

A curved Hénon–Heiles system and its integrable perturbations

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Abstract. The constant curvature analogue on the two-dimensional sphere and the hyperbolic space of the integrable Hénon–Heiles Hamiltonian \mathcal{H} given by

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) + \Omega(q_1^2 + 4q_2^2) + \alpha(q_1^2 q_2 + 2q_2^3),$$

where Ω and α are real constants, is revisited. The resulting integrable curved Hamiltonian, \mathcal{H}_κ , depends on a parameter κ which is just the curvature of the underlying space and allows one to recover \mathcal{H} under the smooth flat/Euclidean limit $\kappa \rightarrow 0$. This system can be regarded as an integrable cubic perturbation of a specific curved 1 : 2 anisotropic oscillator, which was already known in the literature. The Ramani series of potentials associated to \mathcal{H}_κ is fully constructed, and corresponds to the curved integrable analogues of homogeneous polynomial perturbations of \mathcal{H} that are separable in parabolic coordinates. Integrable perturbations of \mathcal{H}_κ are also fully presented, and they can be regarded as the curved counterpart of integrable rational perturbations of the Euclidean Hamiltonian \mathcal{H} . It will be explicitly shown that the latter perturbations can be understood as the ‘negative index’ counterpart of the curved Ramani series of potentials. Furthermore, it is shown that the integrability of the curved Hénon–Heiles Hamiltonian \mathcal{H}_κ is preserved under the simultaneous addition of curved analogues of ‘positive’ and ‘negative’ families of Ramani potentials.

1. Introduction

The Hénon–Heiles Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) + \lambda \left(q_1^2 q_2 - \frac{1}{3} q_2^3 \right)$$

was introduced in [1] in order to model a Newtonian axially-symmetric galactic system. Nevertheless, it was soon considered as the paradigm of a two-dimensional (2D) system that exhibited chaotic behaviour (see, for instance, [2, 3, 4]). Later on, when the following generalization containing adjustable parameters was introduced

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) + \Omega_1 q_1^2 + \Omega_2 q_2^2 + \alpha(q_1^2 q_2 + \beta q_2^3),$$

it was proven that the only Liouville-integrable [5] members of this family of generalized Hénon–Heiles Hamiltonians were given by *three* specific choices of the real parameters Ω_1 , Ω_2 , α and β (see [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] and references therein):



- The Sawada–Kotera system, given by $\beta = 1/3$ and $\Omega_1 = \Omega_2 = \Omega$:

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) + \Omega(q_1^2 + q_2^2) + \alpha\left(q_1^2 q_2 + \frac{1}{3}q_2^3\right). \quad (1)$$

- The Korteweg–de Vries (KdV) system, with $\beta = 2$ and (Ω_1, Ω_2) arbitrary parameters:

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) + \Omega_1 q_1^2 + \Omega_2 q_2^2 + \alpha(q_1^2 q_2 + 2q_2^3). \quad (2)$$

- The Kaup–Kuperschmidt system, with $\beta = 16/3$ and $\Omega_2 = 16\Omega_1 = 16\Omega$:

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) + \Omega(q_1^2 + 16q_2^2) + \alpha\left(q_1^2 q_2 + \frac{16}{3}q_2^3\right). \quad (3)$$

Beyond the former integrable cases, it is worthy to emphasize that there exists a very interesting family of integrable homogeneous potentials, deeply connected to the particular KdV system (2) that arises when $\Omega_2 = 4\Omega_1$. These are the so-called Ramani series of integrable potentials [19, 20], which can be freely superposed by preserving integrability, since they are just the polynomial potentials on the Euclidean plane that can be separated in parabolic coordinates [11, 21]. Furthermore, this classical separability property underlies the fact that a large collection of *integrable rational perturbations* can be added to the Ramani potentials by preserving the integrability of the complete Hamiltonian (see [21, 22, 23, 24, 25] and references therein).

In this paper we firstly review the integrable *curved* analogue on the 2D sphere \mathbf{S}^2 and the hyperbolic (or Lobachevski) space \mathbf{H}^2 of the *flat* KdV Hénon–Heiles Hamiltonian (2) with $\Omega_2 = 4\Omega_1$, which has been recently presented in [26], together with the full curved counterpart of the integrable Ramani series of potentials. In this approach, all the results depend on the Gaussian curvature κ of the underlying space in an explicit form, so that all the flat/Euclidean expressions can be recovered performing the zero-curvature limit (contraction) $\kappa \rightarrow 0$ from the curved expressions. Alternatively, the curvature κ can also be understood as a *deformation parameter* providing the curved systems as deformed versions from the flat/Euclidean ones that preserve the integrability of the former. Secondly, we present new integrable perturbations of the curved KdV system, thus generalizing the results of [26].

The structure of the paper is as follows. In the next section, we review all the *flat* integrable Hamiltonian background, that is, the properties and structure of the KdV Hénon–Heiles Hamiltonian \mathcal{H} (hereafter with $\Omega_2 = 4\Omega_1$) and the Ramani series of potentials on the Euclidean plane \mathbf{E}^2 . In section 3, we construct the known integrable rational perturbations of \mathcal{H} as Ramani potentials with *negative* indices. In section 4, we briefly review the ambient (or Weierstrass) and Beltrami (projective) canonical variables for \mathbf{S}^2 and \mathbf{H}^2 , which are needed in the curved framework. The resulting curved KdV Hénon–Heiles Hamiltonian \mathcal{H}_κ and curved Ramani potentials are addressed in section 5. Finally, section 6 is devoted to present the new integrable perturbations of \mathcal{H}_κ , that can be understood as the ‘negative index’ counterpart of the curved Ramani potentials, as well as the result that the superposition of all the curved Ramani terms does preserve the integrability of the system.

2. An integrable Hénon–Heiles system on the plane

Let us consider the following tuning $\Omega_2 = 4\Omega_1 = 4\Omega$ in the KdV Hénon–Heiles Hamiltonian (2) defined on \mathbf{E}^2 :

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) + \Omega(q_1^2 + 4q_2^2) + \alpha(q_1^2 q_2 + 2q_2^3), \quad (4)$$

where (q_1, q_2) are Cartesian coordinates and (p_1, p_2) their conjugate momenta satisfying the usual canonical Poisson bracket $\{q_i, p_j\} = \delta_{ij}$. This system is known to be endowed with a constant of motion *quadratic* in the momenta and given by

$$\mathcal{I} = p_1(q_1 p_2 - q_2 p_1) + q_1^2 \left(2\Omega q_2 + \frac{\alpha}{4}(q_1^2 + 4q_2^2) \right), \quad (5)$$

that is, $\{\mathcal{H}, \mathcal{I}\} = 0$. Hence \mathcal{H} is integrable in the Liouville sense. Notice that the Hamiltonian (4) can be regarded as an integrable cubic perturbation added to the 1 : 2 oscillator with frequencies $(\omega, 2\omega)$ once the identification $\omega^2 = 2\Omega$ is performed.

It is worth making a carefully analysis of the potentials composing both the Hamiltonian (4) and its invariant (5). For this, let us recall that the so-called *Ramani series of integrable potentials* consists of homogeneous polynomial potentials of degree n given by [19, 20]

$$\mathcal{V}_n(q_1, q_2) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2i} \binom{n-i}{i} q_1^{2i} q_2^{n-2i}, \quad n = 1, 2, \dots \quad (6)$$

In this respect, we remark that the quadratic potential (the 1 : 2 oscillator) and the cubic potential in \mathcal{H} , say \mathcal{V}_2 and \mathcal{V}_3 , are just the second- and the third-order Ramani potentials, respectively. Moreover, the integral \mathcal{I} contains the linear \mathcal{V}_1 and the quadratic \mathcal{V}_2 Ramani potentials; namely

$$\mathcal{V}_1 = 2q_2, \quad \mathcal{V}_2 = q_1^2 + 4q_2^2, \quad \mathcal{V}_3 = 4q_1^2 q_2 + 8q_2^3.$$

In general, it can be straightforwardly proven that a Hamiltonian \mathcal{H}_n containing the Ramani potential \mathcal{V}_n is Liouville integrable, and its integral of the motion \mathcal{L}_n involves the \mathcal{V}_{n-1} potential, namely

$$\mathcal{H}_n = \frac{1}{2}(p_1^2 + p_2^2) + \alpha_n \mathcal{V}_n, \quad \mathcal{L}_n = p_1(q_1 p_2 - q_2 p_1) + \alpha_n q_1^2 \mathcal{V}_{n-1}, \quad \{\mathcal{H}_n, \mathcal{L}_n\} = 0. \quad (7)$$

The formula (6) requires the definition of the 0-th order Ramani potential as a trivial constant $\mathcal{V}_0 := 1$, that is, the first Hamiltonian system within the Ramani series reads

$$\mathcal{H}_1 = \frac{1}{2}(p_1^2 + p_2^2) + \alpha_1 (2q_2), \quad \mathcal{L}_1 = p_1(q_1 p_2 - q_2 p_1) + \alpha_1 q_1^2.$$

A crucial mathematical property of the Ramani potentials is the fact that they can be freely superposed without losing integrability [20, 24, 25]. More explicitly:

Proposition 1. *The Hamiltonian written in Cartesian canonical variables (p_1, p_2, q_1, q_2) as*

$$\mathcal{H}_{(M)} = \frac{1}{2}(p_1^2 + p_2^2) + \sum_{n=1}^M \alpha_n \mathcal{V}_n = \frac{1}{2}(p_1^2 + p_2^2) + \sum_{n=1}^M \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_n 2^{n-2i} \binom{n-i}{i} q_1^{2i} q_2^{n-2i}, \quad (8)$$

where $M \in \mathbb{N}^+$ and α_n are arbitrary real constants, is endowed with the following integral of the motion

$$\begin{aligned} \mathcal{L}_{(M)} &= p_1(q_1 p_2 - q_2 p_1) + q_1^2 \sum_{n=1}^M \alpha_n \mathcal{V}_{n-1} \\ &= p_1(q_1 p_2 - q_2 p_1) + q_1^2 \left(\sum_{n=1}^M \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \alpha_n 2^{n-1-2i} \binom{n-1-i}{i} q_1^{2i} q_2^{n-1-2i} \right). \end{aligned} \quad (9)$$

Therefore, the relationship between the KdV Hénon–Heiles Hamiltonian \mathcal{H} (4) and its constant of motion \mathcal{I} (5) with the Hamiltonian $\mathcal{H}_{(M)}$ (8) and the integral $\mathcal{L}_{(M)}$ (9) comes out as a byproduct of proposition 1, since by setting

$$M = 3, \quad \alpha_1 = 0, \quad \alpha_2 = \Omega, \quad \alpha_3 = \alpha/4, \quad (10)$$

we obtain that

$$\begin{aligned} \mathcal{H} &\equiv \mathcal{H}_{(3)} = \frac{1}{2}(p_1^2 + p_2^2) + \alpha_2 \mathcal{V}_2 + \alpha_3 \mathcal{V}_3, \\ \mathcal{I} &\equiv \mathcal{L}_{(3)} = p_1(q_1 p_2 - q_2 p_1) + q_1^2 (\alpha_2 \mathcal{V}_1 + \alpha_3 \mathcal{V}_2). \end{aligned}$$

3. Integrable rational perturbations of a KdV Hénon–Heiles system on the plane

It is worth stressing that the Ramani potentials can be extended in order to provide integrable rational perturbations of the Hamiltonian (4) by starting from the 0-th order Ramani potential and going ‘backwards’, that is, by considering *negative* indices n .

In order to make this statement explicit, let us start from the trivial Hamiltonian defined by the 0-th order potential $\mathcal{V}_0 := 1$. This is clearly integrable as Poisson-commutes with the integral \mathcal{L}_0 given by

$$\mathcal{H}_0 = \frac{1}{2}(p_1^2 + p_2^2) + \alpha_0 \mathcal{V}_0, \quad \mathcal{L}_0 = p_1(q_1 p_2 - q_2 p_1) + \alpha_0 1. \quad (11)$$

Therefore, according to (7), it seems natural to define

$$\mathcal{L}_0 := p_1(q_1 p_2 - q_2 p_1) + \alpha_0 q_1^2 \mathcal{V}_{-1}, \quad \text{where} \quad \mathcal{V}_{-1} := \frac{1}{q_1^2} \equiv \frac{\mathcal{V}_0}{q_1^2}.$$

It is quite remarkable that \mathcal{V}_{-1} is just a Rosochatius or Winternitz potential [27, 28, 29]. From it, we can construct the corresponding $n = -1$ potential which is again integrable; namely

$$\mathcal{H}_{-1} = \frac{1}{2}(p_1^2 + p_2^2) + \alpha_{-1} \mathcal{V}_{-1}, \quad \mathcal{L}_{-1} = p_1(q_1 p_2 - q_2 p_1) + \alpha_{-1} q_1^2 \mathcal{V}_{-2}, \quad \mathcal{V}_{-2} := -\frac{2q_2}{q_1^4} \equiv -\frac{\mathcal{V}_1}{q_1^4}.$$

In this way, the complete series of rational perturbations of the Hamiltonian (4), which can be understood as the ‘negative’ counterparts of the Ramani potentials (6), are found to be [21, 23]:

$$\begin{aligned} \mathcal{H}_{-n} &= \frac{1}{2}(p_1^2 + p_2^2) + \alpha_{-n} \mathcal{V}_{-n}, \quad \mathcal{L}_{-n} = p_1(q_1 p_2 - q_2 p_1) + \alpha_{-n} q_1^2 \mathcal{V}_{-(n+1)}, \\ \mathcal{V}_{-n} &= (-1)^{n+1} \frac{\mathcal{V}_{n-1}}{q_1^{2n}}, \quad \{\mathcal{H}_{-n}, \mathcal{L}_{-n}\} = 0, \quad n = 1, 2, \dots \end{aligned} \quad (12)$$

Nevertheless, for the sake of clarity, let us consider real parameters with positive indices, λ_n , defined by

$$\lambda_n := (-1)^{n+1} \alpha_{-n}, \quad n = 1, 2, \dots \quad (13)$$

which allow us to rewrite (12) as

$$\mathcal{H}_{-n} = \frac{1}{2}(p_1^2 + p_2^2) + \lambda_n \frac{\mathcal{V}_{n-1}}{q_1^{2n}}, \quad \mathcal{L}_{-n} = p_1(q_1 p_2 - q_2 p_1) - \lambda_n \frac{\mathcal{V}_n}{q_1^{2n}}, \quad n = 1, 2, \dots \quad (14)$$

Moreover, the Ramani potentials (6), the 0-th potential $V_0 = 1$ and the rational perturbations (14) can be freely superposed leading to an integrable Hamiltonian that generalizes the results given in proposition 1 as follows.

Proposition 2. [25] *The Hamiltonian given by*

$$\begin{aligned}\mathcal{H}_{(M,R)} &= \frac{1}{2} (p_1^2 + p_2^2) + \sum_{n=1}^M \alpha_n \mathcal{V}_n + \alpha_0 \mathcal{V}_0 + \sum_{n=1}^R \lambda_n \frac{\mathcal{V}_{n-1}}{q_1^{2n}} \\ &= \frac{1}{2} (p_1^2 + p_2^2) + \sum_{n=1}^M \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_n 2^{n-2i} \binom{n-i}{i} q_1^{2i} q_2^{n-2i} + \alpha_0 \\ &\quad + \sum_{n=1}^R \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \lambda_n 2^{n-1-2i} \binom{n-1-i}{i} \frac{q_2^{n-1-2i}}{q_1^{2(n-i)}},\end{aligned}\quad (15)$$

where α_n , α_0 and λ_n are arbitrary real constants, is integrable for any indices $M, R \in \mathbb{N}^+$. The corresponding integral of the motion reads

$$\mathcal{L}_{(M,R)} = p_1(q_1 p_2 - q_2 p_1) + q_1^2 \sum_{n=1}^M \alpha_n \mathcal{V}_{n-1} + \alpha_0 \mathcal{V}_0 - \sum_{n=1}^R \lambda_n \frac{\mathcal{V}_n}{q_1^{2n}},$$

where \mathcal{V}_n are given in (6) and $\mathcal{V}_0 = 1$.

For instance, if we set $M = 3$ and $R = 4$ we obtain the following integrable generalization of the KdV Hénon–Heiles Hamiltonian (4):

$$\begin{aligned}\mathcal{H}_{(3,4)} &= \frac{1}{2} (p_1^2 + p_2^2) + \alpha_1 \mathcal{V}_1 + \alpha_2 \mathcal{V}_2 + \alpha_3 \mathcal{V}_3 + \alpha_0 \mathcal{V}_0 + \lambda_1 \frac{\mathcal{V}_0}{q_1^2} + \lambda_2 \frac{\mathcal{V}_1}{q_1^4} + \lambda_3 \frac{\mathcal{V}_2}{q_1^6} + \lambda_4 \frac{\mathcal{V}_3}{q_1^8} \\ &= \frac{1}{2} (p_1^2 + p_2^2) + \alpha_1 (2q_2) + \alpha_2 (q_1^2 + 4q_2^2) + \alpha_3 (4q_1^2 q_2 + 8q_2^3) + \alpha_0 \\ &\quad + \lambda_1 \frac{1}{q_1^2} + \lambda_2 \frac{2q_2}{q_1^4} + \lambda_3 \frac{q_1^2 + 4q_2^2}{q_1^6} + \lambda_4 \frac{4q_1^2 q_2 + 8q_2^3}{q_1^8},\end{aligned}$$

which Poisson-commutes with

$$\begin{aligned}\mathcal{L}_{(3,4)} &= p_1(q_1 p_2 - q_2 p_1) + q_1^2 (\alpha_1 \mathcal{V}_0 + \alpha_2 \mathcal{V}_1 + \alpha_3 \mathcal{V}_2) + \alpha_0 \mathcal{V}_0 \\ &\quad - \left(\lambda_1 \frac{\mathcal{V}_1}{q_1^2} + \lambda_2 \frac{\mathcal{V}_2}{q_1^4} + \lambda_3 \frac{\mathcal{V}_3}{q_1^6} + \lambda_4 \frac{\mathcal{V}_4}{q_1^8} \right).\end{aligned}$$

Recall that the λ_1 -potential behaves as a centrifugal barrier on \mathbf{E}^2 when $\lambda_1 > 0$ [29].

4. Ambient and Beltrami canonical variables

In order to achieve the generalization of the above results to the 2D sphere \mathbf{S}^2 and hyperbolic space \mathbf{H}^2 , let us consider the one-parameter family of 3D real Lie algebras $\mathfrak{so}_\kappa(3)$ with commutation relations and Casimir invariant given by [29, 30]:

$$[J_{12}, J_{01}] = J_{02}, \quad [J_{12}, J_{02}] = -J_{01}, \quad [J_{01}, J_{02}] = \kappa J_{12}, \quad (16)$$

$$\mathcal{C} = J_{01}^2 + J_{02}^2 + \kappa J_{12}^2, \quad (17)$$

where κ is a real parameter. The 2D homogeneous space $\text{SO}_\kappa(3)/\text{SO}(2)$, where $\text{SO}_\kappa(3)$ is the Lie group of $\mathfrak{so}_\kappa(3)$ and $\text{SO}(2) = \langle J_{12} \rangle$, has constant Gaussian curvature equal to κ . This generic family of homogeneous space comprises the three relevant cases with constant curvature:

$$\begin{array}{lll} \kappa > 0 : \text{ Sphere} & \kappa = 0 : \text{ Euclidean plane} & \kappa < 0 : \text{ Hyperbolic space} \\ \mathbf{S}^2 = \text{SO}(3)/\text{SO}(2) & \mathbf{E}^2 = \text{ISO}(2)/\text{SO}(2) & \mathbf{H}^2 = \text{SO}(2,1)/\text{SO}(2) \end{array}$$

These 2D spaces can be embedded in $\mathbb{R}^3 = (x_0, x_1, x_2)$ where the *ambient* or Weierstrass coordinates must satisfy

$$x_0^2 + \kappa(x_1^2 + x_2^2) = 1.$$

Next, if we apply a central projection with pole $(0, 0, 0) \in \mathbb{R}^3$ from $(x_0, x_1, x_2) \in \mathbb{R}^3$ to the 2D projective space, we obtain the *Beltrami* coordinates $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$ given by

$$x_0 = \frac{1}{\sqrt{1 + \kappa \mathbf{q}^2}}, \quad \mathbf{x} = \frac{\mathbf{q}}{\sqrt{1 + \kappa \mathbf{q}^2}}, \quad \mathbf{q} = \frac{\mathbf{x}}{x_0}, \quad (18)$$

such that $\mathbf{x} = (x_1, x_2)$, the conjugate Beltrami momenta are $\mathbf{p} = (p_1, p_2)$ and hereafter we denote

$$\mathbf{q}^2 = q_1^2 + q_2^2, \quad \mathbf{p}^2 = p_1^2 + p_2^2, \quad \mathbf{q} \cdot \mathbf{p} = q_1 p_1 + q_2 p_2.$$

A symplectic realization of $\mathfrak{so}_\kappa(3)$ (16) in terms of the Beltrami canonical variables (\mathbf{q}, \mathbf{p}) turns out to be [28, 29, 30]

$$J_{0i} = p_i + \kappa(\mathbf{q} \cdot \mathbf{p})q_i, \quad i = 1, 2; \quad J_{12} = q_1 p_2 - q_2 p_1. \quad (19)$$

In this framework, the curved kinetic energy \mathcal{T}_κ for a particle moving on these spaces comes from the Casimir (17) under the above realization:

$$\mathcal{T}_\kappa \equiv \frac{1}{2}\mathcal{C} = \frac{1}{2}(J_{01}^2 + J_{02}^2 + \kappa J_{12}^2) = \frac{1}{2}(1 + \kappa \mathbf{q}^2)(\mathbf{p}^2 + \kappa(\mathbf{q} \cdot \mathbf{p})^2). \quad (20)$$

Notice that the flat/Euclidean limit $\kappa \rightarrow 0$ of the above expressions leads to

$$x_0 = 1, \quad \mathbf{x} = \mathbf{q}, \quad J_{0i} = p_i, \quad J_{12} = q_1 p_2 - q_2 p_1, \quad \mathcal{T} = \frac{1}{2}\mathbf{p}^2.$$

5. A KdV Hénon–Heiles system on the sphere and the hyperbolic space

In this section we summarize the construction of the curved counterpart of the KdV Hénon–Heiles system (4), that has been recently presented in [26] by making use of of ambient and Beltrami dynamical variables. Such construction requires to obtain, firstly, the curved Ramani potentials $\mathcal{V}_{\kappa,n}$ and, secondly, their superposition, so generalizing proposition 1 to the curved case. In this way, the definition of the curved integrable KdV Hénon–Heiles Hamiltonian \mathcal{H}_κ comes out as a byproduct, and the main object in the construction here presented turns out to be the curved Ramani potentials.

Proposition 3. [26] *The Ramani potentials on the sphere \mathbf{S}^2 and the hyperbolic space \mathbf{H}^2 are defined in Beltrami coordinates (q_1, q_2) (18) as*

$$\mathcal{V}_{\kappa,n} = \left(\frac{1 + \kappa \mathbf{q}^2}{1 - \kappa q_2^2} \right)^2 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2i} \binom{n-i}{i} \left(\frac{q_1}{\sqrt{1 + \kappa \mathbf{q}^2}} \right)^{2i} \left(1 - \frac{i}{n-i} \left[\frac{\kappa q_1^2}{1 + \kappa \mathbf{q}^2} \right] \right) \left(\frac{q_2}{1 + \kappa \mathbf{q}^2} \right)^{n-2i} \quad (21)$$

with $n = 1, 2 \dots$. Each Ramani Hamiltonian

$$\mathcal{H}_{\kappa,n} = \mathcal{T}_{\kappa} + \alpha_n \mathcal{V}_{\kappa,n},$$

is integrable, as is endowed with a constant of motion $\mathcal{L}_{\kappa,n}$ which is quadratic in the momenta

$$\mathcal{L}_{\kappa,n} = J_{01}J_{12} + \alpha_n \frac{q_1^2}{1 + \kappa \mathbf{q}^2} \mathcal{V}_{\kappa,n-1}, \quad \{\mathcal{H}_{\kappa,n}, \mathcal{L}_{\kappa,n}\} = 0, \quad (22)$$

where $\mathcal{V}_{\kappa,0}$ is defined by

$$\mathcal{V}_{\kappa,0} := \frac{(1 + \kappa q_2^2)(1 + \kappa \mathbf{q}^2)}{(1 - \kappa q_2^2)^2}, \quad (23)$$

and J_{01} , J_{12} and \mathcal{T}_{κ} are the functions given by (19) and (20).

We stress that the flat limit $\kappa \rightarrow 0$ of the above expressions leads to the Euclidean Ramani potential \mathcal{V}_n (6), the Hamiltonian \mathcal{H}_n and its integral of motion \mathcal{L}_n (7) along with $\mathcal{V}_0 = 1$. Recall also that, under the flat limit, Beltrami coordinates reduce to Cartesian ones. However, note that in the curved case $\mathcal{V}_{\kappa,0}$ is no longer a trivial potential. We also remark that the quadratic Ramani Hamiltonian, $\mathcal{H}_{\kappa,2} = \mathcal{T}_{\kappa} + \alpha_2 \mathcal{V}_{\kappa,2}$, is just the *superintegrable* curved 1 : 2 oscillator, formerly introduced in [31] and further studied in [29, 30].

In terms of the ambient coordinates (x_0, x_1, x_2) (18), the curved Ramani potentials (21) and (23) turn out to be

$$\mathcal{V}_{\kappa,0} = \frac{1 - \kappa x_1^2}{(x_0^2 - \kappa x_2^2)^2}, \quad \mathcal{V}_{\kappa,n} = \frac{1}{(x_0^2 - \kappa x_2^2)^2} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2i} \binom{n-i}{i} x_1^{2i} \left(1 - \frac{i}{n-i} \kappa x_1^2\right) (x_0 x_2)^{n-2i} \quad (24)$$

which affords their parametrization in any coordinate system under the appropriate change of variables.

Furthermore, as in the Euclidean case, the curved Ramani potentials can be freely superposed. Therefore, the expressions (8) and (9) given in proposition 1 can be generalized to the curved case as follows.

Proposition 4. [26] *The Hamiltonian formed by the linear superposition of the curved Ramani potentials (21) and given by*

$$\mathcal{H}_{\kappa,(M)} = \mathcal{T}_{\kappa} + \sum_{n=1}^M \alpha_n \mathcal{V}_{\kappa,n}, \quad M \in \mathbb{N}^+, \quad (25)$$

Poisson-commutes with the function

$$\mathcal{L}_{\kappa,(M)} = J_{01}J_{12} + \frac{q_1^2}{1 + \kappa \mathbf{q}^2} \sum_{n=1}^M \alpha_n \mathcal{V}_{\kappa,n-1},$$

where J_{01} , J_{12} and \mathcal{T}_{κ} are the functions given by (19) and (20).

As a straightforward consequence, the curved counterpart of the Hénon–Heiles KdV Hamiltonian (4) on \mathbf{S}^2 and \mathbf{H}^2 along with its integral (5), written in Beltrami variables, can be obtained from proposition 4 for the particular case $\mathcal{H}_{\kappa,(3)}$ by setting (10); namely

$$\begin{aligned} \mathcal{H}_{\kappa} &= \mathcal{T}_{\kappa} + \mathcal{V}_{\kappa} = \mathcal{T}_{\kappa} + \Omega \mathcal{V}_{\kappa,2} + \frac{\alpha}{4} \mathcal{V}_{\kappa,3}, \\ \mathcal{V}_{\kappa} &= \Omega \frac{q_1^2(1 + \kappa q_2^2) + 4q_2^2}{(1 - \kappa q_2^2)^2} + \alpha \frac{q_1^2 q_2(1 + \kappa \mathbf{q}^2 - \frac{1}{2} \kappa q_1^2) + 2q_2^3}{(1 - \kappa q_2^2)^2(1 + \kappa \mathbf{q}^2)}. \end{aligned} \quad (26)$$

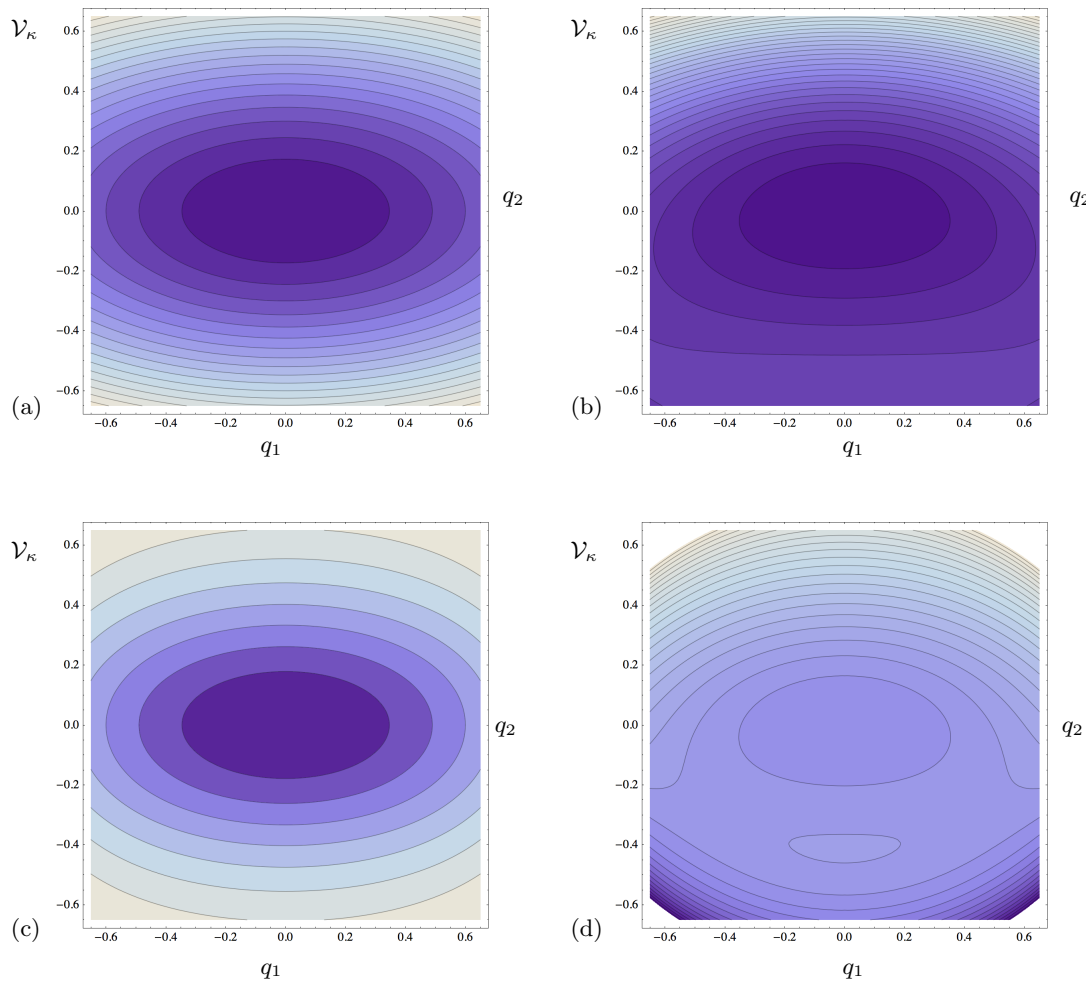


Figure 1. Level plots for the potential \mathcal{V}_κ (26) in Beltrami projective coordinates (q_1, q_2) for four different values of the constants (κ, Ω, α) involved. Values of the potential function \mathcal{V}_κ are coded as follows: dark blue represents low values, and higher values are red-shifted. Figure (a) corresponds to $(\kappa, \Omega, \alpha) = (0, 1, 0)$, (b) to $(0, 1, 2)$, (c) to $(-1, 1, 0)$ and (d) to $(-1, 1, 2)$.

And, therefore, the Hamiltonian \mathcal{H}_κ Poisson-commutes with the corresponding integral of the motion that comes from $\mathcal{L}_{\kappa,(3)}$:

$$\begin{aligned} \mathcal{I}_\kappa &= J_{01}J_{12} + \frac{q_1^2}{1 + \kappa \mathbf{q}^2} \left(\Omega \mathcal{V}_{\kappa,1} + \frac{\alpha}{4} \mathcal{V}_{\kappa,2} \right) \\ &= (p_1 + \kappa(\mathbf{q} \cdot \mathbf{p})q_1) (q_1p_2 - q_2p_1) + \frac{q_1^2}{1 + \kappa \mathbf{q}^2} \left(\Omega \frac{2q_2(1 + \kappa \mathbf{q}^2)}{(1 - \kappa q_2^2)^2} + \alpha \frac{q_1^2(1 + \kappa q_2^2) + 4q_2^2}{4(1 - \kappa q_2^2)^2} \right). \end{aligned}$$

Level plots for the potential \mathcal{V}_κ (26) are shown in figure 1, where the Euclidean ($\kappa = 0$) and hyperbolic ($\kappa < 0$) KdV Hénon–Heiles potentials are represented and compared. We stress that both the curvature κ and the constant α can be considered as deformation parameters that preserve the integrability of the initial Hamiltonian. Case (a) represents the flat 1 : 2 (superintegrable) anharmonic oscillator potential on \mathbf{E}^2 with no Hénon–Heiles term, in particular with $\kappa = 0$, $\Omega = 1$ and $\alpha = 0$. In case (b) the Hénon–Heiles cubic term is added on \mathbf{E}^2 through $\alpha = 2$ (we keep $\Omega = 1$), while the underlying space is still flat ($\kappa = 0$). Here we see that

the Hénon–Heiles term breaks the $q_2 \rightarrow -q_2$ symmetry of the anharmonic oscillator. The (superintegrable) curved 1 : 2 anharmonic oscillator potential on \mathbf{H}^2 with no Hénon–Heiles term ($\kappa = -1$, $\Omega = 1$ and $\alpha = 0$) is represented in case (c). Now we realize that the non-vanishing curvature modifies the values of the potential, although its general shape around the origin is quite similar to the flat case. Finally, when the curved Hénon–Heiles term is added ($\kappa = -1$, $\Omega = 1$ and $\alpha = 2$) the plot (d) is obtained on \mathbf{H}^2 . Note that although superintegrability is broken when the Hénon–Heiles term is considered, both (b) and (d) potentials always generate integrable motions on the hyperbolic plane.

6. Integrable perturbations of a curved KdV Hénon–Heiles system

In this last section we present, as new results, the curved analogues on \mathbf{S}^2 and \mathbf{H}^2 of the ‘negative’ counterparts of the Ramani potentials (12) and of their integrable superposition (15).

As in section 3, let us start from the curved Hamiltonian $\mathcal{H}_{\kappa,0}$ corresponding to the potential $\mathcal{V}_{\kappa,0}$ (23), which, in contrast with the flat case $\mathcal{V}_0 = 1$, is now a non-trivial function:

$$\mathcal{H}'_{\kappa,0} = \mathcal{T}_{\kappa} + \alpha_0 \mathcal{V}_{\kappa,0} = \mathcal{T}_{\kappa} + \alpha_0 \frac{(1 + \kappa q_2^2)(1 + \kappa \mathbf{q}^2)}{(1 - \kappa q_2^2)^2}.$$

This Hamiltonian is, in fact, integrable as it Poisson-commutes with the function

$$\mathcal{L}'_{\kappa,0} = J_{01}J_{12} + \alpha_0 \frac{2\kappa q_1^2 q_2}{(1 - \kappa q_2^2)^2}.$$

According to (22), let us define

$$\mathcal{L}'_{\kappa,0} := J_{01}J_{12} + \alpha_0 \frac{q_1^2}{1 + \kappa \mathbf{q}^2} \mathcal{V}'_{\kappa,-1}, \quad \mathcal{V}'_{\kappa,-1} := \kappa \frac{2q_2(1 + \kappa \mathbf{q}^2)}{(1 - \kappa q_2^2)^2} \equiv \kappa \mathcal{V}_{\kappa,1}.$$

By induction, it can be easily shown that

$$\mathcal{V}'_{\kappa,-n} = \kappa \mathcal{V}_{\kappa,n}, \quad n = 1, 2, \dots$$

and they vanish when $\kappa = 0$. Consequently, under this procedure we have obtained curved Beltrami potentials with negative index, but they are just proportional to (21) and, therefore, we have not obtained any new result concerning integrable perturbations.

However, it turns out that the curved analogue of the rational perturbations (12) can be obtained by starting from the trivial (free geodesic) curved Hamiltonian with a constant potential α_0 (as in (11)) which is clearly integrable; explicitly

$$\mathcal{H}_{\kappa,0} = \mathcal{T}_{\kappa} + \alpha_0, \quad \mathcal{L}_{\kappa,0} = J_{01}J_{12} + \alpha_0, \quad \{\mathcal{H}_{\kappa,0}, \mathcal{L}_{\kappa,0}\} = 0.$$

If we again assume that recurrence (22) should hold, we are led to define

$$\mathcal{L}_{\kappa,0} := J_{01}J_{12} + \alpha_0 \frac{q_1^2}{1 + \kappa \mathbf{q}^2} \mathcal{V}_{\kappa,-1}, \quad \mathcal{V}_{\kappa,-1} := \frac{1 + \kappa \mathbf{q}^2}{q_1^2}.$$

Surprisingly enough, we find that $\mathcal{V}_{\kappa,-1} \equiv 1/x_1^2$ is just the curved Rosochatius or Winternitz potential, which corresponds to a noncentral oscillator potential on the sphere with center (in ambient coordinates) located at $O_1 = (0, 1, 0)$ (see [28, 29, 31, 32] for a detailed discussion).

Now, if we construct the Hamiltonian $\mathcal{H}_{\kappa,-1} = \mathcal{T}_{\kappa} + \alpha_{-1} \mathcal{V}_{\kappa,-1}$, the potential $\mathcal{V}_{\kappa,-2}$ is then obtained through the corresponding integral of motion:

$$\mathcal{L}_{\kappa,-1} = J_{01}J_{12} + \alpha_{-1} \frac{q_1^2}{1 + \kappa \mathbf{q}^2} \mathcal{V}_{\kappa,-2}, \quad \text{where} \quad \mathcal{V}_{\kappa,-2} := -\frac{2q_2(1 + \kappa \mathbf{q}^2)}{q_1^4} = -\frac{(1 - \kappa q_2^2)^2}{q_1^4} \mathcal{V}_{\kappa,1}.$$

Table 1. The Ramani potentials for $n = \{0, \pm 1, \pm 2, \pm 3\}$ on \mathbf{E}^2 in Cartesian coordinates \mathbf{q} (6) and (12) along with their curved counterpart on \mathbf{S}^2 and \mathbf{H}^2 in ambient coordinates (24) and (28) such that $x_0^2 + \kappa \mathbf{x}^2 = 1$. Recall that $x_0 = 1$ and $\mathbf{x} = \mathbf{q}$ when $\kappa = 0$.

\mathbf{E}^2 : Cartesian coordinates \mathbf{q}	\mathbf{S}^2 and \mathbf{H}^2 : Ambient coordinates (x_0, \mathbf{x})
$\mathcal{V}_{-3} = \frac{q_1^2 + 4q_2^2}{q_1^6}$	$\mathcal{V}_{\kappa,-3} = \frac{x_1^2(1 - \kappa x_1^2) + 4x_0^2 x_2^2}{x_1^6}$
$\mathcal{V}_{-2} = -\frac{2q_2}{q_1^4}$	$\mathcal{V}_{\kappa,-2} = -\frac{2x_0 x_2}{x_1^4}$
$\mathcal{V}_{-1} = \frac{1}{q_1^2}$	$\mathcal{V}_{\kappa,-1} = \frac{1}{x_1^2}$
$\mathcal{V}_0 = 1$	$\mathcal{V}_{\kappa,0} = \frac{1 - \kappa x_1^2}{(x_0^2 - \kappa x_2^2)^2}$
$\mathcal{V}_1 = 2q_2$	$\mathcal{V}_{\kappa,1} = \frac{2x_0 x_2}{(x_0^2 - \kappa x_2^2)^2}$
$\mathcal{V}_2 = q_1^2 + 4q_2^2$	$\mathcal{V}_{\kappa,2} = \frac{x_1^2(1 - \kappa x_1^2) + 4x_0^2 x_2^2}{(x_0^2 - \kappa x_2^2)^2}$
$\mathcal{V}_3 = 4q_1^2 q_2 + 8q_2^3$	$\mathcal{V}_{\kappa,3} = \frac{4x_0 x_1^2 x_2(1 - \frac{1}{2}\kappa x_1^2) + 8x_0^3 x_2^3}{(x_0^2 - \kappa x_2^2)^2}$

From this, the integrable curved Ramani potentials $\mathcal{V}_{\kappa,-n}$ can be defined as follows.

Proposition 5. *Let us define*

$$\mathcal{V}_{\kappa,-1} = \frac{1 + \kappa \mathbf{q}^2}{q_1^2}, \quad \mathcal{V}_{\kappa,-m} = (-1)^{m+1} \frac{(1 - \kappa q_2^2)^2}{q_1^{2m}} (1 + \kappa \mathbf{q}^2)^{m-2} \mathcal{V}_{\kappa,m-1}, \quad (27)$$

with $m = 2, 3, \dots$ and $\mathcal{V}_{\kappa,m-1}$ given by (21). The Hamiltonian defined by $\mathcal{H}_{\kappa,-n} = \mathcal{T}_\kappa + \alpha_{-n} \mathcal{V}_{\kappa,-n}$ ($n = 1, 2, \dots$) Poisson-commutes with

$$\mathcal{L}_{\kappa,-n} = J_{01} J_{12} + \alpha_{-n} \frac{q_1^2}{1 + \kappa \mathbf{q}^2} \mathcal{V}_{\kappa,-(n+1)}.$$

Consequently, the generalization of the Euclidean rational perturbations (12) to a constant curvature framework is achieved. In ambient coordinates (18), the potentials (27) read

$$\mathcal{V}_{\kappa,-1} = \frac{1}{x_1^2}, \quad \mathcal{V}_{\kappa,-m} = (-1)^{m+1} \frac{(x_0^2 - \kappa x_2^2)^2}{x_1^{2m}} \mathcal{V}_{\kappa,m-1}, \quad m = 2, 3, \dots \quad (28)$$

with $\mathcal{V}_{\kappa,m-1}$ given in (24). These results are illustrated in table 1 by writing the first Ramani potentials with positive and negative indices on \mathbf{E}^2 , \mathbf{S}^2 and \mathbf{H}^2 . We also remark that proposition 5 can be written in terms of coefficients with positive indices, λ_n , through the definition (13).

Finally, the potentials $\mathcal{V}_{\kappa,-n}$ can also be added to the Hamiltonian (25) leading to the following full integrable curved superposition, that constitutes the main result of this paper. We write it in ambient coordinates as follows.

Theorem 6. *The Hamiltonian formed by the linear superposition of the curved Ramani potentials (24) and (28)*

$$\mathcal{H}_{\kappa,(M,R)} = \mathcal{T}_\kappa + \sum_{n=1}^M \alpha_n \mathcal{V}_{\kappa,n} + \alpha_0 \mathcal{V}_{\kappa,0} + \lambda_1 \frac{1}{x_1^2} + (x_0^2 - \kappa x_2^2)^2 \sum_{m=2}^R \lambda_m \frac{\mathcal{V}_{\kappa,m-1}}{x_1^{2m}},$$

where α_n , α_0 , λ_1 , λ_m are arbitrary real constants and $M, R \in \mathbb{N}^+$, is integrable. Its constant of the motion is given by the function

$$\mathcal{L}_{\kappa,(M,R)} = J_{01}J_{12} + x_1^2 \sum_{n=1}^M \alpha_n \mathcal{V}_{\kappa,n-1} + \alpha_0 x_1^2 \kappa \mathcal{V}_{\kappa,1} - \lambda_1 \frac{2x_2x_0}{x_1^2} - (x_0^2 - \kappa x_2^2)^2 \sum_{m=2}^R \lambda_m \frac{\mathcal{V}_{\kappa,m}}{x_1^{2m}}.$$

In this way, we have obtained the generalization of proposition 2 to \mathbf{S}^2 and \mathbf{H}^2 . The above result can be straightforwardly written in terms of Beltrami variables through (18).

Finally, we would like to mention that the construction of the curved counterpart of the KdV Hénon–Heiles Hamiltonian (2) with arbitrary parameters (Ω_1, Ω_2) , as well as of the Sawada–Kotera (1) and the Kaup–Kuperschmidt (3) Hénon–Heiles systems is currently in progress.

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References

- [1] Hénon M and Heiles C 1964 *Astron. J.* **69** 73
- [2] Tabor M 1989 *Chaos and integrability in nonlinear dynamics* (New York: Wiley)
- [3] Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics (Interdisciplinary Applied Mathematics vol 1)* (New York: Springer)
- [4] Boccaletti D and Pucacco G 2004 *Theory of Orbits* (Berlin: Springer)
- [5] Perelomov A M 1990 *Integrable systems of classical mechanics and Lie algebras* (Basel: Birkhäuser)
- [6] Bountis T, Segur H and Vivaldi F 1982 *Phys. Rev. A* **25** 1257
- [7] Chang Y F, Tabor M and Weiss J 1982 *J. Math. Phys.* **23** 531
- [8] Grammaticos B, Dorizzi B and Padjen R 1982 *Phys. Lett. A* **89** 111
- [9] Hietarinta J 1983 *Phys. Rev. A* **28** 3670
- [10] Fordy A P 1983 *Phys. Lett. A* **97** 21
- [11] Wojciechowski S 1984 *Phys. Lett. A* **100** 277
- [12] Sahadevan R and Lakshmanan M 1986 *J. Phys. A: Math. Gen.* **19** L949
- [13] Fordy A P 1990 *Physica D* **52** 204
- [14] Sarlet W 1991 *J. Phys. A: Math. Gen.* **24** 5245
- [15] Ravoson V, Gavrilov L and Caboz R 1993 *J. Math. Phys.* **34** 2385
- [16] Lakshmanan M and Sahadevan R 1993 *Phys. Rep.* **224** 1
- [17] Conte R, Fordy A P and Pickering A 1993 *Physica D* **69** 33
- [18] Conte R, Musette M and Verhoeven C 2005 *Theor. Math. Phys.* **144** 888
- [19] Ramani A, Dorizzi B and Grammaticos B 1982 *Phys. Rev. Lett.* **49** 1539
- [20] Hietarinta J 1987 *Phys. Rep.* **147** 87
- [21] Ferapontov E V and Fordy A P 1997 *J. Geom. Phys.* **21** 169
- [22] Hone A N W, Novikov V and Verhoeven C 2006 *Inv. Probl.* **22** 2001
- [23] Hone A N W, Novikov V and Verhoeven C 2008 *Phys. Lett. A* **372** 1440
- [24] Blasco A 2009 *Integrability of non-linear Hamiltonian systems with N degrees of freedom* PhD Thesis (Burgos: Burgos University)
- [25] Ballesteros A and Blasco A 2010 *Ann. Phys.* **325** 2787
- [26] Ballesteros A, Blasco A, Herranz F J and Musso 2014 arXiv:1411.2033
- [27] Fris J, Mandrossov V, Smorodinsky Y A, Uhler M and Winternitz P 1965 *Phys. Lett.* **16** 354
- [28] Ballesteros A and Herranz F J 2007 *J. Phys. A: Math. Theor.* **40** F51
- [29] Ballesteros A, Herranz F J and Musso F 2013 *Nonlinearity* **26** 971
- [30] Ballesteros A, Blasco A, Herranz F J and Musso F 2014 *J. Phys. A: Math. Theor.* **47** 345204
- [31] Rañada M F and Santander M 1999 *J. Math. Phys.* **40** 5026
- [32] Ballesteros A, Herranz F J, Santander M and Sanz-Gil T 2003 *J. Phys. A: Math. Theor.* **36** L93