integration measure of the Jacobi polynomials. Hence we obtain the called ‘‘algebraic Jacobi functions’’ (AJF)

$$\mathcal{J}_j^{m,q}(x) := \sqrt{\frac{\Gamma(j+m+1)\Gamma(j-m+1)}{\Gamma(j+q+1)\Gamma(j-q+1)}} \left(\frac{1-x}{2}\right)^{\frac{m+q}{2}} \left(\frac{1+x}{2}\right)^{\frac{m-q}{2}} J_{j-m}^{(m+q, m-q)}(x). \quad (1)$$

For our group-theoretical purposes we impose to  $(j, m, q)$  that

$$j \geq |m|, \quad j \geq |q|, \quad 2j \in \mathbb{N}, \quad j - m \in \mathbb{N}, \quad j - q \in \mathbb{N}, \quad (2)$$

resulting that  $(j, m, q)$  are all together integers or half-integers. The conditions (2) rewritten in terms of the parameters  $(n, \alpha, \beta)$  are different from the original ones. However the change is motivated by the introduction of the normalization inside the functions and by the algebra structure requirements (2). Moreover the AJF verify that  $\mathcal{J}_j^{m,q}(x) = 0$  for  $j < |q|$  (see eq. (1)) and they can be extended to  $j < |m|$  by considering a limit procedure

$$\hat{\mathcal{J}}_j^{m,q}(x) := \lim_{\varepsilon \rightarrow 0} \mathcal{J}_j^{m+\varepsilon, q}(x) = \begin{cases} \mathcal{J}_j^{m,q}(x) & \forall (j, m, q) \text{ verifying conditions (2)} \\ 0 & j < |m| \end{cases}.$$

Henceforth we remove the hat to the AJF  $\hat{\mathcal{J}}_j^{m,q}(x)$ . They present additional symmetries hidden inside the Jacobi polynomials:

$$\begin{aligned} \mathcal{J}_j^{m,q}(x) &= \mathcal{J}_j^{q,m}(x), & \mathcal{J}_j^{m,q}(x) &= (-1)^{l-m} \mathcal{J}_j^{m,-q}(-x), \\ \mathcal{J}_j^{m,q}(x) &= (-1)^{l-q} \mathcal{J}_j^{-m,q}(-x), & \mathcal{J}_j^{m,q}(x) &= (-1)^{m+q} \mathcal{J}_j^{-m,-q}(x). \end{aligned}$$

The AJF for  $m$  and  $q$  fixed verify the orthogonality relation

$$\int_{-1}^1 \mathcal{J}_j^{m,q}(x) (j+1/2) \mathcal{J}_{j'}^{m,q}(x) dx = \delta_{jj'} \quad (3)$$

and the completeness relation

$$\sum_{j=\sup(|m|, |q|)}^{\infty} \mathcal{J}_j^{m,q}(x) (j+1/2) \mathcal{J}_j^{m,q}(y) = \delta(x-y). \quad (4)$$

The differential Jacobi equation satisfied by the Jacobi polynomials  $J_n^{(\alpha,\beta)}(x)$

$$\left[ (1-x^2) \frac{d^2}{dx^2} - ((\alpha + \beta + 2)x + (\alpha - \beta)) \frac{d}{dx} + n(n + \alpha + \beta + 1) \right] J_n^{(\alpha,\beta)}(x) = 0$$

can be rewritten in terms of the AJF  $\mathcal{J}_j^{m,q}(x)$

$$\left[ -(1-x^2) \frac{d^2}{dx^2} + 2x \frac{d}{dx} + \frac{2mqx + m^2 + q^2}{1-x^2} - j(j+1) \right] \mathcal{J}_j^{m,q}(x) = 0, \quad (5)$$

where the symmetry under the interchange  $m \Leftrightarrow q$  is evident.

An important fact is that the AJFs (1) are essentially the elements of the Wigner  $d_j$ -rotation matrices [22, 23] after the change of variable  $x = \cos \beta$ . Indeed, the relation

$$d_{qm}^j(\beta) = \sqrt{\frac{(j+m)!(j-m)!}{(j+q)!(j-q)!}} (\sin \beta/2)^{m-q} (\cos \beta/2)^{m+q} J_{j-m}^{(m-q, m+q)}(\cos \beta) \quad (6)$$

allows to write

$$d_{qm}^j(\beta) = \mathcal{J}_j^{m,-q}(\cos \beta).$$

### 3. Symmetry algebras of the algebraic Jacobi functions

Let us start by introducing, not only the operators  $X$  and  $D_x$  of the configuration space

$$X f(x) = x f(x), \quad D_x f(x) = f'(x), \quad [X, D_x] f(x) = -f(x),$$

but three other diagonal operators  $J$ ,  $M$  and  $Q$  on the AJF

$$J \mathcal{J}_j^{m,q}(x) = j \mathcal{J}_j^{m,q}(x), \quad M \mathcal{J}_j^{m,q}(x) = m \mathcal{J}_j^{m,q}(x), \quad Q \mathcal{J}_j^{m,q}(x) = q \mathcal{J}_j^{m,q}(x). \quad (7)$$

The procedure consists in transforming the differential-difference equations and difference equations verified by the Jacobi polynomials (that one can find in Refs. [24, 25, 26]) in terms of ladder operators. Since there are many recurrence relations, we start considering the equations (18.9.15) and (18.9.16) of Ref. [24]

$$\begin{aligned} \frac{d}{dx} J_n^{(\alpha,\beta)}(x) &= \frac{1}{2}(n + \alpha + \beta + 1) J_{n-1}^{(\alpha+1,\beta+1)}(x), \\ \frac{d}{dx} \left[ (1-x)^\alpha (1+x)^\beta J_n^{(\alpha,\beta)}(x) \right] &= -2(n+1)(1-x)^{\alpha-1} (1+x)^{\beta-1} J_{n+1}^{(\alpha-1,\beta-1)}(x), \end{aligned}$$

that in terms of the AJF become

$$A_\pm \mathcal{J}_j^{m,q}(x) = \sqrt{(j \mp m)(j \pm m + 1)} \mathcal{J}_j^{m \pm 1, q}(x), \quad (8)$$

where

$$A_\pm := \pm \sqrt{1-X^2} D_x + \frac{1}{\sqrt{1-X^2}} (XM + Q). \quad (9)$$

By inspection only the parameter  $m$  of AJF changes and varies in  $\pm 1$ . Both operators  $A_\pm$  together with  $A_3 := M$  close a  $su(2)$  Lie algebra (denoted by  $su_A(2)$ ) that commutes with  $J$  and  $Q$

$$[A_3, A_\pm] = \pm A_\pm, \quad [A_+, A_-] = 2A_3.$$

In this way the set of AJF such that  $2j \in \mathbb{N}$ ,  $j - m \in \mathbb{N}$  and  $-j \leq m \leq j$  support the  $(2j + 1)$ -dimensional UIR of the Lie group  $SU_A(2)$  independently from the value of  $q$ .

Taking into account the differential realization (9) of  $A_{\pm}$  and the Casimir,  $\mathcal{C}_A$ , of  $su_A(2)$

$$[\mathcal{C}_A - J(J + 1)] \mathcal{J}_j^{m,q}(x) \equiv \left[ A_3^2 + \frac{1}{2} \{A_+, A_-\} - J(J + 1) \right] \mathcal{J}_j^{m,q}(x) = 0, \quad (10)$$

we recover the Jacobi differential equation (5) in operator terms

$$\left[ -(1 - X^2)D_x^2 + 2XD_x + \frac{1}{1 - X^2}(2XMQ + M^2 + Q^2) - J(J + 1) \right] \mathcal{J}_j^{m,q}(x) = 0, \quad (11)$$

which can also be obtained by two factorized equations that reproduce the Jacobi equation

$$\begin{aligned} [A_+ A_- - (J + M)(J - M + 1)] \mathcal{J}_j^{m,q}(x) &= 0, \\ [A_- A_+ - (J - M)(J + M + 1)] \mathcal{J}_j^{m,q}(x) &= 0. \end{aligned} \quad (12)$$

Note that eqs. (10) and (12) are particular cases of a general rule: the defining differential equation can be recovered applying to the ASF the second order Casimir operator of any involved algebra and sub-algebra as well as any diagonal product of ladder operators.

Now taking into account the symmetry  $m \Leftrightarrow q$  of  $\mathcal{J}_j^{m,q}(x)$ , we construct the algebra  $su_B(2)$  of operators  $B_{\pm}, B_3$ , that change  $q$  leaving  $j$  and  $m$  unchanged, from  $A_{\pm}$  and  $A_3$  by the interchange

$$(A_{\pm}, A_3) \xleftrightarrow{M \Leftrightarrow Q} (B_{\pm}, B_3).$$

Obviously the counterparts of the expressions (8)–(12) for  $B_{\pm}, B_3$ , are obtained by the changes  $m \Leftrightarrow q$  and  $M \Leftrightarrow Q$  in these expressions.

Since the operators  $A_{\pm}$  and  $A_3$  commute with  $B_{\pm}$  and  $B_3$  we get, on the space of  $\mathcal{J}_j^{m,q}(x)$  with  $j$  fixed, the symmetry algebra

$$su_A(2) \oplus su_B(2).$$

So, the set of  $\mathcal{J}_j^{m,q}(x)$  (for fixed  $j$  and  $-j \leq m \leq j$ ,  $-j \leq q \leq j$ ) determines a UIR of  $SU_A(2) \otimes SU_B(2)$ .

#### 4. $su(1, 1)$ -ladder operators

Proceeding in a similar way, from the remaining difference and differential-difference relations we can obtain eight new infinitesimal generators. In a first step we construct two ones ( $C_{\pm}$  such that  $C_{\pm}^{\dagger} = C_{\mp}$ ) that determine a  $su(1, 1)$  algebra ( $su_C(1, 1)$ ):

$$\begin{aligned} C_+ &:= + \frac{(1 + X)\sqrt{1 - X}}{\sqrt{2}} D_x - \frac{1}{\sqrt{2(1 - X)}} (X(J + 1) - (J + 1 + M + Q)), \\ C_- &:= - \frac{(1 + X)\sqrt{1 - X}}{\sqrt{2}} D_x - \frac{1}{\sqrt{2(1 - X)}} (XJ - (J + M + Q)). \end{aligned} \quad (13)$$

They act on the space  $\{\mathcal{J}_j^{m,q}\}$  (for  $j, m, q$  integer and half-integer such that  $j \geq |m|, |q|$ )

$$\begin{aligned} C_+ \mathcal{J}_j^{m,q}(x) &= \sqrt{(j + m + 1)(j + q + 1)} \mathcal{J}_{j+1/2}^{m+1/2, q+1/2}(x), \\ C_- \mathcal{J}_j^{m,q}(x) &= \sqrt{(j + m)(j + q)} \mathcal{J}_{j-1/2}^{m-1/2, q-1/2}(x). \end{aligned} \quad (14)$$

They change all the parameters  $(j, m, q)$  by  $\pm 1/2$ . Moreover

$$[C_+, C_-] = -2C_3, \quad [C_3, C_\pm] = \pm C_\pm, \quad C_3 := J + \frac{1}{2}(M + Q) + \frac{1}{2}.$$

The Casimir  $\mathcal{C}_C$  of  $su_C(1, 1)$  is

$$\mathcal{C}_C \mathcal{J}_j^{m,q}(x) \equiv \left[ C_3^2 - \frac{1}{2} \{C_+, C_-\} \right] \mathcal{J}_j^{m,q}(x) = \frac{1}{4} [(m - q)^2 - 1] \mathcal{J}_j^{m,q}(x).$$

The Jacobi equation (11) can be derived from  $\mathcal{C}_C$

$$\begin{aligned} & \left[ \mathcal{C}_C - \frac{1}{4}(M - Q)^2 + \frac{1}{4} \right] \mathcal{J}_j^{m,q}(x) \\ & \equiv \left[ C_3^2 - \frac{1}{2} \{C_+, C_-\} - \frac{1}{4}(M - Q)^2 + 1/4 \right] \mathcal{J}_j^{m,q}(x) = 0, \end{aligned} \quad (15)$$

and also from

$$\begin{aligned} & [C_+ C_- - (J + M)(J + Q)] \mathcal{J}_j^{m,q}(x) = 0, \\ & [C_- C_+ - (J + 1 + M)(J + 1 + Q)] \mathcal{J}_j^{m,q}(x) = 0. \end{aligned}$$

Analyzing (15) we see that since  $(m - q) = 0, \pm 1, \pm 2, \pm 3, \dots$  the IR of  $su(1, 1)$  with Casimir  $\mathcal{C}_C = (m - q)^2/4 - 1/4 = -1/4, 0, 3/4, 2, 15/4, \dots$  are obtained. Hence, the set of AJF supports many infinite-dimensional UIR of the discrete series of  $SU_C(1, 1)$  [27].

We can find six more ladder operators,  $D_\pm, E_\pm, F_\pm$ , whose explicit differential expressions are easily obtained from those of  $C_\pm$  (13) according to the changes suggested by

$$\begin{aligned} D_\pm(X, D_x, J, M, Q) &= C_\pm(-X, -D_x, J, M, -Q), \\ E_\pm(X, D_x, J, M, Q) &= C_\pm(-X, -D_x, J, -M, Q), \\ F_\pm(X, D_x, J, M, Q) &= -C_\pm(X, D_x, J, -M, -Q). \end{aligned} \quad (16)$$

All the relations where the operators  $C_\pm$  are involved can be rewritten for these new operators taking into account (16).

The generators  $A_\pm, B_\pm, C_\pm, D_\pm, E_\pm, F_\pm, J, M, Q$  close the Lie algebra  $su(2, 2)$  [3]. Its quadratic Casimir has the form

$$\begin{aligned} \mathcal{C}_{su(2,2)} &= \frac{1}{2} (\{A_+, A_-\} + \{B_+, B_-\} - \{C_+, C_-\} - \{D_+, D_-\} - \{E_+, E_-\} - \{F_+, F_-\}) \\ &+ 2J(J + 1) + M^2 + Q^2 + \frac{1}{2} \equiv -\frac{3}{2}. \end{aligned}$$

Once more taking into account the differential realization of the operators involved in the Casimir we recover the Jacobi equation.

The AJF support a UIR of the group  $SU(2, 2)$  with the value  $-3/2$  of  $\mathcal{C}_{su(2,2)}$  such that the integer and half-integer values of  $(j, m, q)$  are putted all together.

A more detailed discussion can be found in Ref. [3]

### 5. Symmetries for fermion and boson states

The operators  $A_{\pm}$  ( $B_{\pm}$ ) change only a label  $m$  ( $q$ ) of  $\pm 1$  but the operators  $C_{\pm}$ ,  $D_{\pm}$ ,  $E_{\pm}$ ,  $F_{\pm}$  change all the three labels  $(j, m, q)$  by  $\pm 1/2$ . Hence, we can obtain operators that only change  $j$  by  $\pm 1$  and leave the other two parameters invariant by composing the action of pairs of operators, for instance  $F_{\pm}C_{\pm}$  (or  $C_{\pm}F_{\pm}$ ,  $D_{\pm}E_{\pm}$ ,  $E_{\pm}D_{\pm}$ ). They are second order differential operators but they can be reduced to first order ones when act on  $\mathcal{J}_j^{m,q}$  by means of the Jacobi equation. We have seen that the  $\mathcal{J}_j^{m,q}$  have  $j \geq |m|$  and  $j \geq |q|$ , but now we have to consider separately specific values of  $m$  and  $q$ .

Thus we define two conjugate hermitian operators

$$K_+ := F_+C_+ \frac{1}{\sqrt{(J+1)^2 - Q^2}}, \quad K_- := F_-C_- \frac{1}{\sqrt{J^2 - Q^2}},$$

that can be written as

$$\begin{aligned} K_+ &:= \left( -(1 - X^2) D_x + X (J + 1) + \frac{MQ}{J+1} \right) \frac{J+1}{\sqrt{(J+1)^2 - Q^2}}, \\ K_- &:= \left( (1 - X^2) D_x + X J + \frac{MQ}{J} \right) \frac{J}{\sqrt{J^2 - Q^2}}, \end{aligned} \quad (17)$$

with the condition that they act on  $\mathcal{J}_j^{m,q}$  with  $j \geq |m| > |q|$ . Their explicit action on  $\mathcal{J}_j^{m,q}$  is

$$\begin{aligned} K_+ \mathcal{J}_j^{m,q}(x) &= \sqrt{(j+1)^2 - m^2} \mathcal{J}_{j+1}^{m,q}(x), \\ K_- \mathcal{J}_j^{m,q}(x) &= \sqrt{j^2 - m^2} \mathcal{J}_{j-1}^{m,q}(x). \end{aligned}$$

The operators  $K_{\pm}$  together with  $K_3 := J + 1/2$  close the Lie algebra  $su_K(1,1)$ . Hence  $\{\mathcal{J}_j^{m,q}; m, q \text{ fixed}, j \geq |m| > |q|\}$  is a basis of the  $SU(1,1)$ -UIRs with Casimir  $\mathcal{C}_K = m^2 - 1/4$ .

In the case  $|m| < |q|$  it is enough to interchange  $M \Leftrightarrow Q$  (or  $m \Leftrightarrow q$ ) in  $K_{\pm}$ . However, when  $|m| = |q|$  the action of  $K_-$  is not well defined in eq. (17) for  $j = |m| = |q|$ , but we can extend its definition to this case by considering the limit

$$K_- := \lim_{\epsilon \rightarrow 0} \left[ \left( (1 - X^2) D_x + X J + \frac{(M + \epsilon)(Q + \epsilon)}{J} \right) \frac{J}{\sqrt{J^2 - (Q + \epsilon)^2}} \right].$$

In any case the  $\mathcal{J}_j^{m,q}$  determine, for fixed  $m$  and  $q$ , a UIR of  $SU(1,1)$ . Thus, if  $|m| \geq |q|$  we have eqs. (5) with  $j = |m|, |m| + 1, |m| + 2 \dots$  and Casimir  $\mathcal{C}_K = m^2 - 1/4$ , while for  $|m| < |q|$  we have to exchange  $q$  and  $m$  everywhere. These  $SU(1,1)$ -UIRs contain only states with integer or half-integer values of its labels. From a physical point of view the Wigner  $d_j$ -matrices (6) do not mix integer and half-integer spins, i.e. bosons and fermions.

These  $SU(1,1)$  groups cannot, in general, be extended to larger groups except when  $q = 0$  or  $m = 0$ . Then, the  $\mathcal{J}_j^{m,0}$  (with  $j, m \in \mathbb{Z}$  and  $j \geq |m|$ ) are related to the associated Legendre polynomials  $P_{n+\alpha}^{\alpha}$  and support a UIR of  $SO(3,2)$ . In this case we get not only  $K_{\pm}$  but also  $A_{\pm}$  that allow us to obtain the whole algebra  $so(3,2)$  described in [1]. For  $\mathcal{J}_j^{0,q}$  the results are similar.

The ‘‘fermions’’ states  $\{\mathcal{J}_j^{m,\pm 1/2}\}$  and  $\{\mathcal{J}_j^{\pm 1/2,q}\}$  are related to the same algebra  $so(3,2)$  but to a representation of its covering group  $Spin(3,2)$  [28] with all the parameters half-integer.

## 6. $L^2$ -functions spaces and $\mathcal{J}_j^{m,q}(x)$

It is well known that the orthogonal polynomials are bases on  $L^2$ -functions spaces [29]. In this case according to the relations (3) and (4) the set  $\{\mathcal{J}_j^{m,q}(x) \mid m, q \text{ fixed}\}_{j=\sup(|m|, |q|)}^\infty$  is a basis of the  $L^2$ -functions defined in the interval  $\mathbb{E} = (-1, 1)$ . Moreover the AJF are transition matrices between continuous and discrete bases in representation spaces.

Effectively, let us consider the simplest case. From expressions (3) and (4) we are able to consider  $j$  and  $x$  as conjugate variables on the same space after fixed  $m$  and  $q$ . Let  $\{|j, (m, q)\rangle; m, q \text{ fixed}, j = \sup(|m|, |q|)\}$  be the set of eigenvectors of  $K_3$  and  $\mathcal{C}_K$

$$\begin{aligned} K_3 |j, (m, q)\rangle &= (j + \frac{1}{2}) |j, (m, q)\rangle, \\ \mathcal{C}_K |j, (m, q)\rangle &= (m^2 - \frac{1}{4}) |j, (m, q)\rangle. \end{aligned} \quad (18)$$

The action of  $K_\pm$  on these vectors gives

$$\begin{aligned} K_+ |j, (m, q)\rangle &= \sqrt{(j+1)^2 - m^2} |j+1, (m, q)\rangle, \\ K_- |j, (m, q)\rangle &= \sqrt{j^2 - m^2} |j-1, (m, q)\rangle. \end{aligned} \quad (19)$$

So the set  $\{|j, (m, q)\rangle\}$  is a basis of the space support of the IUR of  $SU(1, 1)$  determined by expressions (18) and (19), i.e.

$$\langle j, (m, q) | j', (m, q) \rangle = \delta_{j j'}, \quad \sum_{j=\sup(|m|, |q|)}^\infty |j, (m, q)\rangle \langle j, (m, q)| = \mathcal{I}. \quad (20)$$

The orthogonality and completeness of  $\{|j, (m, q)\rangle\}$  (20) and the AJF allows us to define the vectors

$$|x, (m, q)\rangle := \sum_{j=\sup(|m|, |q|)}^\infty |j, (m, q)\rangle \sqrt{j+1/2} \mathcal{J}_j^{m,q}(x)$$

which are a basis of the space  $\mathbb{E} = (-1, 1)$

$$\langle x, (m, q) | x', (m, q) \rangle = \delta(x - x'), \quad \int_{-1}^{+1} |x, (m, q)\rangle dx \langle x, (m, q)| = \mathcal{I}.$$

This implies that the  $\{\mathcal{J}_j^{m,q}(x)\}$  are the transition matrices between the two bases, i.e.

$$\mathcal{J}_j^{m,q}(x) = \frac{1}{\sqrt{j+1/2}} \langle x, (m, q) | j, (m, q) \rangle = \frac{1}{\sqrt{j+1/2}} \langle j, (m, q) | x, (m, q) \rangle,$$

and

$$|j, (m, q)\rangle = \int_{-1}^{+1} |x, (m, q)\rangle \sqrt{j+1/2} \mathcal{J}_j^{m,q}(x) dx.$$

Note that there is a basis associated to each triple  $(m, q, \mathfrak{g} \in SU(1, 1))$ .

The case of two discrete variables,  $(j, m)$  or  $(j, q)$ , is related to the group  $SO(3, 2)$  and has been discussed in [1], so it will be not reconsidered here.

The case of three variables  $(j, m, q)$  corresponds to  $SU(2, 2)$  and now both  $m$  and  $q$  are modified by the group action. Like in the previous case let us consider the set of eigenvectors of  $J, M, Q$  and also of the Casimir operators  $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C, \mathcal{C}_D, \mathcal{C}_E$  and  $\mathcal{C}_F$

$$\{|j, m, q\rangle; m, q \in \mathbb{Z}/2, j \geq |m|, j \geq |q|, j - m \in \mathbb{N}, j - q \in \mathbb{N}\},$$

where  $\mathbb{Z}/2 = \{0, \pm 1/2, \pm 1, \pm 3/2, \dots\}$ . Hence the action, for instance, of the operators  $C_{\pm}$  on these vectors is easy to compute obtaining

$$C_+|j, m, q\rangle = \sqrt{(j+m+1)(j+q+1)} |j+1/2, m+1/2, q+1/2\rangle,$$

$$C_-|j, m, q\rangle = \sqrt{(j+m)(j+q)} |j-1/2, m-1/2, q-1/2\rangle,$$

and similarly for the other generators of  $su(2, 2)$ . These vectors support the IUR of  $SU(2, 2)$  above described. So

$$\langle j, m, q | j', m', q' \rangle = \delta_{jj'} \delta_{mm'} \delta_{qq'}, \quad \sum_{j, m, q} |j, m, q\rangle \langle j, m, q| = \mathcal{I}.$$

We can define a new set of vectors

$$\{|x, m, q\rangle; x \in \mathbb{E} = (-1, 1) \subset \mathbb{R}, q \in \mathbb{Z}/2, m - q \in \mathbb{Z}\}$$

in terms of the vectors  $|j, m, q\rangle$  and the AJF  $\mathcal{J}_j^{m, q}(x)$  by

$$|x, m, q\rangle := \sum_{j=\sup(|m|, |q|)}^{\infty} |j, m, q\rangle \sqrt{j+1/2} \mathcal{J}_j^{m, q}(x).$$

Orthonormality and completeness are easily obtained

$$\langle x, m, q | x', m', q' \rangle = \delta(x-x') \delta_{mm'} \delta_{qq'}, \quad \sum_{m, q} \int_{-1}^{+1} |x, m, q\rangle dx \langle x, m, q| = \mathcal{I}.$$

The space in this case can be identified as  $\mathbb{E} \times \mathbb{Z} \times \mathbb{Z}/2$  and is the direct sum of the spaces  $\mathbb{E}_{m, q}$  (associated to the configuration space  $E = (-1, 1) \subset \mathbb{R}$ ) with  $m$  and  $q$  fixed,

$$\mathbb{E} \times \mathbb{Z} \times \mathbb{Z}/2 = \bigcup_{m-q \in \mathbb{Z}} \bigcup_{q \in \mathbb{Z}/2} \mathbb{E}_{m, q},$$

where  $\mathbb{Z} \times \mathbb{Z}/2$  is related to the set of pairs  $\{(m - q, q)\}$  since  $m$  and  $q$  are together integer or half-integer.

The  $\{\mathcal{J}_j^{m, q}(x)\}$  are the transition matrices between the two bases  $\{|j, m, q\rangle\}$  and  $\{|x, m, q\rangle\}$

$$\mathcal{J}_j^{m, q}(x) = \frac{1}{\sqrt{j+1/2}} \langle x, m, q | j, m, q \rangle = \frac{1}{\sqrt{j+1/2}} \langle j, m, q | x, m, q \rangle.$$

Like in Ref. [1, 2] the role of the AJF as transition matrices reflects the fact that the algebra generators can be seen as differential operators in  $\mathbb{E} \times \mathbb{Z} \times \mathbb{Z}/2$  or algebraic operators in the space of labels  $\mathbb{N}/2 \times \mathbb{N} \times \mathbb{N}$  related to the set of triplets  $\{(j, j - m, j - q)\}$ . This allows to make explicit the Lie algebra structure in contrast with Ref. [11, 12, 14].

An arbitrary vector  $|f\rangle \in L^2(\mathbb{E}, \mathbb{Z}, \mathbb{Z}/2)$  can be written as

$$|f\rangle = \sum_{m, q=-\infty}^{\infty} \int_{-1}^{+1} dx |x, m, q\rangle f^{m, q}(x) = \sum_{m, q=-\infty}^{+\infty} \sum_{j=\sup(|m|, |q|)}^{\infty} |j, m, q\rangle f_j^{m, q},$$

where

$$f^{m,q}(x) = \langle x, m, q | f \rangle = \sum_{j=\sup(|m|,|q|)}^{\infty} \sqrt{j+1/2} \mathcal{J}_j^{m,q}(x) f_j^{m,q},$$

$$f_j^{m,q} = \langle j, m, q | f \rangle = \int_{-1}^{+1} dx \sqrt{j+1/2} \mathcal{J}_j^{m,q}(x) f^{m,q}(x).$$

So, the  $L^2$ -functions,  $f^{m,q}(x)$ , defined on  $(\mathbb{E}, \mathbb{Z}, \mathbb{Z}/2)$  can be developed as

$$f^{m,q}(x) = \sum_{m,q=-\infty}^{\infty} \sum_{j=\sup(|m|,|q|)}^{\infty} \sqrt{j+1/2} \mathcal{J}_j^{m,q}(x) f_j^{m,q}.$$

Note that since  $\{\mathcal{J}_j^{m,q}\}$  is a basis of a UIR of  $SU(2, 2)$  and, at the same time, a basis of the  $L^2$ -functions defined on  $(\mathbb{E}, \mathbb{Z}, \mathbb{Z}/2)$  then  $L^2(\mathbb{E}, \mathbb{Z}, \mathbb{Z}/2)$  supports the same UIR of  $SU(2, 2)$ . This implies that every change of basis in  $L^2(\mathbb{E}, \mathbb{Z}, \mathbb{Z}/2)$  is related to an element  $\mathfrak{g}$  of  $SU(2, 2)$  and that the operators acting on  $L^2(\mathbb{E}, \mathbb{Z}, \mathbb{Z}/2)$  can be written inside the UEA[ $su(2, 2)$ ].

## 7. Conclusions

The algebraic special functions are relevant since they constitute a bridge between second order differential equations in one dimension and Lie algebras. Moreover they allow to establish a homomorphism between the UEA of the corresponding symmetry Lie algebra and the vector space of the operators defined on the  $L^2$ -functions. In particular the algebraic Jacobi functions coincide with the Wigner  $d_j$ -matrices, which play an important role in Quantum Mechanics. This fact enhances the interest of the algebraic special functions in physical applications.

## References

- [1] Celeghini E and del Olmo M A 2013 *Ann. of Phys.* **333** 90
- [2] Celeghini E and del Olmo M A 2013 *Ann. of Phys.* **335** 78
- [3] Celeghini E, del Olmo M A and Velasco M A 2014 Lie groups of Jacobi polynomials and Wigner  $d$ -matrices (*Preprint* arXiv: 1402.5217)
- [4] Calzada J A, Negro J and del Olmo M A 2006 *J. Math. Phys.* **47** 043511
- [5] Calzada J A, Kuru S, Negro J and del Olmo M A 2008 *J. Phys. A: Math. Theor.* **41** 255201
- [6] Calzada J A, Kuru S, Negro J and del Olmo M A 2012 *Ann. Phys.* **327** 808
- [7] Celeghini E, Kuru S, Negro J and del Olmo M A 2013 *Ann. Phys.* **332** 27
- [8] Wigner E P 1955 *The application of group theory to the special functions of mathematical physics* (Princeton: Princeton Univ. Press)
- [9] Talman J D 1968 *Special functions: a group theoretic approach* (New York: Benjamin)
- [10] Truesdell C 1948 *Annals in Math. Studies* vol **18** (Princeton: Princeton Univ. Press)
- [11] Vilenkin N Ja 1968 *Special Functions and the Theory of Group Representations* (Providence: Amer. Math. Soc.)
- [12] Vilenkin N Ja and Klimyk A U 1991, 1992, 1993 *Representation of Lie Groups and Special Functions* vol 1, 2, 3 (Dordrecht: Kluwer)
- [13] Vilenkin N Ja and Klimyk A U 1995 *Representation of Lie Groups and Special Functions: Recent Advances* (Dordrecht: Kluwer)
- [14] Miller Jr W 1968 *Lie Theory and Special Functions* (New York: Academic Press)
- [15] Miller Jr W 1977 *Symmetry and Separation of Variables* (Reading, MA: Addison-Wesley)
- [16] Schrödinger E 1941 *Proc. Roy. Irish Acad.* **A46**, 183; **A47** 53
- [17] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
- [18] Miller Jr W 1972 *SIAM J. Math. Anal.* **3** 31; *J. Math. Phys.* **13** 1393
- [19] Miller Jr W 1973 *SIAM J. Math. Anal.* **4** 314; *SIAM J. Appl. Math.* **25** 226
- [20] Floreanini R and Vinet L 1996 *J. Math. Phys.* **37** 4135
- [21] Kalnins E G and Miller W 1989 *Rocky Mountain J.* **19** 223
- [22] Wigner E P 1927 *Zeitschrift für Physik* **43** 624
- [23] Biedenharn L C and Louck J D 1981 *Angular Momentum in Quantum Mechanics* (Reading, MA: Addison-Wesley)

- [24] Olver F W J, Lozier D W, Boisvert R F and Clark C W (edit.) 2010 *NIST Handbook of Mathematical Functions* (New York: Cambridge Univ. Press)
- [25] Luke Y L 1969 *The Special Functions and Their Approximations* Vol.1 (San Diego: Academic Press)
- [26] Abramowitz M and Stegun I 1972 *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* (San Diego: Dover)
- [27] Bargmann V 1947 *Ann. of Math.* **48** 368
- [28] Lawson H B and Michelsohn M L 1989 *Spin Geometry* (Princeton: Princeton Univ. Press).
- [29] Celeghini E and del Olmo M A 2014 Quantum physics and signal processing in rigged Hilbert spaces by means of special functions, Lie algebras and Fourier and Fourier-like transforms (to be publish in J. Phys.: Conf. Ser. 2015; *Preprint* arXiv: 1411.3263)