

# Quantum physics and signal processing in rigged Hilbert spaces by means of special functions, Lie algebras and Fourier and Fourier-like transforms

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**Abstract.** Quantum physics and signal processing in the line  $\mathbb{R}$  are strictly related to Fourier transform and Weyl-Heisenberg algebra. We discuss here the addition of a new discrete variable that measures the degree of the Hermite functions and allows to obtain the projective algebra  $io(2)$ . A rigged Hilbert space is found and a new discrete basis in  $\mathbb{R}$  obtained. All operators defined on  $\mathbb{R}$  are shown to belong to the universal enveloping algebra of  $io(2)$  allowing, in this way, their algebraic treatment. Introducing in the half-line a Fourier-like transform, the procedure is extended to  $\mathbb{R}^+$  and can be easily generalized to  $\mathbb{R}^n$  and to spherical coordinate systems.

## 1. Introduction

The starting points of quantum mechanics in the line  $\mathbb{R}$  and its wave functions counterpart  $L^2(\mathbb{R})$  are the Weyl-Heisenberg algebra  $h(1) \equiv \langle X, P, \mathbb{I} \rangle$  [1] that defines position and momentum and the Fourier transform FT [2] that connects them. This is true also in signal processing, also if there the operatorial structure is scarcely taken in consideration.

We consider here a more predictive approach, adding to  $h(1)$  a new operator  $N$  related to the degree of the Hermite functions [3] that became the transition matrices between discrete and continuous bases in quantum space [4]. A projective algebra  $io(2)$  [6, 7] is constructed and a rigged Hilbert space (RHS)[5] found, where  $N$  defines a new discrete basis in  $\mathbb{R}$  and  $L^2(\mathbb{R})$ . Moreover the spaces of the operators defined on  $\mathbb{R}$ ,  $\{\mathcal{O}[\mathbb{R}]\}$ , and on  $L^2(\mathbb{R})$ ,  $\{\mathcal{O}[L^2(\mathbb{R})]\}$ , are isomorphic to the universal enveloping algebra  $UEA[io(2)]$ .

Three bases are indeed obtained, two continuous  $\{|x\rangle\}$  and  $\{|p\rangle\}$  related to position  $X$  and momentum  $P$ - and a new one discrete,  $\{|n\rangle\}$ , connected to the Hermite functions. As in a Hilbert space all bases have the same cardinality, the mathematical structure is a RHS [5], the true space of quantum mechanics. A state in quantum mechanics was indeed originally defined as a ray in a RHS, an intricate concept involving a Gelfand triple  $\phi \subseteq \mathcal{H} \subseteq \phi'$ , where  $\mathcal{H}$  is a Hilbert space,  $\phi$  (dense subset of  $\mathcal{H}$ ) is the “ket” space and  $\phi'$  (dual of  $\phi$ ) is the “bra” space. However quickly RHS has been considered an unnecessary complication as all results can be found (also if by means of elaborated limits on functions with compact support) in the Hilbert space obtained representing, by means of the axiom of choice, each entire ray with a vector of norm one and phase zero.



We attempt to show here that RHS are not so complicated to justify its elimination from all recent textbooks and can be more powerful than standard Hilbert spaces because they allow the operatorial description (and the inclusion inside the same algebra) of variables with different cardinality. Hoping that it could convince more than a general discussion, that can be found in [5], we discuss here the 1-dimensional case.

The projective algebra  $io(2) \equiv \langle X, P, N, \mathbb{I} \rangle$  above introduced is isomorphic to the so called algebra of the harmonic oscillator  $\langle H, a, a^\dagger, \mathbb{I} \rangle$  [6, 7], usually described inside the UEA[ $h(1)$ ]. As the  $\infty$ -dimensional UEA[ $io(2)$ ] is a complicated object, a set of relevant subspaces emerges in  $\mathbb{R}$  and  $\{\mathcal{O}[\mathbb{R}]\}$  as well as in  $L^2(\mathbb{R})$  and  $\{\mathcal{O}[L^2(\mathbb{R})]\}$ . In particular, as  $\infty$ -many copies of  $io(2)$  are contained into UEA[ $io(2)$ ],  $\infty$ -many operators that close  $io(2)$  are found.

Note that in  $L^2(\mathbb{R})$  (but not in  $\mathbb{R}$ ,  $\{\mathcal{O}[\mathbb{R}]\}$  and  $\{\mathcal{O}[L^2(\mathbb{R})]\}$ ) the subspaces derived from the algebra can be obtained also from the fractional Fourier transform (FrFT) [8]. Indeed the eigenvectors of the FrFT are, as the ones of FT, the Hermite functions and for each  $k \in \mathbb{N}$  a FrFT can be constructed such that its eigenvalues are the  $k$ th roots of unity, each one corresponding to one subspace.

The results stated in  $\mathbb{R}$  can be easily extended to orthogonal and pseudo-orthogonal spaces of any dimension by means of a tensorial construction.

Finally to exhibit that the approach is not peculiar of  $\mathbb{R}$ , the discussion is repeated on the half line  $\mathbb{R}^+$ . For this purpose, starting from the Hermite functions and the relations between the Hermite functions and the generalized Laguerre functions, two Fourier-like transforms are constructed each one related to a different definition of conjugation of bases.

## 2. The line $\mathbb{R}$

To describe the line we start from the unitary irreducible representations of the translation group  $T^1$

$$P|p\rangle = p|p\rangle, \quad U^p(x)|p\rangle = e^{-ipx}|p\rangle.$$

The regular representation  $\{|p\rangle\}$  ( $-\infty < p < \infty$ ) is such that

$$\langle p|p'\rangle = \sqrt{2\pi} \delta(p-p'), \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |p\rangle dp \langle p| = \mathbb{I}$$

and the conjugate basis  $\{|x\rangle\}$ , defined by the operator  $X$ , is obtained from the FT

$$|x\rangle := \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dp e^{-ipx} \right] |p\rangle, \quad \langle x|x'\rangle = \sqrt{2\pi} \delta(x-x'), \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |x\rangle dx \langle x| = \mathbb{I}.$$

The operators  $X$  and  $P$  close, together with  $\mathbb{I}$ , the Weyl-Heisenberg algebra [1]. At this point, we move to a non standard approach related to the ray representations [6, 7, 9] of the inhomogeneous orthogonal algebra  $io(2)$ . In addition to  $X$  and  $P$  (which spectra have the cardinality of the continuum  $\aleph_1$ ) an operator  $N$  (with spectrum of cardinality  $\aleph_0$ ), related to the index  $n$  of the Hermite polynomials  $H_n(x)$ , is indeed introduced. As the Hermite functions

$$\psi_n(x) := \frac{e^{-x^2/2}}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x),$$

are a basis of the space of real/complex square integrable functions on the line  $L^2((-\infty, \infty)) \equiv L^2(\mathbb{R})$  [2]:

$$\int_{-\infty}^{\infty} \psi_n(x) \psi_{n'}(x) dx = \delta_{n,n'}, \quad \sum_{n=0}^{\infty} \psi_n(x) \psi_n(x') = \delta(x-x').$$

The basic idea is now to introduce the set of vectors  $\{|n\rangle\}$

$$|n\rangle := (2\pi)^{-1/4} \int_{-\infty}^{\infty} dx \psi_n(x) |x\rangle \quad n \in \mathbb{N}, \quad (1)$$

that, by inspection, is an orthonormal and complete set in  $\mathbb{R}$

$$\langle n | n' \rangle = \delta_{nn'}, \quad \sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbb{I}.$$

Also  $\{|n\rangle\}$  is thus a discrete basis in the real line  $\mathbb{R}$ , i.e.  $\mathbb{R} \equiv \{|p\rangle\} \equiv \{|x\rangle\} \equiv \{|n\rangle\}$ .

Relations among the three bases are easily established, as  $\{\psi_n(x)\}$  are eigenvectors of FT,

$$[\text{FT}] \psi_n(x) = \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{ipx} \right] \psi_n(x) = \mathbf{i}^n \psi_n(p), \quad (2)$$

$$\begin{aligned} |x\rangle &= (2\pi)^{1/4} \sum_{n=0}^{\infty} \psi_n(x) |n\rangle, & |p\rangle &= \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{ipx} \right] |x\rangle, \\ |n\rangle &= \mathbf{i}^n (2\pi)^{-1/4} \int_{-\infty}^{\infty} dp \psi_n(p) |p\rangle, & |p\rangle &= (2\pi)^{1/4} \sum_{n=0}^{\infty} \mathbf{i}^n \psi_n(p) |n\rangle. \end{aligned}$$

For an arbitrary vector  $|f\rangle \in \mathbb{R}$  we thus have

$$\begin{aligned} |f\rangle &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx f(x) |x\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dp f(p) |p\rangle = \sum_{n=0}^{\infty} f_n |n\rangle, \\ f(x) &:= \langle x|f\rangle = (2\pi)^{1/4} \sum_{n=0}^{\infty} \psi_n(x) f_n, & f(p) &:= \langle p|f\rangle = (2\pi)^{1/4} \sum_{n=0}^{\infty} (-\mathbf{i})^n \psi_n(p) f_n, \\ f_n &:= \langle n|f\rangle = (2\pi)^{-1/4} \int_{-\infty}^{+\infty} dx \psi_n(x) f(x) = \mathbf{i}^n (2\pi)^{-1/4} \int_{-\infty}^{+\infty} dp \psi_n(p) f(p) \end{aligned}$$

and the wave functions  $f(x), f(p)$  and the sequence  $\{f_n\}$  describe  $|f\rangle$  in the three bases.

All seems trivial, but  $\{|n\rangle\}$  has the cardinality of the natural numbers  $\mathbb{N}_0$  and, as all bases in a Hilbert space have the same cardinality, the structure we have constructed (the quantum space on the line  $\mathbb{R}$ ) is not a Hilbert space but a rigged Hilbert space.

In this way, by means of FT and Hermite functions, we went back to the foundations of quantum mechanics, where a physical state was defined in a Hilbert space as a ray and not as a vector, as we are used considering RHS as an unnecessary complication. However the problem is apparent also in the simple case of the harmonic oscillator: the algebraic description consider  $H (\equiv N + 1/2)$ ,  $a$ ,  $a^\dagger$  and  $\mathbb{I}$ . But  $X = \frac{1}{\sqrt{2}}(a + a^\dagger)$  and  $P \equiv -\mathbf{i} D_x = \frac{-\mathbf{i}}{\sqrt{2}}(a - a^\dagger)$  do not belong to the algebra because they are not operators in a Hilbert space. In a RHS, instead,  $X$  and  $P$  are standard operators also if their spectra are continuous and we have no problems to consider them as generators of a Lie algebra together with the number operator  $N$  [4, 5]. This allows us to include all differential operators inside the Lie algebra and to extend the set of operators defined in the universal enveloping algebra.

### 3. Algebra of the line $\mathbb{R}$

To describe the structure of the operators on the line we introduce now in  $L^2(\mathbb{R})$  the operators  $X$ ,  $D_x(= \mathbf{i}P)$ ,  $N$  and  $\mathbb{I}$

$$X\psi_n(x) := x\psi_n(x), \quad D_x\psi_n(x) := \psi'_n(x), \quad N\psi_n(x) := n\psi_n(x), \quad \mathbb{I}\psi_n(x) := \psi_n(x).$$

Recurrence relations of Hermite polynomials

$$H'_n(x) = 2n H_{n-1}(x), \quad H'_n(x) - 2x H_n(x) = H_{n+1}(x)$$

can be rewritten as

$$a\psi_n(x) = \sqrt{n}\psi_{n-1}(x), \quad a^\dagger\psi_n(x) = \sqrt{n+1}\psi_{n+1}(x),$$

where the operators  $a$  and  $a^\dagger$  are defined in terms of the operators  $X$  and  $P$

$$a := \frac{1}{\sqrt{2}}(X + \mathbf{i}P), \quad a^\dagger := \frac{1}{\sqrt{2}}(X - \mathbf{i}P).$$

The algebra contains the rising and lowering operators on the Hermite functions [10, 11]

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a, a^\dagger] = \mathbb{I}, \quad [\mathbb{I}, \bullet] = 0 \quad (3)$$

and is isomorphic to the projective algebra  $io(2)$  [6, 7]:

$$[N, X] = -\mathbf{i}P, \quad [N, P] = \mathbf{i}X, \quad [X, P] = \mathbf{i}\mathbb{I}, \quad [\mathbb{I}, \bullet] = 0. \quad (4)$$

Because of Eq.(1),  $\mathbb{R}$  and  $L^2(\mathbb{R})$  are isomorphic and, as discussed in [9], they are representations with zero value of the Casimir operator of the projective algebra (4) (or equivalently (3)),

$$C \equiv (X^2 - D_x^2)/2 - (N + 1/2)\mathbb{I} = \{a, a^\dagger\}/2 - (N + 1/2)\mathbb{I} = 0. \quad (5)$$

On the representations we can assume  $\mathbb{I} = 1$  and write

$$C|n\rangle = \left[\{a, a^\dagger\}/2 - N - 1/2\right]|n\rangle = 0$$

that in  $L^2(\mathbb{R})$  can be alternatively written as

$$C\psi_n(x) = [(X^2 - D_x^2)/2 - N - 1/2]\psi_n(x) = 0.$$

Eq.(5) gives indeed the operatorial identity that defines  $\mathbb{R}$  and  $L^2(\mathbb{R})$

$$N \equiv (X^2 - D_x^2 - 1)/2 \equiv \{a, a^\dagger\}/2 - 1/2, \quad (6)$$

by inspection equivalent to the Hermite differential equation:

$$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0.$$

#### 4. Universal enveloping algebra of $io(2)$

The representations are irreducible, so that, on both spaces  $L^2(\mathbb{R})$  and  $\mathbb{R}$ , all operators of the  $UEA[io(2)]$  are defined and an isomorphism exists between the  $UEA[io(2)]$  and the space of the operators  $\{\mathcal{O}[L^2(\mathbb{R})]\}$  and  $\{\mathcal{O}[\mathbb{R}]\}$  :

$$\{\mathcal{O}[L^2(\mathbb{R})]\} \equiv UEA[io(2)] \equiv \{\mathcal{O}[\mathbb{R}]\} , \quad (7)$$

i.e. each operator  $\mathcal{O}$  can be written

$$\mathcal{O} = \sum c_{\alpha\beta\gamma} X^\alpha D_x^\beta N^\gamma = \sum d_{\alpha\beta\gamma} a^{\dagger\alpha} N^\beta a^\gamma.$$

From the analytical point of view, an ordered monomial  $X^\alpha D_x^\beta N^\gamma \in UEA[io(2)]$  is an order  $\beta$  differential operator but, because of the operatorial identity (6), we have

$$D_x^2 \equiv X^2 - 2N - 1,$$

and any operator in  $\{\mathcal{O}[L^2(\mathbb{R})]\}$  and in  $\{\mathcal{O}(\mathbb{R})\}$  can be written

$$\mathcal{O} = f_0(X) g_0(N) + f_1(X) D_x g_1(N)$$

that on the basis vector  $\psi_n(x)$  becomes

$$\mathcal{O} \psi_n(x) = f_0(x) g_0(n) \psi_n(x) + f_1(x) g_1(n) \psi'_n(x).$$

#### 5. Substructures in the line

The  $UEA[io(2)]$  is a rich structure and it contains  $\infty$ -many  $io(2)$  algebras that allow to construct an intricate structure of subspaces. Moreover for any  $k \in \mathbb{N}$  we have

$$UEA[io(2)] \supset \bigoplus_{q=0}^{k-1} io_{k,q}(2) ,$$

where  $k$  and  $q$  are parameters that define each copy  $io_{k,q}(2)$  of the algebra  $io(2)$ . Indeed, starting from  $n$  and  $k$ , we can define two other integers  $r := \text{Quotient}[n, k]$  and  $q := \text{Mod}[n, k]$  so that  $n = kr + q$  and the corresponding operators  $R$  and  $Q$  are, of course, diagonal on  $\{|kr + q\rangle\}$

$$R |kr + q\rangle = r |kr + q\rangle , \quad Q |kr + q\rangle = q |kr + q\rangle .$$

By inspection, the operators  $A_{k,q}^\dagger, A_{k,q} \in UEA[io(2)]$

$$A_{k,q}^\dagger := (a^\dagger)^k \frac{\sqrt{N+k-q}}{\sqrt{k \prod_{j=1}^k (N+j)}} , \quad A_{k,q} := \frac{\sqrt{N+k-q}}{\sqrt{k \prod_{j=1}^k (N+j)}} (a)^k \quad (8)$$

are defined on the whole set  $\{|kr + q\rangle\}$  and give

$$A_{k,q}^\dagger |kr + q\rangle = \sqrt{r+1} |k(r+1) + q\rangle , \quad A_{k,q} |kr + q\rangle = \sqrt{r} |k(r-1) + q\rangle .$$

Each couple  $(k, q)$  gives thus a representation with  $C = 0$  of the algebra  $io(2)$ , we denote  $io_{k,q}(2)$  :

$$[R, A_{k,q}^\dagger] = +A_{k,q}^\dagger \quad [R, A_{k,q}] = -A_{k,q} \quad [A_{k,q}, A_{k,q}^\dagger] = \mathbb{I} \quad [\mathbb{I}, \bullet] = 0.$$

In particular, for  $k = 4$  and  $0 \leq q < 4$ , we can define

$$\mathbb{R}_{4,q} := \{|4r + q\rangle\} , \quad L_{4,q}^2(\mathbb{R}) := \{\psi_{4r+q}(x)\} \quad (r = 0, 1, 2, \dots) ,$$

so that the operators  $\{\mathcal{O}[L^2_{4,q}(\mathbb{R})]\}$  and  $\{\mathcal{O}[\mathbb{R}_{4,q}]\}$  acting inside each of them belong to  $UEA[io_{4,q}(2)]$

$$\{\mathcal{O}[L^2(\mathbb{R}_{4,q})]\} \equiv UEA[io_{4,q}(2)] \equiv \{\mathcal{O}[\mathbb{R}_{4,q}]\}.$$

The spaces  $L^2(\mathbb{R})$  and  $\mathbb{R}$  can be thus divided in four subspaces each one supporting the corresponding representation of  $io_{4,q}(2)$ :

$$L^2(\mathbb{R}) = \oplus_{q=0}^3 L^2_{4,q}(\mathbb{R}) = \oplus_{q=0}^3 \{\psi_{4r+q}(x)\}, \quad (9)$$

$$\mathbb{R} = \oplus_{q=0}^3 \mathbb{R}_{4,q} = \oplus_{q=0}^3 \{|4r+q\rangle\}.$$

These results are general as Eqs. (8) allow to write for all  $k \in \mathbb{N}$ ,  $0 \leq q < k$  and  $r = 0, 1, 2, \dots$

$$L^2_{k,q}(\mathbb{R}) := \{\psi_{kr+q}(x)\}, \quad \mathbb{R}_{k,q} := \{|kr+q\rangle\},$$

$$\{\mathcal{O}[L^2(\mathbb{R}_{k,q})]\} \equiv UEA[io_{k,q}(2)] \equiv \{\mathcal{O}[\mathbb{R}_{k,q}]\},$$

$$L^2(\mathbb{R}) = \oplus_{q=0}^{k-1} L^2_{k,q}(\mathbb{R}) = \oplus_{q=0}^{k-1} \{\psi_{kr+q}(x)\}, \quad \mathbb{R} = \oplus_{q=0}^{k-1} \mathbb{R}_{k,q} = \oplus_{q=0}^{k-1} \{|kr+q\rangle\}.$$

Eq.(9) can also be obtained from FT (2) that divides  $L^2(\mathbb{R})$  in four subspaces each one corresponding to one eigenvalue of FT (but does not give any information about the operators). Besides its generalization to all  $k \in \mathbb{N}$  allows to divide  $L^2(\mathbb{R})$  in  $k$  subspaces (again disregarding the operators) and can be obtained from the fractional Fourier transform (FrFT) [8]:

$$[\text{FrFT}]_{\alpha} \psi_n(x) = e^{i\alpha n} \psi_n(x') \quad \alpha \in \mathbb{C}.$$

Indeed specializing  $\alpha$  to  $\alpha = 2\pi/k$  we get

$$[\text{FrFT}]_{2\pi/k} \psi_n(x) = e^{i\frac{2\pi n}{k}} \psi_n(x'), \quad (10)$$

$$[\text{FrFT}]_{2\pi/k} f(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{ixx'} e^{2\pi i(1/k-1/4)N} f(x) = [\text{FT}] \left[ e^{2\pi i(1/k-1/4)N} \right] f(x)$$

where  $N$  -diagonal on each  $\psi_n(x)$ - on the whole space is rewritten as  $N \equiv (X^2 - D_x^2 - 1)/2$ . As an example, for  $k = 3$  Eq. (10) gives

$$[\text{FrFT}]_{2\pi/3} \psi_n(x) = e^{i\frac{2\pi n}{3}} \psi_n(x'),$$

that, similarly to Eq.(9), exhibits three subspaces of  $L^2(\mathbb{R})$

$$L^2(\mathbb{R}) = L^2_{3,0}(\mathbb{R}) \oplus L^2_{3,1}(\mathbb{R}) \oplus L^2_{3,2}(\mathbb{R}) = \{\psi_{3r}\} \oplus \{\psi_{3r+1}\} \oplus \{\psi_{3r+2}\},$$

but, like FT, does not give any information about the operators acting inside each of them.

## 6. The half-line $\mathbb{R}^+$

The results obtained on the line  $\mathbb{R}$  can be rewritten in  $\mathbb{R}^+$ , the vector space defined by the operator  $Y$ , with eigenvectors  $\{|y\rangle\}$  where  $y$  is into the open set  $(0 < y < \infty)$ :

$$Y |y\rangle = y |y\rangle, \quad \langle y | y' \rangle = \delta(y - y'), \quad \int_0^{\infty} |y\rangle dy \langle y| = \mathbb{I}.$$

The generalized Laguerre polynomials  $L_n^\alpha(y)$  ( $n \in \mathbb{N}$ ) play now the role of the Hermite polynomials. For each value of  $\alpha$  an alternative discrete basis  $\{|n\rangle\}$  is obtained as follows. From  $L_n^\alpha(y)$  the normalized generalized Laguerre functions  $M_n^\alpha(y)$  are

$$M_n^\alpha(y) := \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}} y^{\alpha/2} e^{-y/2} L_n^\alpha(y) ,$$

that satisfy

$$\int_0^\infty M_n^\alpha(y) M_m^\alpha(y) dy = \delta_{nm} , \quad \sum_{n=0}^\infty M_n^\alpha(y) M_n^\alpha(y') = \delta(y-y')$$

so that, like the set  $\{\psi_n(x)\}$  in  $L^2(\mathbb{R})$ , the set  $\{M_n^\alpha(x)\}$  is -for fix and arbitrary  $\alpha$ - a basis in  $L^2(\mathbb{R}^+)$  [2]. Again, like in (1), we define -for  $n \in \mathbb{N}$ - the vector  $|n\rangle$

$$|n\rangle := \int_0^\infty dy M_n^\alpha(y) |y\rangle$$

and we have

$$\langle n|n'\rangle = \delta_{nm} , \quad \sum_{n=0}^\infty |n\rangle\langle n| = \mathbb{I} ,$$

i.e.  $\{|n\rangle\}_{n \in \mathbb{N}}$  is an orthonormal basis of the half-line  $\mathbb{R}^+$  and  $M_n^\alpha(y)$  are the transformations matrices that relate the two bases  $\{|y\rangle\}$  and  $\{|n\rangle\}$  :

$$M_n^\alpha(y) = \langle y|n\rangle = \langle n|y\rangle .$$

Like in the line, an arbitrary vector  $|f\rangle \in \mathbb{R}^+$  can be written

$$|f\rangle = \int_0^\infty dy f(y) |y\rangle = \sum_{n=0}^\infty |n\rangle f_n ,$$

$$f(y) := \langle y|f\rangle = \sum_{n=0}^\infty M_n^\alpha(y) f_n , \quad f_n := \langle n|f\rangle = \int_0^\infty dy M_n^\alpha(y) f(y) ,$$

so that the wave function  $f(y)$  and the sequence  $\{f_n\}$  describe the vector  $|f\rangle$  in the two bases.

As one discrete basis  $\{|n\rangle\}$  and one continuous one  $\{|y\rangle\}$  have been found also the half-line is a RHS. The third basis will be constructed in Sect. 9.

## 7. Algebra of the half-line

We introduce now the associated algebra. For  $L_n^\alpha(y)$  the recurrence relations are

$$y L_n^\alpha(y)' + (n + \alpha + 1 - y) L_n^\alpha(y) = (n + 1) L_{n+1}^\alpha(y) ,$$

$$-y L_n^\alpha(y)' + n L_n^\alpha(y) = (n + \alpha) L_{n-1}^\alpha(y)$$

that, defining the operators  $Y$ ,  $D_y$ ,  $\mathbb{N}$  and  $\mathbb{I}$ ,

$$Y M_n^\alpha(y) := y M_n^\alpha(y) , \quad D_y M_n^\alpha(y) := M_n^\alpha(y)' , \quad N M_n^\alpha(y) := n M_n^\alpha(y) , \quad \mathbb{I} M_n^\alpha(y) := M_n^\alpha(y) ,$$

can be rewritten

$$\begin{aligned} J_+ M_n^\alpha(y) &:= \left( Y D_y + N + 1 + \frac{\alpha - Y}{2} \right) M_n^\alpha(y) = \sqrt{(n+1)(n+\alpha+1)} M_{n+1}^\alpha(y) , \\ J_- M_n^\alpha(y) &:= \left( -Y D_y + N + \frac{\alpha - Y}{2} \right) M_n^\alpha(y) = \sqrt{n(n+\alpha)} M_{n-1}^\alpha(y) . \end{aligned} \quad (11)$$

By inspection we see that  $J_\pm = J_\mp^\dagger$ , i.e.  $J_\mp$  is the hermitian conjugate operator of  $J_\pm$ . Note that, unlike the full line, to define the rising and lowering operators the operator  $N$  is needed.

To complete eqs. (11), we introduce now  $J_3 := N + (\alpha + 1)/2$

$$J_3 M_n^\alpha(y) = [N + (\alpha + 1)/2] M_n^\alpha(y) = (n + (\alpha + 1)/2) M_n^\alpha(y) , \quad (12)$$

that, together with  $J_\pm$ , close the  $su(1,1)$  algebra

$$[J_3, J_\pm] = \pm J_\pm , \quad [J_+, J_-] = -2 J_3$$

of which, as shown by Eqs.(11,12),  $\{M_n^\alpha(y)\}$  are an unitary irreducible representation. Furthermore, as  $\{|n\rangle\}$  is isomorphic to  $\{M_n^\alpha(y)\}$ , we have

$$\begin{aligned} J_+ |n\rangle &= \sqrt{(n+1)(n+\alpha+1)} |n+1\rangle , \\ J_3 |n\rangle &= (n + (\alpha + 1)/2) |n\rangle , \\ J_- |n\rangle &= \sqrt{n(n+\alpha)} |n-1\rangle . \end{aligned}$$

The Casimir operator  $\mathcal{C}$  is :

$$\mathcal{C} = \left( J_3^2 - \frac{1}{2} \{J_+, J_-\} \right) = \frac{\alpha^2 - 1}{4} ,$$

that, remembering Eqs.(11,12), can be written as

$$\left[ Y D_y^2 + D_y + N + \frac{\alpha + 1}{2} - \frac{\alpha^2}{2Y} - \frac{Y}{4} \right] M_n^\alpha(y) = 0 , \quad (13)$$

reformulation of the associated Laguerre differential equation:

$$y L_n^\alpha(y)'' + (\alpha + 1 - y) L_n^\alpha(y)' + n L_n^\alpha(y) = 0 .$$

Eq.(13) defines the operatorial identity of the half-line, similar to Eq.(6) of the line,

$$N \equiv -Y D_y^2 - D_y - \frac{\alpha + 1}{2} + \frac{\alpha^2}{4Y} + \frac{Y}{4} . \quad (14)$$

## 8. UEA and substructures on the half-line

Also in the half line, the representation is irreducible so that the operators acting on the  $L^2(\mathbb{R}^+)$  and  $\mathbb{R}^+$  belong to the  $UEA[su(1,1)]$ , i.e. can be written as

$$\mathcal{O} = \sum c_{\alpha\beta\gamma} J_+^\alpha J_3^\beta J_-^\gamma = \sum d_{\alpha\beta\gamma} Y^\alpha D_y^\beta N^\gamma$$



and the space of the operators  $\{\mathcal{O}[L^2(\mathbb{R}^+)]\}$  and  $\{\mathcal{O}[\mathbb{R}^+]\}$  are isomorphic to the  $\text{UEA}[su(1,1)]$ ;

$$\{\mathcal{O}[L^2(\mathbb{R}^+)]\} \equiv \text{UEA}[su(1,1)] \equiv \{\mathcal{O}[\mathbb{R}^+]\}.$$

The monomials  $J_+^\alpha J_3^\beta J_-^\gamma \in \text{UEA}[su(1,1)]$  look to be differential operators of order  $\alpha + \gamma$  but the identity (14) can also be read

$$D_y^2 \equiv -\frac{1}{Y} \left( D_y + N + \frac{\alpha+1}{2} - \frac{\alpha^2}{2Y} - \frac{Y}{4} \right)$$

so that all the off-diagonal operators can be written as zero or one order differential operators and all the diagonal ones as zero order or equivalent to Eq.(13).

Also the  $\text{UEA}[su(1,1)]$  contains, in analogy with the  $\text{UEA}[io(2)]$ ,  $\infty$ -many  $su(1,1)$ ,  $su_{k,q}(1,1)$ . We can define  $L_{k,q}^2(\mathbb{R}^+) := \{M_{kr+q}(\mathbb{R}^+)\}$  and  $\mathbb{R}_{k,q}^+ := \{|kr+q\rangle\}$  so that for each  $k \in \mathbb{N}$  the spaces  $L^2(\mathbb{R}^+)$  and  $\mathbb{R}^+$  are sums of  $k$  subspaces

$$L^2(\mathbb{R}^+) = \bigoplus_{q=0}^{k-1} L_{k,q}(\mathbb{R}^+), \quad \mathbb{R}^+ = \bigoplus_{q=0}^{k-1} \mathbb{R}_{k,q}^+.$$

The same subspaces structure is found in the spaces of operators  $\{\mathcal{O}[L^2(\mathbb{R}^+)]\}$  and  $\{\mathcal{O}(\mathbb{R}^+)\}$ , i.e. we have

$$\{\mathcal{O}[L^2(\mathbb{R}_{k,q}^+)]\} \equiv \text{UEA}[su_{k,q}(1,1)] \equiv \{\mathcal{O}[\mathbb{R}_{k,q}^+]\}.$$

## 9. Fourier-like transforms on $\mathbb{R}^+$

Finally to find the conjugate basis of  $\{|y\rangle\}$ , we need something that plays in  $\mathbb{R}^+$  the role of the FT in  $\mathbb{R}$ , i.e. an integral transform that has as eigenvectors the  $\{M_n^\alpha(y)\}$ . Remembering the fact that Hermite functions are eigenstates of FT and the well known relations [3]

$$\psi_{2n}(x) = (-1)^n (x^2)^{1/4} M_n^{-1/2}(x^2), \quad \psi_{2n+1}(x) = (-1)^n x (x^2)^{-1/4} M_n^{+1/2}(x^2)$$

two transforms  $T^\pm$  are found

$$\begin{aligned} [T^+] f(y) &:= \left[ \frac{(-1)^n}{\sqrt{2\pi}} \int_0^\infty dy \frac{\sin[\sqrt{yy'}]}{(yy')^{1/4}} \right] f(y), \\ [T^-] f(y) &:= \left[ \frac{(-1)^n}{\sqrt{2\pi}} \int_0^\infty dy \frac{\cos[\sqrt{yy'}]}{(yy')^{1/4}} \right] f(y), \end{aligned}$$

that have  $M_n^{\pm 1/2}(y)$  as eigenvectors and  $(-1)^n$  as eigenvalues

$$[T^\pm] M_n^{\pm 1/2}(y) = (-1)^n M_n^{\pm 1/2}(y').$$

In this way, we find two alternative conjugate basis of  $\{|y\rangle\}$

$$|q\rangle^+ = \frac{(-1)^n}{\sqrt{2\pi}} \int_0^\infty dy \frac{\sin[\sqrt{qy}]}{(qy)^{1/4}} |y\rangle, \quad |q\rangle^- = \frac{(-1)^n}{\sqrt{2\pi}} \int_0^\infty dy \frac{\cos[\sqrt{qy}]}{(qy)^{1/4}} |y\rangle$$

and two different related substructures

$$L^2(\mathbb{R}^+) = L_0^2(\mathbb{R}^+)^{\pm} \oplus L_1^2(\mathbb{R}^+)^{\pm} = \{M_{2n}^{\pm 1/2}\} \oplus \{M_{2n+1}^{\pm 1/2}\} \quad \forall n \in \mathbb{N}. \quad (15)$$

Like in  $\mathbb{R}$ , the subspaces of Eq.(15) (but again not the operators acting on them) can be generalized introducing the fractional transforms  $[\text{Fr}T^\pm]$

$$[\text{Fr}T^\pm]_{2\pi/k} f(y) := [T^\pm] \left[ e^{2\pi i(1/k-1/2)N} \right] f(y),$$

where  $N$  is given by the identity (14) and is diagonal on  $M_n^{\pm 1/2}(y)$

$$[\text{Fr}T^\pm]_{2\pi/k} M_n^{\pm 1/2}(y) = e^{i2\pi n/k} M_n^{\pm 1/2}(y').$$

## 10. Conclusions

Rigged Hilbert spaces are shown to be more predictive than Hilbert spaces, both in quantum physics and signal processing in optics and informatics, as operators of different cardinality can be considered together.

In rigged Hilbert spaces continuous and discrete bases exist with special functions as transformation matrices between them.

In a RHS, operators of different cardinality can be together generators of a Lie algebra and/or elements of an universal enveloping algebra.

An elaborate algebraic structure is found, inside both the quantum space and the space of operators defined on it. In particular, an infinite set of substructures emerges both in the space of the states and in the space of operators acting on them.

An alternative definition of integral transform, as an operator that has special functions as eigenvectors, has been introduced allowing -unlike for the case of  $\mathbb{R}$ - to give for  $\mathbb{R}^+$  two alternative definitions of conjugate basis.

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