

Boundary conditions, semigroups, quantum jumps, and the quantum arrow of time

Arno Bohm

Department of Physics, University of Texas at Austin, Austin TX 78712, USA

E-mail: bohm@physics.utexas.edu

Abstract. Experiments on quantum systems are usually divided into preparation of states and the registration of observables. Using the traditional mathematical methods (the Hilbert space and Schwartz space of distribution theory), it is not possible to distinguish mathematically between observables and states. The Hilbert space as well as Schwartz space boundary conditions for the dynamical equations lead by mathematic theorems (Stone-von Neumann) to unitary group with $-\infty < t < \infty$. But in the experimental set-up, one clearly distinguishes between the preparation of a state and the registration of an observable in that state. Furthermore, a state must be prepared first before an observable can be measured in this state (causality). This suggests time asymmetric boundary conditions for the dynamical equations of quantum theory. Such boundary conditions have been provided by Hardy space in the Lax-Phillips theory for electromagnetic and acoustic scattering phenomena. The Paley-Wiener theorem for Hardy space then leads to semi-group and time asymmetry in quantum physics. It introduces a finite “beginning of time” t_0 for a time asymmetric quantum theory, which have been observed as an ensemble of finite times $t_0^{(i)}$, the onset times of dark periods in the quantum jump experiments on a single ion.

1. Introduction

The choice of the right boundary conditions for the dynamical equation of quantum physics (e.g. Schrödinger or Heisenberg equation) is as important as the equation, or even more so. The boundary condition specify the space of allowed solutions. The first boundary condition introduced by von Neumann was the Hilbert space boundary condition; it was followed by the Schwartz space boundary condition which justifies the Dirac formalism. Both (Hilbert space \mathcal{H} and Schwartz space Φ) lead by mathematical theorems [1][2] to (unitary for \mathcal{H}) group evolutions with the time extending over $-\infty < t < \infty$. But heuristic arguments (causality) requires a beginning of time, namely a time t_0 at which the quantum state of a (scattering) experiment has been prepared and after which the registration (or detection) of an observable can begin. Thus time t in the quantum theory should extend over $t_0 < t < \infty$. This beginning of time t_0 for an ensemble of quanta, is observed as an ensemble of finite beginnings of time $\{t_0^{(i)}\} \longleftrightarrow t_0$, and t_0 is thus the time at which the state ϕ^+ describing this ensemble has been prepared, and after which the observable can be registered by a detector (i.e. can be detected). The existence of such time t_0 described by an ensemble of times $\{t_0^{(i)}\}$ has been established in experiments on single quantum systems (like the onset times $\{t_0^{(i)}\}$ of the i -th dark period in Dehmelt's and others [3] quantum jump experiments on a single ion). The existence of this t_0 suggests that one replaces the Hilbert space boundary condition, as well as the Schwartz space boundary condition



for the dynamical equations, by some other boundary conditions for the dynamical (Schrödinger or Heisenberg) equations, which do not lead to the standard unitary group evolution of [1] or to the group evolution in the Schwartz space Φ [2], with t extending over $-\infty < t < \infty$.

Such a pair of new boundary condition for the dynamical equation is provided by the Hardy spaces boundary condition (cf. also in the Lax-Phillips theory [4]). Using this pair of Hardy space boundary conditions for the solutions of the dynamical equation (Schrödinger and Heisenberg equation) of quantum theory leads, by another mathematical theorem, the Paley-Wiener theorem, to a quantum mechanical semi-group evolution with $t_0 < t < \infty$. The Hardy space boundary condition thus leads to time asymmetry of a new quantum theory with finite “beginnings of time” $t_0^{(i)}$. This ensemble of individual beginnings of time for an individual micro-system is exactly what is observed by the onset times $t_0^{(i)}$ of the dark-period in numerous quantum jump experiments [3]. The Hardy space axiom, using a pair of spaces, Φ_- for the states and Φ_+ for the observables, also provides the means to distinguish mathematically between (accelerator prepared) states and (detector registered) observables, in scattering experiments like e.g. Fig.1 below.

The main new aspects of the Hardy space boundary conditions are:

- 1.) the finite beginnings of time $\{t_0^{(i)}\}$ and
- 2.) the equality of inverse width \hbar/Γ of the Breit-Wigner resonance (e.g. corresponding to a pole of the S -matrix) and the lifetime τ for its exponentially decaying state, which fulfill $\tau = \hbar/\Gamma$.

The Hardy space axiom provides the most promising mathematical theory for quantum physics. It combines resonance scattering and decay phenomena into one unified theory of scattering and decay phenomena. This time asymmetric quantum theory can be extended into the relativistic domain by replacing the non-relativistic energy E by the relativistic invariant mass squared: $E \rightarrow s^2 = E^2 - \vec{p}^2$ [5].

2. Boundary Conditions and Dynamical Equations

Quantum physics separates roughly into two areas which are characterized by the two groups that govern the space-time motion, the Galileo group for non-relativistic motions, and the Poincare group for motion in relativistic space-time. For the non-relativistic theory which we shall discuss here only, one usually works with the solutions of the dynamical differential equation:

The Schrödinger equation

$$i\hbar \frac{\partial \phi^+(t)}{\partial t} = H \phi^+(t) \quad (1)$$

for the state vector $\phi^+(t)$.

or

One could also consider the von Neumann equation for a more complicated state described by the density operator $\rho(t)$:

$$i\hbar \frac{\partial \rho(t)}{\partial t} = [H, \rho(t)] \quad (2)$$

Or the Heisenberg equation

$$i\hbar \frac{\partial A(t)}{\partial t} = -[H, A(t)] \quad (3)$$

for the observables (operator) $A(t)$

And one could consider the Heisenberg equation for the special case of the simple observable $A_{\psi^-} = |\psi^-(t)\rangle \langle \psi^-(t)|$; then one can use in place of A_{ψ^-} the “observable vector” ψ^- and in place of (3) the “vector-Heisenberg” equation:

$$i\hbar \frac{\partial \psi^-(t)}{\partial t} = -H \psi^-(t) \quad (4)$$

We shall use for our discussion here the simplest situation: the Schrödinger equation (1) for the state vector $\phi(t)$; and for the observable $A(t)$, we restrict ourself to the Heisenberg equation (4) using the “observable vector” $\psi^-(t)$ of the special case $A_{\psi^-} = |\psi^-(t)\rangle \langle \psi^-(t)|$. The state

vector ϕ^+ of (1), as well as the “observable” vector ψ^- of (4), represent physical apparatuses in laboratory experiments. This is depicted in Fig.1 for the example of the preparation of the “simplest” state $\phi^+(t)$ and the registration of the “simplest” observable $|\psi^-\rangle\langle\psi^-|$, in terms of detector counts of the $K^0 \rightarrow \pi^+ \pi^-$. (Remark: For a relativistic theory, one replaces the Lippmann-Schwinger eigenkets by the eigenkets labeled by the eigenvalues of relativistic 4-velocity operator $P_\mu M^{-1}$ ($\mu = 0, 1, 2, 3$), where $M^2 = P^\mu P_\mu$ is the mass-square operator and P_μ is the 4-momentum operator [5].)

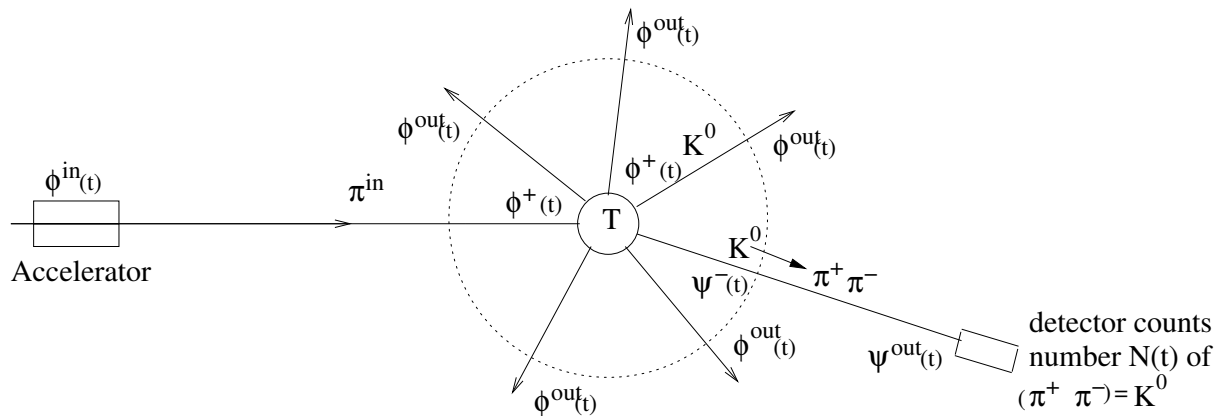
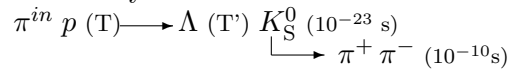


Figure 1. Preparation and decay of K^0 in the reaction



The theoretical predictions which need to be compared with the experimental data (given by $N(t)$, the detector counts in Fig.1), are the Born probabilities:

$$\mathcal{P}_\rho(A(t)) \equiv \text{Tr}(A(t) \rho) \quad \text{Heisenberg} \quad = \quad \text{Tr}(A \rho(t)) \quad \text{Schrödinger picture} \quad (5)$$

In the special case (discussed here for simplicity) the probability to detect the observable $A_{\psi^-} = |\psi^-(t)\rangle\langle\psi^-(t)|$ in the prepared state $|\phi^+(t)\rangle\langle\phi^+(t)|$, is given by

$$\mathcal{P}_{\phi^+}(\psi^-(t)) = \text{Tr}(|\psi^-(t)\rangle\langle\psi^-(t)| |\phi^+\rangle\langle\phi^+|) = |\langle\psi^-|\phi^+(t)\rangle|^2. \quad (6)$$

To solve a differential equation (like e.g., the Schrödinger eq. (1) or the Heisenberg eq. (4)) requires the choice of a **Boundary Condition** (that means one needs to choose the mathematical spaces, to which our solutions of the equations (1) or of eq. (4) should belong). The right choice of a Boundary Condition is very important for quantum physics. The boundary condition specifies the mathematical spaces which the solutions of the differential equation (1)-(4) need to belong to. In particular the boundary conditions specify the topological properties of the spaces of solution (e.g., the definition of convergence of infinite sequences in the spaces of solutions of (1) or of (4)). This choice of a specific boundary conditions has important consequences for the physical properties described by the theory, like, e.g., the existence of Dirac kets (not possible under Hilbert space boundary condition but possible under Schwartz space axiom). Or the mathematical meaning of Lippmann-Schwinger kets [6] (they do not exist as functional in the dual of the Schwartz space), or the mathematical requirements on the spaces containing Gamow vectors with exponential time evolution [8], and the possible relation of the Gamow vectors of the lifetime τ to the Breit-Wigner resonances (of Lorentzian line-shape) [9].

The boundary conditions are usually formulated as the choice of the mathematical spaces to which the solution of the dynamical differential equation (1)-(4) must belong. For instance, one

has chosen in the past as admitted solutions of the dynamical equation (1) the Hilbert space \mathcal{H} , and one chooses as admitted solution for ψ of equation (4) also the Hilbert space boundary condition: $\phi(t) \in \mathcal{H}$, $\psi(t) \in \mathcal{H}$. One could equally well have chosen the Schwartz space (7d) as boundary condition for the dynamical equations; then one would obtain, as a bonus, that the Dirac kets are continuous antilinear functionals on the Schwartz space Φ , i.e. the scattering states would be mathematically defined as elements of the Schwartz space $|E\rangle \in \Phi^\times$. But in Schwartz space one would still not obtain Gamow kets [8] or Lippmann-Schwinger kets [6], and one could not distinguish in Schwartz space Φ between states ϕ^+ and observables ψ^- .

Since it is pretty obvious that the state $\phi^+(t)$ in Fig.1 has totally different physical meaning than the observable ψ^- , one should expect that the $\phi^+(t)$ of (1) and the $\psi^-(t)$ of (4), should not both be from the Hilbert space \mathcal{H} and also not from the Schwartz space Φ .

Our goal will therefore be to arrive at new boundary conditions for the dynamical equation (1), (4) which:

1. mathematically distinguish between state ϕ and observable ψ
2. lead to a mathematical theory which unifies resonance and decay phenomena.

In the historical development of quantum theory, the following boundary conditions have been used for the dynamical equation (1) and (4):

- (i) Hilbert space boundary condition of von Neumann ("Hilbert space axiom"):

Set of state vectors $\{\phi\}$ = Set of observables $\{\psi\} \doteq \mathcal{H}$ = mathematical Hilbert space
(meaning the use of Lebesgue integrals which most physicists ignore.)

(7a)

- (ii) Schwartz space boundary condition will provide us with the wonderful Dirac formalism, with the Dirac kets being $|E\rangle \in \Phi^\times$ (continuous, antilinear functionals on the Schwartz space Φ). Using the Schwartz space axiom, every vector $\phi \in \Phi$ has a basis vector expansion

$$\phi = \int_0^\infty |E\rangle \langle E|\phi\rangle, \quad (7b)$$

but convergence is defined not by one norm as in \mathcal{H} : $\phi_n \xrightarrow{\text{in } \mathcal{H}} \phi$, iff $\|\phi_n - \phi\| \rightarrow 0$, but by a countable number of norms

$$\phi_n \xrightarrow{\Phi} \phi \quad \text{iff} \quad (\phi_n, (A+1)^p \phi_n) \rightarrow 0 \text{ for every power } p = 0, 1, 2, \dots \quad (7c)$$

The operator A in the definition of the countable number of scalar products is a distinguished observable in Φ , e.g. the Hamiltonian or the Nelson operator in more complicated systems. And in Schwartz space, states fulfilling (1) or (2) and observables fulfilling (3) or (4) will be mathematically identified

$$\begin{array}{ll} \text{Set of state vectors } \{\phi\} & = \text{Set of observables } \{\psi\} \doteq \Phi = \text{Schwartz space } \Phi. \\ \text{fulfilling (1)} & \text{fulfilling (4)} \end{array} \quad (7d)$$

This would identify the physically different entities, accelerator prepared state $\{\phi^+\}$ and registered observable $\{\psi^-\}$ detected by the counter in Fig.1, and describe them by the same mathematical Schwartz space Φ .

Furthermore, from the boundary condition (7a), as well as from (7d) follows (by the Stone-von Neumann theorem [1] for the Hilbert space \mathcal{H} , and by a similar theorem for the Schwartz space Φ [2]), that the time evolution in both cases is given by a *group evolution*, similar to the unitary group:

The solutions of the Schrödinger equation (1) under the Hilbert space condition $\phi \in \mathcal{H}$ are: (Stone-von Neumann)

$$\phi(t) = U^\dagger(t)\phi = e^{-iHt/\hbar}\phi, \quad \text{with } -\infty < t < +\infty \text{ for } \phi \in \mathcal{H}. \quad (8a)$$

And solution of (1) under the Schwartz space boundary condition $\phi \in \Phi$ are also given by:

$$\phi(t) = U_\Phi^\dagger(t) \phi = e^{-iHt/\hbar}\phi; \quad -\infty < t < \infty \text{ for } \phi \in \Phi. \quad (8b)$$

$(U_\Phi^\dagger(t) = U^\dagger(t)|_\Phi$ is the restriction of U^\dagger to the Schwartz subspace $\Phi \subset \mathcal{H}$.)

Similar results hold for the solutions of the Heisenberg equation under the Hilbert boundary conditions:

$$\psi(t) = U(t) \psi = e^{iHt/\hbar}\psi, \quad \text{with } -\infty < t < +\infty \text{ for } \psi \in \mathcal{H}, \quad (9a)$$

and for the Schwartz space boundary condition $\psi \in \Phi$:

$$\psi(t) = U_\Phi(t) \psi = e^{iHt/\hbar}\psi, \quad -\infty < t < \infty \text{ for } \psi \in \Phi. \quad (9b)$$

These result: the (unitary) group evolution (8a) and (8b) for the state $\phi(t)$, as well as (9a) and (9b) for the observable $\psi(t)$, have been well-known for the Hilbert space axiom (Stone-von Neumann theorem [1]). The group evolution (8b) and (9b) have also been proven for the Schwartz space boundary condition $\psi, \phi \in \Phi$. [2]

As a consequence of these theorems (Stone-von Neumann and Schwartz space) it follows, that if one uses the *Hilbert space boundary condition*, or if one uses the *Schwartz space boundary condition* (Dirac's formalism), one predicts the Born probability $\mathcal{P}_\rho(A(t)) = |\langle \psi^-(t) | \phi^+ \rangle|^2$ to detect an observable $A(t) = |\psi^-(t)\rangle\langle \psi^-(t)|$ in the state ϕ^+ for all time t : $-\infty < t < +\infty$.

This contradicts our feeling of *causality*, since $-\infty < t < +\infty$ in (9a) (9b) means that the probability (6) to detect the observable $A(t) = |\psi^-(t)\rangle\langle \psi^-(t)|$ in the prepared state $\phi^+(t_0)$ is also *predicted*, for times of the past i.e. for $t < t_0$, where t_0 is the time in the future of t at which the state ϕ^+ will be prepared. Such prediction does not make physical sense.

Thus, accepting the Hilbert space or the Schwartz space boundary condition (7a) or (7d) would lead to a conflict with our feeling for causality, because:

A detector which is represented by the observable ψ^- (i.e. by $|\psi^-(t)\rangle\langle \psi^-(t)|$ or by an operator $A(t)$, as in Fig.1) cannot detect anything connected to this scattering process until after the (finite) time t_0 , at which the accelerator has been turned on and a state ϕ^+ has been prepared by the preparation part of the experiment, as seen from Fig.1. Thus, our standard quantum theory which makes predictions for all times $-\infty < t < +\infty$ (like the theory based on the Hilbert space axiom (9a) and also the theory based on the Schwartz space axiom (9b)) can not be quite right.

At best such boundary condition (7a)(7d) would provide an inaccurate, and deficient theoretical description of quantum processes, as shown and discussed using the experiment of Fig.1, because predictions for $-\infty < t < +\infty$ would contradict the causality principle, which asserts that:

A state ϕ^+ needs to be prepared first, by a time t_0 , *before* an observable (detector) $|\psi^-(t)\rangle\langle \psi^-(t)| \rightarrow |\psi^{out}\rangle\langle \psi^{out}|$ can be measured in the state ϕ^+ at times $t \geq t_0$ by detector counts, $N(t)/N \sim |\langle \psi^-(t) | \phi(t_0) \rangle|^2$: There cannot be any $K_S^0 \rightarrow \pi^+ \pi^-$ decays detected, before the π^{in} (Fig.1) hits the target T. Event by event (i), the K^0 has to be produced by the finite time $t_0^{(i)}$ at the target T before $\pi^+ \pi^-$ can be registered by the detector $|\psi^-(t)\rangle\langle \psi^-(t)|$ at times $t > t_0^{(i)}$. (10a)

The beginning of time t_0 for an ensemble of i micro-systems, is of course not a specific time on the clocks in the laboratory where the experiment takes place, because this experiment involves an **ensemble** of quantum projectiles that come into being at an **ensemble** of beginnings of time $t_0^{(i)}$; it usually also involves an ensemble of target particles T. Thus the time t_0 , after which the decay products can be counted, would also be an ensemble of times; unless one can do experiments on single particles, like Dehmelt's quantum jump experiment [3]. The lifetime is a weighted average of individual time intervals with an ensemble of "beginnings of time" $t_0^{(i)}$ and an ensemble of (usually) different "individual lifetime $\tau^{(i)}$ ". This ensemble of beginnings of time $t_0^{(i)}$ is the finite "beginning of time" t_0 for the quantum system (particle) on the metastable level, that thereafter decays.

Thus, the well-known unitary-group time evolution (8a) and (8b) of standard quantum mechanics (7a) or (7d) needs to be replaced by a semigroup time evolution $t_0 < t < \infty$, where the **finite** semi-group time t_0 represents the ensemble of the individual beginning of time $t_0^{(i)}$ for each individual micro-object, $i = 1, 2 \dots: \{t_0^{(i)}\} = t_0$ of the prepared state ϕ^+ [10],

$$\phi(t) = e^{-iH(t-t_0)}\phi(t_0), \quad t > t_0. \quad (10b)$$

In the quantum jump experiment [3], the lifetime is a weighted average of individual lifetimes $\tau^{(i)}$ of single ions (i) on a meta-stable level. This is the way how the lifetime for the long-lifetime state in the quantum jumps experiments were measured [3]. The exponential decay law for the time evolution of this ensemble of **single** meta-stable (excited) states has been observed with sufficient accuracy in [3] to justify the interpretation that this ensemble of "single meta-stable ion states" obeys the exponential law: The i -th meta-stable particle has a finite beginning of time t_0 which is observed as the onset times of the dark periods. The ensemble of these onset times of the dark periods $\{t_0^{(i)}\}$ is the finite time $t_0 (= 0)$ representing the ensemble $\{t_0^{(i)}\}$ of onset times of dark periods. In order to accommodate this ensemble of finite beginnings of time $\{t_0^{(i)}\}$, one needs a state (operator or state vector) with a finite beginning of time t_0 , representing the ensembles of the beginnings of time for the dark periods $\{t_0^{(i)}\} \equiv t_0 \equiv 0 \leq t < \infty$. Here $t_0 = 0$ represents the ensemble observed as onset times of the dark period, starting with the i -th quantum jump onto the metastable level at $\{t_0^{(i)}\}$.

Since the standard Hilbert space (or the Schwartz space) theory for the dynamical equations leads (by the mathematical theorem [1]) to the unitary group evolution $U(t)$ and $U^\dagger(t)$ with $-\infty < t < \infty$, these spaces (the Hilbert space or the Schwartz space) can **not** be used for boundary conditions that lead to a causal evolution with a finite beginning of time t_0 , as it is observed by the onset times of the dark periods in [3].

In order to obtain a quantum theory with a finite beginning of time t_0 , and which describes a state of an ensemble with finite beginning of times $\{t_0^{(i)}\} \equiv t_0 (= 0)$ as it has been observed in [3], one needs new boundary conditions for the dynamical equations, which replace the standard (Hilbert space (7a), or also the Schwartz space (7d)) boundary conditions for the dynamical equations. The new mathematical boundary condition must lead to a finite beginning of time t_0 , that represents the time at which the state has been prepared and after which the observable can be registered by the detectors. Since quantum theory deals with an ensemble of micro-particles, we can expect that this beginning of time t_0 is observed as an ensemble of times $t_0^{(i)}$, like the onset times (on the clock in the lab) of the dark periods, with durations that obey the exponential law, as shown in the experiment of [3].

Summarizing: Causality suggests that the Born probabilities of the observable $A(t) = |\psi^-(t)\rangle\langle\psi^-(t)|$ in the prepared state ϕ^+ make sense only after a finite time, the preparation time t_0 of the state. This finite t_0 is observed as an ensemble of finite times $\{t_0^{(i)} = \text{finite}\} \equiv t_0$.

The Born probabilities

$$\mathcal{P}_{\phi^+}(|\psi^-(t)\rangle\langle\psi^-(t)|) = |\langle\psi^-(t)|\phi^+(t_0)\rangle|^2 = |\langle\psi^-(t)|\phi^+\rangle|^2, \quad (11)$$

are physically defined only for time $t > t_0$.

Here t_0 is the *time at which the state ϕ^+ is prepared and after which the observable ψ^- can be registered (i.e. detected) in the state ϕ^+* . Since a quantum state represents an ensemble of micro systems in the lab, the beginning of time t_0 represents (usually) an ensemble of finite times, $t_0 \leftrightarrow \{t_0^{(i)}\}$; these $t_0^{(i)}$ are different times on a clock in the lab, but they represent the same time t_0 for the metastable state. These $t_0^{(i)}$ have been observed as the onset times of the dark periods in Dehmelt's and others quantum jump experiments [3] with single ions in a Paul trap; where each excited single-ion (i) of the ensemble, has its own "beginning of time $t_0^{(i)}$ " by the clocks in the lab [10].

3. Lippmann-Schwinger Kets and Hardy Space Boundary Conditions

3.1. From Time Symmetric to Time Asymmetric Boundary Conditions

The above described observations [3] suggest, that there is a finite quantum mechanical beginning of time; therefore one must solve the dynamical differential equation not under the standard Hilbert -or the Schwartz space boundary conditions, because the Hilbert space axiom and the Schwartz space axioms (due to the Stone-von Neumann theorem) does not allow for a finite beginning of time t_0 [1]. The insight provided by the experiments on single ion [3] indicates that one needs *new boundary conditions* which distinguish mathematically between the space of states and the space of observables [10] and this leads to finite "beginnings of time" $t_0^{(i)}$, and therewith to a *semi-group evolution* with $\{t_0^{(i)}\} \cong t_0 \leq t < \infty$.

It is incredible that a mathematical theorem (Stone-von Neumann for the Hilbert space which requires Lebesgue integrals that most physicists do not even use [3]), could have distracted us from the time asymmetry in the quantum world.

In quantum physical experiments, the time asymmetry is an expression of causality: a state needs to be prepared first before an observable can be registered in the state, as it is displayed in Fig.1.

To obtain such causal (or time-asymmetric) solutions of the dynamical equations:

$$\psi^-(t) = \mathcal{U}(t - t_0) \psi(t_0) \quad \text{with the beginning of time } t_0 : \quad t_0 \leq t < +\infty, \quad (12)$$

one does not require new dynamical equations; but one needs to use *new boundary conditions for the solutions* of the standard dynamical equations (1)-(4). That means one has to look for new mathematical spaces for the state vectors $\{\phi^+\}$ (representing the preparation apparatus (accelerator)) and for new mathematical spaces describing the observables $\{\psi^-\}$ (representing the registration apparatus i.e, the detector in the scattering experiment of Fig.1).

Since the standard boundary conditions for the dynamical equation (1) or (4), which use

$$\text{the Hilbert space axiom:} \quad \{\phi\} = \{\psi\} \div \mathcal{H} \quad \text{of von Neumann,} \quad (13a)$$

or which use:

$$\text{the Schwartz space axiom:} \quad \{\phi\} = \{\psi\} \div \Phi \quad \text{of the Dirac formulation,} \quad (13b)$$

lead to the time symmetric (unitary) group evolution (8a)(8b), (9a)(9b) with $-\infty < t < +\infty$, we must look for new boundary conditions which respect causality, in place of the Hilbert space and the Schwartz space axioms (7a) and (7d) for the two dynamical differential equations (1)

and (4). These new boundary conditions, for the Schrödinger equation of the states $\{\phi^+\}$ and for the Heisenberg equation of the observables $\{\psi^-\}$, need to differ from each other.

Furthermore, since the $\{\phi^+\}$ represents the prepared states, and the $\{\psi^-\}$ represents the detected observable, as it is suggested by the arrangement of the scattering experiments in Fig.1, it is more natural to use two different mathematical spaces: One space, say Φ_- , will represent the space of states $\{\phi^+\}$ and the other space, say Φ_+ , will represent the space of observables $\{\psi^-\}$. The question then is: Which mathematical spaces or which pair of mathematical spaces should be chosen for these two sets of physical entities, the set of states $\{\phi^+\}$ obeying (1) and the set of observables $\{\psi^-\}$ obeying (2)?

The best answer at this time in history, based on experimental results in quantum resonance scattering and decay phenomena is, that these spaces are the pair of Hardy spaces (Hardy spaces have been used before by Lax-Phillips [4] in classical scattering theory, and also by Y. Strauss *et al* [11] in quantum scattering).

The new Hardy space boundary conditions for the dynamical equations (1)-(4) has the advantage that it will employ different mathematical spaces for the two different physical entities, one space Φ_- for the states $\{\phi^+\}$ and the other space which we call Φ_+ for the observables $\{\psi^-\}$. There is no reason to use one and the same mathematical space Φ (e.g. the Schwartz space) to represent for such different physical entities as the set of states $\{\phi^+\}$ representing the preparation apparatus, and the set of observables $\{\psi^-\}$ representing the detector.

Thus, the new rule that relates the physical quantities to the mathematical representation space is given by a new axiom of quantum theory, the Hardy space axiom.

3.2. The Hardy Space Axiom

The set of state vectors $\{\phi^+\}$ representing the accelerator-prepared states (see Fig.1 above) and fulfilling the Schrödinger equation (1), is the Hardy space Φ_- of the lower complex energy semi-plane (2nd sheet of the analytic S -matrix).

$$\text{physically prepared in-states } \{\phi^+\} \doteq \Phi_- \quad \text{mathematical Hardy space for the solutions of the Schrödinger eq.(1)} \quad (14a)$$

The set of observable vectors $\{\psi^-\}$ representing the detector-registered observables (Fig.1) and fulfilling the Heisenberg equation (4), is the Hardy space Φ_+ of the upper complex energy semi-plane (2nd sheet of the analytic S -matrix):

$$\text{experimentally detected out-observables } \{\psi^-\} \doteq \Phi_+ \quad \text{mathematical Hardy space for the solutions of the Heisenberg eq.(4)} \quad (14b)$$

Here we have used a notation that labels the prepared in-state vector (in-state of a scattering experiment in Fig.1) by ϕ^+ , as it is the custom in scattering theory; this label $+$ has its origin in the standard notation for the in-state kets $|E^+\rangle$ of the Lippmann-Schwinger equation (as will be defined below in (16) [6] [7]). For the out-vectors representing the detector, we use the notation ψ^- (which here represents the observable $|\psi^-\rangle\langle\psi^-|$). The observable-vector ψ^- needs to obey the Heisenberg equation (4). The label $-$ for the observable ψ^- and the label $+$ for the in-state ϕ^+ have their origin in the standard notation for the Lippmann-Schwinger kets (see below in equation (16) (17)).

This miss-match (14a) and (14b)) of the notation for vectors and for spaces:

$$\phi^+ \in \Phi_- \quad (\text{lower complex plane}), \quad \text{for the prepared in- state} \quad (15a)$$

$$\psi^- \in \Phi_+ \quad (\text{upper complex plane}) \quad \text{for the detected observable} \quad (15b)$$

has its origin in the different conventions used in physics for the vectors ϕ^+ and ψ^- , and the convention used in mathematics for the Hardy space Φ_- and the Hardy space Φ_+ .

The notation used for the famous Lippmann-Schwinger kets [6] in the standard physics literature (c.f equation (17) below), results in the mismatch of \pm in (15a) and (15b) between the state vectors ϕ^+ and their mathematical space $\{\phi^+\} = \Phi_-$ (in the mathematicians' convention for Hardy spaces), and for the Lippmann-Schwinger kets and the labels \mp for the dual spaces Φ_-^\times , Φ_+^\times (which are the dual space of the Hardy spaces in (14a) and (14b)).

Therefore, we want to give the heuristic Lippmann-Schwinger ket [6] a mathematical meaning, by defining them as the antilinear continuous functionals on the Hardy space (14a) and (14b):

$$|E^+\rangle \in \Phi_-^\times = \text{dual of the space } \Phi_- \quad \text{and} \quad |E^-\rangle \in \Phi_+^\times = \text{dual of the space } \Phi_+. \quad (15c)$$

The Lippmann-Schwinger kets $|E^+\rangle$ and $|E^-\rangle$ were initially introduced by the Lippmann-Schwinger equation of (16) below. They have with (15c) obtained the mathematical meaning as the continuous antilinear functionals on the Hardy space Φ_- and Φ_+ , respectively.

Our Hardy space Φ_- is realized by the “smooth Hardy” function (the intersection of Hardy class \mathcal{H}_-^2 with the Schwartz space S). In analogy to the components $\langle E|\phi\rangle$ of the Dirac basis vector expansion (7b), the components of the Hardy space vectors (14a) (14b) are very well-behaved functions

$$\phi^+(E) = \langle^+ E|\phi^+\rangle \in (\mathcal{H}_-^2 \cap S)_{\mathbb{R}_+} \text{ on } \mathbb{C}_-. \quad (15d)$$

This means $\phi^+(E)$ are smooth functions that can also be continued into the lower complex energy semiplane \mathbb{C}_- second sheet of the S -matrix. This definition is compatible with the physicists convention for the Lippmann-Schwinger kets [6]. The energy wave function $\phi^+(E) = \langle^+ E|\phi^+\rangle$ of the accelerator prepared in-states ϕ^+ are analytic function in the lower complex energy plane \mathbb{C}_- ; for this \mathbb{C}_- one choose the second sheet of the S -matrix (where the resonance poles of the S -matrix are located).

Similarly, the Hardy space Φ_+ is realized by the smooth Hardy function

$$\psi^-(E) = \langle^- E|\psi^-\rangle \in (\mathcal{H}_+^2 \cap S)_{\mathbb{R}_+} \text{ on } \mathbb{C}_+, \quad (15e)$$

i.e. smooth functions on the upper complex energy semi-plane \mathbb{C}_+ (second sheet of the S -matrix). As a consequence, their complex conjugates $\bar{\psi}^-(E) = \langle\psi^-|E^-\rangle$ are also analytical functions on the second sheet of the S -matrix lower complex semi-plane, which is important for the analytic continuation of the S -matrix (e.g. to the places where the resonance poles of the S -matrix are located).

From (15d),(15e) follows that the product of these smooth Hardy functions $\langle\psi^-|E^-\rangle\langle^+ E|\phi^+\rangle$ can be analytically continued into the lower complex semi-plane second sheet of the S -matrix element $S(E)$, where the resonance poles of the S -matrix are located. (in general there are also other interesting features on the second sheet of the S -matrix, e.g, higher order poles which will not be discussed in this paper.) First order poles of the S -matrix correspond to Breit-Wigner resonances.

The Hardy space boundary conditions (14a) and (14b) (or (15d) and (15e)) for the dynamical equation (1) and (4) were already foreshadowed by the Lippmann-Schwinger equations (16), though Hardy space mathematicians [4] and physicists using Lippmann-Schwinger equations did not seem to have been aware of each other.

The Lippmann-Schwinger kets [6] were postulated earlier (in analogy to the Dirac kets) as the in-plane wave “states” $|E^+\rangle$, and the out-plane wave “states” $|E^-\rangle$ which:

1. are energy “eigenkets” of the “exact Hamiltonian $H = K + V$; and of the “interaction-free Hamiltonian K

$$H |E^\pm\rangle = E |E^\pm\rangle, \quad K |E\rangle = E |E\rangle$$

that fulfill the Lippmann-Schwinger equations [6]:

$$|E^\pm\rangle = |E\rangle + \frac{1}{E - K \pm i\epsilon} V |E^\pm\rangle \quad H = K + V, \quad (16)$$

2. whose formal “solutions” are given by:

$$|E \pm i\epsilon\rangle = |E^\pm\rangle = |E\rangle + \frac{1}{E - H \pm i\epsilon} V |E\rangle = \Omega^\pm |E\rangle, \quad \epsilon \rightarrow +0. \quad (17)$$

Now, with (15d) and (15e), the Lippmann-Schwinger ket $|E^\pm\rangle$ have been given a mathematical meaning. They are defined as the continuous antilinear functionals on the pair of the Hardy spaces Φ_- and Φ_+ , $|E^\pm\rangle \in \Phi_\mp^\times$. These functionals $F_E = |E^\pm\rangle$ are also the generalized eigenvectors F_E with eigenvalue E of the Hamiltonian H ; i.e

$$\langle H\phi^+ | E^+ \rangle \equiv \langle \phi^+ | H^\times | E^+ \rangle = E \langle \phi^+ | E^+ \rangle, \quad (18a)$$

and

$$\langle H\psi^- | E^- \rangle \equiv \langle \psi^- | H^\times | E^- \rangle = E \langle \psi^- | E^- \rangle. \quad (18b)$$

(H^\times in (18a)(18b) is the definition of the operator H^\times conjugate to H in Φ_- and Φ_+ .)

In the mathematics literature, one uses the notation \mathcal{H}_-^2 for the Hardy *class* function (L^2 -integrable) of the lower complex semi-plane, and the notation \mathcal{H}_+^2 for the Hardy class function (L^2 -integrable) of the upper complex semi-plane. The energy wave functions that we are interested in here for the quantum mechanical scattering and resonance theory, are not the L^2 -integrable function classes analytic in the lower complex plane $f_-(E) \subset \mathcal{H}_-^2$ (Hardy class functions in \mathbb{C}_-) or the L^2 -integrable function classes analytic in the upper complex plane $f_+(E) \subset \mathcal{H}_+^2$ (Hardy class in \mathbb{C}_+) [4].

For the theory of resonance scattering and decay phenomena, we use the space of smooth Hardy functions (15d) and (15e). They are defined as the space of (L^2 -integrable) Hardy functions \mathcal{H}_-^2 which have also the properties of Schwartz functions S on the lower complex energy semi-plane \mathbb{C}_- (second sheet of the S -matrix) [10]:

$$\phi^+(E) = \langle^+ E | \phi^+ \rangle \in (\mathcal{H}_-^2 \cap S)_{\mathbb{R}_+}. \quad (15d)$$

Here $\mathcal{H}_-^2 \cap S$ is the intersection of \mathcal{H}_-^2 with the Schwartz space S . Similarly we choose the set of functions:

$$\psi^-(E) = \langle^- E | \psi^- \rangle \in (\mathcal{H}_+^2 \cap S)_{\mathbb{R}_+} \quad (15e)$$

on the upper complex energy semi-plane \mathbb{C}_+ (second sheet of the S -matrix).

The $\mathcal{H}_+^2 \cap S$ are the spaces of smooth Hardy function $\langle^- E | \psi^- \rangle \in \mathcal{H}_+^2 \cap S$ on the upper complex energy plane, and the $\langle^+ E | \phi^+ \rangle \in \mathcal{H}_-^2 \cap S$ are the smooth Hardy functions on the lower complex plane (second sheet of the S -matrix), i.e. the energy wave function $\langle^+ E | \phi^+ \rangle$ and $\langle^- E | \psi^- \rangle$ are the smooth Hardy space functions that can be analytically continued into the lower (for ϕ^+) and upper (for ψ^-) complex energy plane, second sheet of the S -matrix. Therefore the product $\langle \psi^- | E^- \rangle \langle^+ E | \phi^+ \rangle = \langle^- E | \psi^- \rangle \langle^+ E | \phi^+ \rangle$ are smooth Hardy functions that can be analytically continued into the lower complex energy plane (second sheet of the S -matrix), where (as depicted in Fig.2 at z_R) the resonance poles of the S -matrix $S(E)$ (and other interesting physical entities) are located.

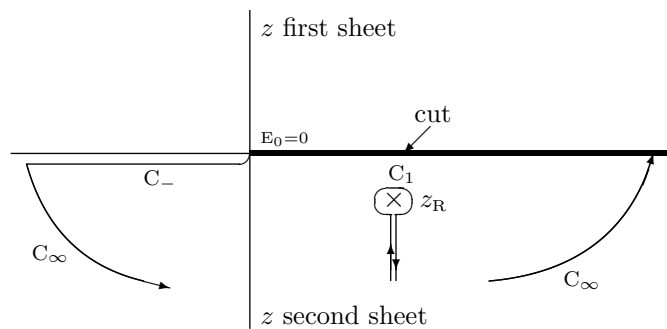


Figure 2: First upper complex plane and second sheet lower complex plane with cut and resonance pole of the S -matrix at $z_R = E_R - i\Gamma/2$, $\tau = \hbar/\Gamma$ = lifetime.

To simplify the notation, we have so far omitted the additional quantum number j , j_3 , η in the energy kets for which one usually choose the angular momentum j and its component j_3 and uses η for the other degeneracy quantum numbers. Using the complete set of observables H , J^2 , J_3 , $\hat{\eta}$, the energy eigenkets for the spaces Φ_{\pm}^{\times} are $|E^{\pm}\rangle = |E, j, j_3, \eta^{\pm}\rangle$ and the energy wave function $\langle^{\pm}E|\phi^{\pm}\rangle = \langle^{\pm}E, j, j_3, \eta|\phi^{\pm}\rangle$ are given by the Hardy functions (15d) and (15e). (H , J^2 , J_3 , $\hat{\eta}$ are a complete system of commuting observables and η denote additional species quantum numbers, not important here.)

The Hardy space kets (i.e continuous antilinear functionals of Φ_{-}^{\times}) $|E^{+}\rangle = |E, j, j_3, \eta^{+}\rangle$ are taken as basis systems for a Dirac basis ket expansion (using the “nuclear spectral theorem” for the “nuclear” linear topological space Φ_{-}) of the in-state vectors $\phi^{+} \in \Phi_{-}$ (representing the preparation apparatus, e.g. the accelerator, in the scattering experiment of Fig.1):

$$\phi^{+} = \sum_{j, j_3, \eta} \int_0^{\infty} dE |E, j, j_3, \eta^{+}\rangle \langle^{+}E, j, j_3, \eta| \phi^{+} = \int_0^{\infty} dE |E^{+}\rangle \langle^{+}E| \phi^{+} = \int dE |E^{+}\rangle \phi^{+}(E) \quad (19)$$

Nuclear space here means that Φ_{-} fulfills the conditions of the nuclear spectral theorem which means the conditions for a Dirac basis vector expansion like (19), which the Schwartz -and the Hardy space do.

Similarly, the $|E^{-}\rangle = |E, j, j_3, \eta^{-}\rangle$ are taken as basis systems for out-vectors $\{\psi^{-}\}$ (the ψ^{-} represent the observables $|\psi^{-}\rangle \langle \psi^{-}|$ registered (or “detected”) by the detector of Fig.1):

$$\psi^{-} = \sum_{j, j_3, \eta} \int_0^{\infty} dE' |E', j, j_3, \eta^{-}\rangle \langle^{-}E', j, j_3, \eta| \psi^{-} = \int_0^{\infty} dE' |E'^{-}\rangle \langle^{-}E'| \psi^{-} = \int dE' |E'^{-}\rangle \psi^{-}(E') \quad (20)$$

4. The Time Evolution Semigroup

The Lippmann-Schwinger ket expansion (19) and (20) were originally postulated in analogy to the Dirac basis vector expansion, just replacing the Dirac kets by the Lippmann-Schwinger ket (16) (17), which were also just postulated [6]. The difference between (19),(20) and the Dirac ket expansions is, that the original Dirac ket expansion is justified as the “nuclear spectral theorem” for the Schwartz-Rigged Hilbert space Φ [12].

To justify the Lippmann-Schwinger ket expansions (19) and (20), and to provide a mathematic definition of the Lippmann-Schwinger kets, required the construction of a *new pair of Rigged Hilbert spaces*, which are based on the pair of “smooth”-Hardy functions (15d) and (15e) of the upper complex energy plane 1st sheet, and lower complex energy plane [2] second sheet of the analytic S -matrix and along the positive real energy axis, cf. Fig.2.

Thus

$$\text{the in-states } \phi^{+} \in \Phi_{-} \quad \text{and the out-observables } \psi^{-} \in \Phi_{+} \quad (21a)$$

$$\text{have the energy wave functions} \quad \text{have the energy wave functions} \quad (21b)$$

$$\phi^{+}(E) = \langle^{+}E|\phi^{+}\rangle \in (\mathcal{H}_{-}^2 \cap S)_{\mathbb{R}_{+}} \quad \psi^{-}(E) = \langle^{-}E|\psi^{-}\rangle \in (\mathcal{H}_{+}^2 \cap S)_{\mathbb{R}_{+}}$$

In (21b), \mathcal{H}_{\pm}^2 is the standard notation of the Hardy class functions (L^2 -integrable) on the lower complex semiplane (for $\phi^+(E)$), and on the upper complex semi-plane (for $\psi^-(E)$), second sheet of the S -matrix. The function spaces $(\mathcal{H}_{\pm}^2 \cap S)_{\mathbb{R}_+}$ are the intersections of \mathcal{H}_{\pm}^2 with the Schwartz function space S restricted to the positive energy. $\mathcal{H}_{\pm}^2 \cap S$ are the spaces of Hardy functions of the lower $(-)$ (in (15d)) and the upper $(+)$ (in (15e)) complex energy plane, second sheet of the S -matrix; thus $\langle \psi^- |^- E \rangle \langle + E | \phi^+ \rangle \in (\mathcal{H}_{\pm}^2 \cap S)$ on the second sheet lower complex E -plane, where the resonance poles of the S -matrix are located.

The Dirac kets $|E\rangle$ are mathematically defined as continuous antilinear functionals on the **Schwartz** space Φ : $|E\rangle \in \Phi^\times$ (the space of “tempered distribution”). This is as far as the distribution theory using the Schwartz space could go. The shortcoming of the Schwartz space is that the states $\{\phi\}$ and the observables $\{\psi\}$ can not be mathematically distinguished from each other since both are mathematically described by the same Schwartz space cf (8b) and (9b). But physically, it is clear that states and observables are different physical entities, the accelerator defines the states and the detector defines the observables.

The new idea, drawn from the *pair* of phenomenological Lippmann-Schwinger kets (16), is to relate the set of prepared in-states ϕ^+ (experimentally defined by the preparation apparatus (e.g, accelerator etc of Fig.1) to the mathematical Hardy class space \mathcal{H}_{\pm}^2 ; (this then leads to the smooth Hardy functions (15d)). And also to relate the observable ψ^- to the smooth Hardy functions (15e).

This is expressed by *the Hardy space axiom* of time-asymmetric quantum theory (15d) and (15e) and in terms of the abstract vector spaces:

$$\text{Set of accelerator prepared states } \{\phi^+\} \doteq \Phi_- = \text{Hardy space on } \mathbb{C}_- \text{ (lower complex energy plane, 2nd sheet of the } S\text{-matrix).} \quad (14a)$$

$$\text{Set of detector registered observables } \{\psi^-\} \doteq \Phi_+ = \text{Hardy space on } \mathbb{C}_+ \text{ (upper complex energy plane, 2nd sheet of the } S\text{-matrix).} \quad (14b)$$

The accelerator prepared energy wave functions $\phi^+(E)$ are the smooth Hardy functions (15d), and the detector registered energy wave function are the smooth Hardy functions $\psi^-(E)$ (15e).

Taking as the new axiom of quantum physics, the Hardy space axiom (14a), (14b) or (15d), (15e), the Schrödinger and the Heisenberg equation have to be solved under these Hardy space boundary conditions.

The great improvement of the Hardy space axiom (14a), (14b) over the standard Hilbert space axiom (7a) or the Schwartz space axiom (7d) is that the Hilbert space and the Schwartz space boundary condition for the dynamical equations lead to the unitary group evolution (8a) and (9a), and (8b) and (9b), for which t extends over $-\infty < t < \infty$ (by the Stone-von Neumann theorem *etc*).

In contrast, the Hardy space boundary conditions (14a), (14b) leads (by *the Paley-Wiener theorem* [13] for Hardy functions) to two different *semi-groups evolutions*, one for the accelerator prepared states $\phi^+ \in \Phi_-$ obeying the Schrödinger equation, and the other for the detector-registered observable $\psi^- \in \Phi_+$ obeying the Heisenberg equation. Their time evolutions are given by the two semigroups:

<p>for solution of the Schrödinger equation for states in the space Φ_- $\phi^+(t) = e^{-iHt/\hbar} \phi^+$ with $0 \leq t < \infty$</p>	<p>for solution of the Heisenberg equation for observables in the space Φ_+ $\psi^-(t) = e^{iHt/\hbar} \psi^-$ with $0 \leq t < \infty$</p>
---	--

(22)

Remember that the change in time evolution from the unitary group evolution (8a), (8b) and (9a), (9b) to the semigroup in (22), is due to our change from the Hilbert space and

from the Schwartz space boundary condition for the dynamical equations to the new Hardy space boundary conditions (14a) and (14b). As a consequence of the change for the boundary condition from (the Hilbert space with the Stone-von Neumann $\rightarrow -\infty < t < +\infty$) to (the Hardy space with the Paley-Wiener $\rightarrow 0 < t < +\infty$), we obtain a **new quantum theory** with strictly causal time evolution:

The probability to detect the time evolved observable $\psi^-(t)$ in the state ϕ^+ is predicted as the Born probability:

$$\begin{aligned} \mathcal{P}_{\phi^+}(\psi^-(t)) &= |\langle \psi^-(t) | \phi^+ \rangle|^2 = |\langle e^{iHt/\hbar} \psi^- | \phi^+ \rangle|^2 \\ &= |\langle \psi^- | e^{-iHt/\hbar} | \phi^+ \rangle|^2 = |\langle \psi^- | \phi^+(t) \rangle|^2 \quad \text{for } t \geq 0 \text{ only.} \end{aligned} \quad (23)$$

This mathematical prediction is a consequence of the new Hardy space axioms (14a) and (14b) from which (22) and thus (23) follows as a consequence of the Paley-Wiener theorem [13].

This is a new theoretical prediction which follows from (14a),(14b). It is in agreement with the experimental observation that the detector counts $N(t)/N$:

$$N(t)/N \sim \text{probability for detected observable } \mathcal{P}_{\phi^+}(\psi^-(t)), \text{ are detected only for } t \geq t_0 = 0. \quad (24)$$

This result (24) also establishes agreements of the calculated probabilities (23) and the experimental data $N(t)/N$. In (24), t_0 is the time at which the state ϕ^+ had been prepared by the accelerator in Fig.1, and after which the registration of the observable makes sense. The theoretical prediction (23) follows from the Hardy space boundary condition in perfect agreement with our general feeling about causality. But the result cannot be obtained from the standard Hilbert space or Schwartz space axioms (8a)-(9b), which “predict the probabilities for all time $-\infty < t < \infty$. The Hardy space prediction (23) is in agreement with causality (24) whereas the Hilbert space and Schwartz space predictions (8a)-(9b) are not causal (they also predict for the past $-\infty < t < 0$). The question then is: how can one see such beginnings of time t_0 ?

The agreement between the physicists’ set of state vectors $\{\phi^+\}$ and the mathematical space Φ_- , as well as the agreement between the physicists’ set of observables $\{\psi^-\}$ and the mathematical space Φ_+ is perfect, except for the mix-up in the notation of $\phi^+ \in \Phi_-$ and $\psi^- \in \Phi_+$ of the + and – signs.

The reason for this “mis-labeling” between the \pm labels of the pair $\phi^+ \in \Phi_-$ and $\psi^- \in \Phi_+$ are just different notational conventions; the Lippmann-Schwinger kets $|E^\pm\rangle$ were labeled by physicists [6][7] such that the energy wave function $\langle^+ E | \phi^+ \rangle$ and the $\langle \psi^- | E^- \rangle = \overline{\langle^+ E | \psi^- \rangle}$ are analytic function on the lower complex plane (2nd sheet of the S -matrix), where resonance poles of the S -matrix and other interesting physical properties (singularities) are located.

The new statement of *physical contents* is the Hardy space boundary condition (14a) and (14b) which are expressed in terms of the energy wave functions (on the second sheet of the S -matrix below the cut from $0 \leq E < +\infty$). These wave functions have the property of the smooth Hardy functions

$$\langle^+ E | \phi^+ \rangle \in (\mathcal{H}_-^2 \cap S)_{\mathbb{R}_+} \quad \text{and} \quad \langle^- E | \psi^- \rangle = \overline{\langle \psi^- | E^- \rangle} \in (\mathcal{H}_+^2 \cap S)_{\mathbb{R}_+}. \quad (25)=(15d), (15e)$$

This is an axiom which one can only conjecture if one looks at many aspects of resonance and decay phenomena. The result of such search process was that

$$\begin{aligned} &\text{the prepared state } \phi^+ \in \Phi_- \text{ and the registered observable } \psi^- \in \Phi_+ \text{ are represented} \\ &\text{by Hardy spaces of the lower and upper complex energy plane (2nd sheet of the} \\ &\text{\textit{S}-matrix } S_j(E)); \text{ thus } |E^+ \rangle \in \Phi_-^\times \text{ and } |E^- \rangle \in \Phi_+^\times \text{ are continuous antilinear} \\ &\text{functionals on the dual of the Hardy spaces } \Phi_- \text{ describing states, and } \Phi_+ \\ &\text{describing observables.} \end{aligned} \quad (26)$$

The mathematical entities which in quantum mechanics had been associated with the prepared “state” ϕ^+ fulfilling eq.(1) and the detected observable $|\psi^-\rangle\langle\psi^-|$ fulfilling eq.(4) are represented in the new mathematical theory by a pair of (mathematically well-defined) Hardy spaces, Φ_- for the state ϕ^+ , and Φ_+ for the observable ψ^- whose energy wave functions are given by the smooth Hardy functions (14a)(14b), (15d)(15e).

The Hilbert space, and Schwartz space theory with Dirac kets, have done a fair job in the description of scattering processes, but they did not succeed to provide an adequate description of resonance and decay phenomena. For that one needed the Lippmann-Schwinger kets and also Gamow vectors with exponential time evolution for the decaying states, and one needed energy wave function $\phi^+(E) = \langle^+E|\phi^+\rangle$, $\overline{\psi^-(E)} = \langle^-E|\psi^-\rangle$ which can be analytically continued into the complex energy plane (e.g. second sheet of the S -matrix where the resonance poles are located). And all this could be done so far with the “smooth Hardy function (25).

If one wants a theory that unifies the Breit-Wigner resonance of width Γ with an exponentially decaying state of lifetime $\tau = \hbar/\Gamma$ (as an exact relation), one needs the Hardy space axiom (14a) (14b) (15d) (15e). Also the Hardy space axiom leads to a strictly causal theory for the experimental predictions (24) with (23).

In order to achieve all this, new boundary condition of the dynamical equations of quantum mechanics were required: instead of solving the dynamical equation under the standard Hilbert space -(or Schwartz space)- boundary conditions, one has to use the (pair of) Hardy spaces. Then one obtains, again from a mathematical theorem of Paley-Wiener the time asymmetric dynamical evolution (22), (23) which agrees with our intuition of causality and it explains experimental observations like the quantum jump experiments [3].

The two dynamical equations of quantum mechanics (e.g. (1) and (4)) are usually considered as describing two pictures (Schrödinger or Heisenberg) of the same physical situation. This is indeed the case if one chooses the same Hilbert space boundary conditions as boundary condition of these differential equations. In most experimental situations (e.g. the scattering experiment of Fig.1), however, (for instance the scattering experiment of Fig.1), this is not justified: detected observable $A = |\psi^-\rangle\langle\psi^-|$ and prepared state $\rho = |\phi^+\rangle\langle\phi^+|$ are utterly different from each other. Therefore, the standard axiom (e.g. Hilbert space but also Schwartz space axiom (for the mathematical Dirac kets) which predicts the Stone-von Neumann unitary group) do not apply.

A new boundary condition, which was foreshadowed by the Lippmann-Schwinger equation, is the Hardy space boundary conditions (14a)(14b). It was conjectured from the requirement that, the width of the resonance $\Gamma = \hbar/\tau$ the inverse lifetime of decaying state. This equality of resonance width $\Gamma = \hbar/\tau$ inverse lifetime has recently been experimentally confirmed to high accuracy [14].

References

- [1] Stone M H 1932 *Ann. Math.* **33** No. 3 643; Neumann V J 1932 *Ann. Math.* **33** No. 3 567
- [2] Bohm A and Gadella M 1989 *Dirac Kets, Gamow Vectors and Gelfand Triplet* Lecture Note in Physics Vol **348** (New York and Berlin: Springer-Verlag), specifically Ch4 P.82
- [3] Dehmelt H, Nagourney W and Sandberg J 1986 *Phys. Rev. Lett.* **56** 2797; Sauter Th, Neuhauser W, Blatt R and Toschek P E 1986 *Phys. Rev. Lett.* **57** 1696; Bergquist J C, Hulet R G, Itano W M and Wineland D J 1986 *Phys. Rev. Lett.* **57** 1699; Peik E, Hollemann G and Walther H 1994 *Phys. Rev. A* **49** 402; Zhao X, Yu N, Dehmelt H and Nagourney W 1995 *Phys. Rev. A* **51** 4483
- [4] Lax P D and Phillips R S 1967 *Scattering Theory* 1st Edition (New York: Academic Press)
- [5] Bohm A, Kaldass H and Wickramasekara S 2003 *Fortsch. Phys.* **51** No 6 569-603, 604-634
- [6] Lippmann B A and Schwinger J 1950 *Phys. Rev.* **79** 469480
- [7] The same convention is also used in Bohm A 1986, 1993, 2001 *Quantum Mechanics* (Springer-Verlag) Chapter XV and XXI
- [8] Gamow G 1928 *Z. Phys.* **51** 204-12
- [9] Breit G and Wigner E P 1936 *Phys. Rev.* **49** 519
- [10] Bohm A, Kielanoski P and Wickramasekara S 2006 *Ann. Phys.* **32** 2299

- [11] Strauss Y, Horwitz L P and Eisenberg E 2000 *J. Math. Phys.* **41** No. 12 8050; Strauss Y 2003 *Int. J. Theor. Phys.* **42** 2285
- [12] Gelfand I M and Vilenkin G F 1967 *Generalised Functions* Vol IV (New York: Accademic Press) Chapter 1
- [13] Paley R and Wiener N 1934 *Fourier Transform in the Complex Domain* (New York: American Mathematical Society)
- [14] Oates C W, Vogel K R and Hall J L 1996 *Phys. Rev. Lett.* **76** 2866; Volz U, Majerus M, Liebel H, Schmitt A and Schmoranzner A 1996 *Phys. Rev. Lett.* **76** 2862