

On the raising and lowering difference operators for eigenvectors of the finite Fourier transform

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Abstract. We construct explicit forms of raising and lowering difference operators that govern eigenvectors of the finite (discrete) Fourier transform. Some of the algebraic properties of these operators are also examined.

1. Introduction

We choose to set out by recalling first some mathematical aspects of the classical Fourier transform (FT). The orthonormalized wave functions $\psi_n(x)$ of the linear harmonic oscillator in non-relativistic quantum mechanics,

$$\int_{\mathbb{R}} \psi_m(x) \psi_n(x) dx = \delta_{mn}, \quad m, n = 0, 1, 2, \dots, \quad (1.1)$$

are explicitly given as

$$\psi_n(x) := c_n^{-1} H_n(\xi) \exp(-\xi^2/2), \quad c_n = \sqrt{\sqrt{\pi} 2^n n!}, \quad (1.2)$$

where $H_n(\xi)$ are the classical Hermite polynomials and $\xi := \sqrt{m\omega/\hbar} x$ is a dimensionless coordinate (see, for example, [1]). In quantum mechanics they emerge as eigenfunctions of the Hamiltonian \mathbf{H} for the linear harmonic oscillator,

$$\mathbf{H} \psi_n(x) \equiv \frac{\hbar\omega}{2} \left(\xi^2 - \frac{d^2}{d\xi^2} \right) \psi_n(x) = \hbar\omega (n + 1/2) \psi_n(x), \quad (1.3)$$

which is a self-adjoint differential operator of the second order. The wave functions $\psi_n(x)$ (we recall that the functions $H_n(x)e^{-x^2/2}$ are usually referred to as *Hermite functions* in the mathematical literature) represent an important explicit example of an orthonormal and complete system in the Hilbert space $L^2(\mathbb{R}, dx)$ of square-integrable functions on the full real line $x \in \mathbb{R}$. It is further well known that the wave functions of the linear harmonic oscillator $\psi_n(x)$ possess the simple transformation property with respect to the Fourier transform: *they are also eigenfunctions of the Fourier transform, associated with the eigenvalues i^n , that is,*

$$(\mathcal{F} \psi_n)(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy} \psi_n(y) dy = i^n \psi_n(x). \quad (1.4)$$



Assume now for the moment that we are not aware of this important mathematical fact and are interested in finding eigenfunctions of the FT operator \mathcal{F} explicitly. Since mutually commuting operators have the same set of eigenfunctions, one may solve this problem by defining such a self-adjoint differential operator with simple spectrum of distinct eigenvalues, that commutes with the FT operator (1.4). Then the eigenfunctions of that differential operator can be found by solving a corresponding to this case differential equation and they will be at same time the eigenfunctions of the FT operator (1.4). So in this way one reduces a problem of finding eigenfunctions of the FT operator \mathcal{F} to one of solving some differential equation.

Indeed, to illustrate this point let us start with the first-order differential operator $\frac{d}{dx}$ and evaluate its action on (1.4):

$$\frac{d}{dx} \int_{\mathbb{R}} e^{ixy} f(y) dy = i \int_{\mathbb{R}} e^{ixy} y f(y) dy, \quad (1.5)$$

where $f(x) \in L^2(\mathbb{R}, dx)$ and for the sake of simplicity from now on we write x instead of the dimensionless variable ξ . Consequently, from the right side of (1.5) one deduces that the next step should be to evaluate

$$x \int_{\mathbb{R}} e^{ixy} f(y) dy = -i \int_{\mathbb{R}} \left(\frac{d e^{ixy}}{dy} \right) f(y) dy = i \int_{\mathbb{R}} e^{ixy} \frac{d f(y)}{dy} dy, \quad (1.6)$$

upon integrating by parts the middle term in (1.6). From (1.5) and (1.6) it thus follows that

$$\left(x \pm \frac{d}{dx} \right) \int_{\mathbb{R}} e^{ixy} f(y) dy = \pm i \int_{\mathbb{R}} e^{ixy} \left(y \pm \frac{d}{dy} \right) f(y) dy. \quad (1.7)$$

In the operator form these identities can be represented as

$$\mathbf{a} \mathcal{F} = i \mathcal{F} \mathbf{a}, \quad \mathbf{a}^\dagger \mathcal{F} = -i \mathcal{F} \mathbf{a}^\dagger, \quad (1.8)$$

where

$$\mathbf{a} := \frac{1}{\sqrt{2}} \left(x + \frac{d}{dx} \right), \quad \mathbf{a}^\dagger := \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx} \right), \quad (1.9)$$

are the lowering and raising first-order differential operators, which obey the standard Heisenberg commutation relation

$$[\mathbf{a}, \mathbf{a}^\dagger] := \mathbf{a} \mathbf{a}^\dagger - \mathbf{a}^\dagger \mathbf{a} \equiv \left[\frac{d}{dx}, x \right] = \mathbf{I}. \quad (1.10)$$

From the identities (1.8) it follows now at once that the self-adjoint second-order differential operators $\mathbf{a} \mathbf{a}^\dagger$ and $\mathbf{a}^\dagger \mathbf{a}$ commute with the FT operator (1.4), i.e., $[\mathbf{a} \mathbf{a}^\dagger, \mathcal{F}] = 0$ and $[\mathbf{a}^\dagger \mathbf{a}, \mathcal{F}] = 0$. So we are naturally led to the desired differential operator of the second order,

$$\mathbf{H} = \frac{1}{2} (\mathbf{a} \mathbf{a}^\dagger + \mathbf{a}^\dagger \mathbf{a}) = \frac{1}{2} \left(x^2 - \frac{d^2}{dx^2} \right), \quad (1.11)$$

which commutes with the Fourier transform operator \mathcal{F} and represents, in effect, the well-known Hamiltonian (1.3) for the linear harmonic oscillator in quantum mechanics.

On the other hand, there exist a finite (or discrete) analogue of the integral Fourier transform (1.4). We recall that the *finite (discrete) Fourier transform* (FFT) based on points N is represented by an $N \times N$ unitary symmetric matrix $\Phi^{(N)} = \left(\Phi_{m,n}^{(N)} \right)$ with elements (see, for example, [2, 3])

$$\Phi_{m,n}^{(N)} := \frac{1}{\sqrt{N}} \exp \left(\frac{2\pi i}{N} m n \right) \equiv \frac{1}{\sqrt{N}} q^{mn}, \quad q := e^{\frac{2\pi i}{N}}, \quad (1.12)$$

where $m, n \in \{0, 1, \dots, N-1\}$. Given a complex valued vector \vec{y} with components $\{y_k\}_{k=0}^{N-1}$, one can compute another vector \vec{z} with components

$$z_m = \sum_{n=0}^{N-1} \Phi_{m,n}^{(N)} y_n, \quad (1.13)$$

referred to as *the finite Fourier transform* of the vector \vec{y} . Those vectors $\vec{f}^{(k)}$, which are solutions of the standard equations

$$\sum_{n=0}^{N-1} \Phi_{m,n}^{(N)} f_n^{(k)} = \lambda_k f_m^{(k)}, \quad k \in \{0, 1, \dots, N-1\}, \quad (1.14)$$

then represent eigenvectors of the FFT operator $\Phi^{(N)}$, associated with the eigenvalues λ_k . Since the fourth power of $\Phi^{(N)}$ is the unit matrix, the only four distinct eigenvalues among λ_k 's are ± 1 and $\pm i$.

Although there exists a good deal of discussion in the literature on eigenvectors of the FFT (see, for example [4]-[11] and references therein), the problem of deriving eigenvectors of FFT analytically still remains to be solved. So our objective is to find, in complete analogy with the continuous case (1.11), such a *self-adjoint difference operator with distinct eigenvalues*, which commutes with the FFT. The ability to solve a difference equation for eigenvectors thus obtained will enable one to define an analytical form of the desired set of eigenvectors for the finite Fourier transform.¹

This presentation is a brief account of the first development along those lines, discussed above: namely, we have succeeded in constructing the explicit forms of raising and lowering difference operators that govern eigenvectors of the finite Fourier transform.

The layout of the paper is as follows. Section 2 collects those known facts about eigenvectors of the FFT, which are needed in section 3 for deriving some explicit forms of difference operators, commuting with the FFT $\Phi^{(N)}$. In section 4 we construct an explicit form of the difference equation for eigenvectors of the FFT. Section 5 closes the paper with a brief remark, which outline some further research directions of interest.

Throughout this exposition we employ standard notations of the theory of special functions, see, for example, [12] or [13].

2. Eigenvectors of the FFT and their properties

It is a simple matter to verify that the matrix elements of $\Phi^{(N)}$ are, by their definition, formally N -periodic, that is,

$$\Phi_{m,m'}^{(N)} = \Phi_{m+N,m'}^{(N)} = \Phi_{m,m'+N}^{(N)}. \quad (2.1)$$

This means that required explicit forms of the all eigenvectors $\vec{f}^{(n)}$ of the FFT operator $\Phi^{(N)}$ should exhibit the same periodicity property

$$f_k^{(n)} = f_{k+N}^{(n)}, \quad n, k = 0, 1, \dots, N-1. \quad (2.2)$$

We now turn to the question of defining how such N -periodic vectors \vec{y} (which are not necessarily eigenvectors of the FFT operator $\Phi^{(N)}$) transform when their components y_k are

¹ We emphasize from the outset that the idea of employing such approach for deriving eigenvectors of the FFT is not new (see, for example, [4, 5]). However, it seems that there never was consistent attempt to carry out this program explicitly for arbitrary N .

multiplied by powers of $q^{\pm k}$ and acted upon by the shift operators $\mathbf{T}^{(\pm)}$. These relations are building blocks of the rest of this work.

Let us define first two operators $\mathbf{Q}^{(\pm)}$ with matrix elements $Q_{kl}^{(\pm)} := q^{\pm k} \delta_{kl}$, where $q = e^{2\pi i/N} \equiv e^{i\theta_N}$, $\theta_N := 2\pi/N$. Then, by this definition, $\mathbf{Q}^{(\pm)} \vec{y} = \vec{z}^{(\pm)}$, where $z_k^{(\pm)} = \sum_{l=0}^{N-1} Q_{kl}^{(\pm)} y_l = \sum_{l=0}^{N-1} q^{\pm k} \delta_{kl} y_l = q^{\pm k} y_k$. This means that under the action of the operators $\mathbf{Q}^{(\pm)}$ the components y_k of an arbitrary vector \vec{y} get multiplied by the factors $q^{\pm k}$, respectively. Consequently, for their linear combinations $\mathbf{C} := \frac{1}{2}(\mathbf{Q}^{(+)} + \mathbf{Q}^{(-)})$ and $\mathbf{S} := \frac{1}{2i}(\mathbf{Q}^{(+)} - \mathbf{Q}^{(-)})$ one obtains that

$$\begin{aligned} (\mathbf{C} \vec{y})_k &= \frac{1}{2} (q^k + q^{-k}) y_k = \cos(k \theta_N) y_k, \\ (\mathbf{S} \vec{y})_k &= \frac{1}{2i} (q^k - q^{-k}) y_k = \sin(k \theta_N) y_k. \end{aligned} \quad (2.3)$$

The next step is to define a pair of the shift operators $\mathbf{T}^{(\pm)}$ with matrix elements $T_{kl}^{(\pm)} := \delta_{k\pm 1, l}$, where $\delta_{-1, l} \equiv \delta_{N-1, l}$ and $\delta_{N, l} \equiv \delta_{0, l}$. Then

$$(\mathbf{T}^{(\pm)} \vec{y})_k = \sum_{l=0}^{N-1} T_{kl}^{(\pm)} y_l = \sum_{l=0}^{N-1} \delta_{k\pm 1, l} y_l = y_{k\pm 1}, \quad (2.4)$$

where $y_{-1} \equiv y_{N-1}$ and $y_N \equiv y_0$.

Lemma. Components of any N -periodic vector \vec{y} satisfy the following identities:

$$\sum_{k=0}^{N-1} q^{j(k\pm 1)} y_k = \sum_{k=0}^{N-1} q^{jk} y_{k\mp 1}, \quad j = 0, 1, 2, \dots, N-1. \quad (2.5)$$

Proof. Begin with the left side of the first identity in (2.5),

$$\sum_{k=0}^{N-1} q^{j(k+1)} y_k = \sum_{m=1}^N q^{jm} y_{m-1} = \sum_{m=1}^{N-1} q^{jm} y_{m-1} + q^{jN} y_{N-1} = \sum_{m=0}^{N-1} q^{jm} y_{m-1}, \quad (2.6)$$

in view of the evident relation $q^{Nj} = 1$. The second identity in (2.5) is argued similarly. So the lemma is proved. \square

The operators $\mathbf{Q}^{(\pm)}$ and $\mathbf{T}^{(\pm)}$ are actually interconnected through the FFT operator $\Phi^{(N)}$ in the following way.

Proposition. The intertwining relations

$$\mathbf{Q}^{(\pm)} \Phi^{(N)} = \Phi^{(N)} \mathbf{T}^{(\mp)}, \quad \mathbf{T}^{(\pm)} \Phi^{(N)} = \Phi^{(N)} \mathbf{Q}^{(\pm)}, \quad (2.7)$$

for the operators $\mathbf{Q}^{(\pm)}$ and $\mathbf{T}^{(\pm)}$ with the FFT operator $\Phi^{(N)}$ are respectively valid.

Proof. Two relations on the left in (2.7) follow directly from the lemma, if one takes into account the definition of the shift operators $\mathbf{T}^{(\pm)}$. So let us consider now matrix elements of two identities on the right in (2.7):

$$\begin{aligned} \left(\mathbf{T}^{(\pm)} \Phi^{(N)} \right)_{mn} &= \sum_{k=0}^{N-1} T_{mk}^{(\pm)} \Phi_{kn}^{(N)} = \sum_{k=0}^{N-1} \delta_{m\pm 1, k} \Phi_{kn}^{(N)} = \Phi_{m\pm 1, n}^{(N)} \\ &= \frac{1}{\sqrt{N}} q^{(m\pm 1)n} = q^{\pm n} \Phi_{m, n}^{(N)} = \sum_{l=0}^{N-1} \Phi_{m, l}^{(N)} q^{\pm l} \delta_{ln} = \left(\Phi^{(N)} \mathbf{Q}^{(\pm)} \right)_{mn}. \end{aligned}$$

This completes the proof of the proposition. \square

Notice that if a vector $\mathbf{y}^{(n)}$ with the components $\{y_j^{(n)}\}_{j=0}^{N-1}$ is N -periodic, then all m vectors $\mathbf{z}^{(n;m)}$, which have, by definition, components $\{z_j^{(n;m)}\}_{j=0}^{N-1} := \{q^{mj} y_j^{(n)}\}_{j=0}^{N-1}$, m is an arbitrary integer number, are also N -periodic. Therefore, applying each line in (2.7) m times in succession, one arrives at the following corollary to the proposition:

$$\begin{aligned} q^{\pm mj} \sum_{k=0}^{N-1} \Phi_{j,k}^{(N)} y_k^{(n)} &= \sum_{k=0}^{N-1} \Phi_{j,k}^{(N)} \mathbf{T}_{\mp}^m y_k^{(n)}, \\ \mathbf{T}_{\pm}^m \sum_{k=0}^{N-1} \Phi_{j,k}^{(N)} y_k^{(n)} &= \sum_{k=0}^{N-1} \Phi_{j,k}^{(N)} q^{\pm mk} y_k^{(n)}, \end{aligned} \quad (2.8)$$

where m is an integer.

3. Difference operators, commuting with $\Phi^{(N)}$

We are now in a position to discuss explicit forms of shift (difference) operators, which commute with the finite Fourier transform operator $\Phi^{(N)}$. Making use of (2.7), we evaluate first, in analogy with (1.5), that

$$\frac{1}{2}(\mathbf{T}^{(+)} + \mathbf{T}^{(-)}) \Phi^{(N)} = \frac{1}{2} \Phi^{(N)} (\mathbf{Q}^{(+)} + \mathbf{Q}^{(-)}) = \Phi^{(N)} \mathbf{C}. \quad (3.1)$$

Since on the other hand, in view of the same proposition (2.7),

$$\mathbf{C} \Phi^{(N)} = \frac{1}{2}(\mathbf{Q}^{(+)} + \mathbf{Q}^{(-)}) \Phi^{(N)} = \frac{1}{2} \Phi^{(N)} (\mathbf{T}^{(+)} + \mathbf{T}^{(-)}), \quad (3.2)$$

one readily concludes from (3.1) and (3.2) that the operator $\mathbf{U}^{(1)}$,

$$\mathbf{U}^{(1)} := \mathbf{C} + \frac{1}{2}(\mathbf{T}^{(+)} + \mathbf{T}^{(-)}), \quad (3.3)$$

with matrix elements $U_{kl}^{(1)} = \cos k\theta_N \delta_{kl} + \frac{1}{2}(\delta_{k+1,l} + \delta_{k-1,l})$, does commute with the FFT operator $\Phi^{(N)}$:

$$\mathbf{U}^{(1)} \Phi^{(N)} = \Phi^{(N)} \mathbf{U}^{(1)}.$$

Moreover, one verifies that in the continuous limit as $N \rightarrow \infty$

$$\lim_{N \rightarrow \infty} \frac{N}{\pi} \left(\mathbf{I} - \frac{1}{2} \mathbf{U}^{(1)} \right) = \frac{1}{2} \left(x^2 - \frac{d^2}{dx^2} \right), \quad (3.4)$$

where x is dimensionless coordinate and \mathbf{I} is the $N \times N$ unit matrix (consult [4, 8] for a detailed discussion of how to treat such convergence of a discrete operator to a continuum one in a rigorous mathematical manner).

To facilitate ease of comparing the operator $\mathbf{U}^{(1)}$ with other previously considered in the literature operators, which commute with $\Phi^{(N)}$, we display its matrix form:

$$\left(U_{m,m'}^{(1)} \right) = \frac{1}{2} \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 2 \cos \frac{2\pi}{N} & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 \cos \frac{4\pi}{N} & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \cos \frac{4\pi}{N} & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \cos \frac{2\pi}{N} \end{pmatrix}.$$

This matrix has repeatedly surfaced in the literature on the finite Fourier transform. For instance, it coincides with the matrix S in [5], see formula (12) therein. An eigenvalue problem for the matrix (or difference operator) $\mathbf{U}^{(1)}$ probably first occurs in [6] and it is therefore known as Harper's equation, whereas the combination $\frac{2\pi}{N} (\mathbf{I} - \frac{1}{2} \mathbf{U}^{(1)})$ represents the Hamiltonian of the so-called *Harper function harmonic oscillator* [7, 8].

However, the operator $\mathbf{U}^{(1)}$ fails to serve to our purposes for generic N (*in spite of the fact that it does commute with the $\Phi^{(N)}$ and reduces to the linear harmonic oscillator Hamiltonian (1.3) in the continuous limit as $N \rightarrow \infty$*) because it has multiple eigenvalues (in particular, when N is divisible by four, it has two zero eigenvalues).

Before proceeding to the problem of finding another difference operator in question (*with distinct eigenvalues for generic N*), we call attention to the following two observations. Firstly, one may generalize $\mathbf{U}^{(1)}$ and derive a family of operators $\mathbf{U}^{(k)}$,

$$\mathbf{U}^{(k)} := \frac{1}{2} \left[\left(\mathbf{T}^{(+)} \right)^k + \left(\mathbf{T}^{(-)} \right)^k + \left(\mathbf{Q}^{(+)} \right)^k + \left(\mathbf{Q}^{(-)} \right)^k \right], \quad (3.5)$$

where k is a positive integer. Then it is not hard to verify, by using relations (2.7), that all operators $\mathbf{U}^{(k)}$, $k = 1, 2, 3, \dots$, commute with the finite Fourier transform $\Phi^{(N)}$. Secondly, since $\mathbf{U}^{(1)}$ commutes with $\Phi^{(N)}$, so does its square, $\left(\mathbf{U}^{(1)} \right)^2$. Combining these two facts, one therefore concludes that the operator

$$\mathbf{V}_1 := \left(\mathbf{U}^{(1)} \right)^2 - \frac{1}{2} \mathbf{U}^{(2)} \quad (3.6)$$

also does commute with the finite Fourier transform $\Phi^{(N)}$. The reason for picking up such a combination of the operators $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$ in (3.6) is that it still contains only linear terms in the shift operators $\mathbf{T}^{(\pm)}$. Indeed, by definition (3.6), one has

$$\begin{aligned} \mathbf{V}_1 &= \left(\mathbf{U}^{(1)} \right)^2 - \frac{1}{2} \mathbf{U}^{(2)} = \frac{1}{4} \left[\left(\mathbf{T}^{(+)} + \mathbf{T}^{(-)} + \mathbf{Q}^{(+)} + \mathbf{Q}^{(-)} \right)^2 \right. \\ &\quad \left. - \left(\mathbf{T}^{(+)} \right)^2 - \left(\mathbf{T}^{(-)} \right)^2 - \left(\mathbf{Q}^{(+)} \right)^2 - \left(\mathbf{Q}^{(-)} \right)^2 \right] \equiv \mathbf{I} + \cos \frac{\pi}{N} \mathbf{V}_2, \end{aligned} \quad (3.7)$$

upon taking into account the readily verified commutation relations

$$\mathbf{T}^{(+)} \mathbf{Q}^{(\pm)} = q^{\pm 1} \mathbf{Q}^{(\pm)} \mathbf{T}^{(+)}, \quad \mathbf{T}^{(-)} \mathbf{Q}^{(\pm)} = q^{\mp 1} \mathbf{Q}^{(\pm)} \mathbf{T}^{(-)} \quad (3.8)$$

between the operators $\mathbf{T}^{(\pm)}$ and $\mathbf{Q}^{(\pm)}$. The operator \mathbf{V}_2 in (3.7) is equal to

$$\mathbf{V}_2 = \left(\cos \frac{\pi}{N} \mathbf{C} - \sin \frac{\pi}{N} \mathbf{S} \right) \mathbf{T}^{(+)} + \left(\cos \frac{\pi}{N} \mathbf{C} + \sin \frac{\pi}{N} \mathbf{S} \right) \mathbf{T}^{(-)}, \quad (3.9)$$

or, in the matrix form,

$$\left((\mathbf{V}_2)_{m,m'} \right) = \frac{1}{2} \begin{pmatrix} 0 & \cos \frac{\pi}{N} & 0 & 0 & \dots & 0 & 0 & \cos \frac{\pi}{N} \\ \cos \frac{\pi}{N} & 0 & \cos \frac{3\pi}{N} & 0 & \dots & 0 & 0 & 0 \\ 0 & \cos \frac{3\pi}{N} & 0 & \cos \frac{5\pi}{N} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \cos \frac{5\pi}{N} & 0 & \cos \frac{3\pi}{N} \\ \cos \frac{\pi}{N} & 0 & 0 & 0 & \dots & 0 & \cos \frac{3\pi}{N} & 0 \end{pmatrix}.$$

In the continuous limit as $N \rightarrow \infty$ one obtains that

$$\lim_{N \rightarrow \infty} \frac{N}{\pi} \left(\mathbf{I} - \frac{1}{2} \mathbf{V}_2 \right) = x^2 - \frac{d^2}{dx^2} . \quad (3.10)$$

Examples (3.5) and (3.6) just reflect the fact that there are infinitely many possibilities for constructing difference operators, which commute with the finite Fourier transform $\Phi^{(N)}$. So we need to find among them *a difference operators with the simplest spectrum of distinct eigenvalues for generic N* . Clearly, in the continuous limit as $N \rightarrow \infty$ this operator should reduce to the second-order differential equation (1.3) for the Hermite functions (1.2).

As we shall establish in the next section, a required difference operator is actually comprised of the two above-introduced operators $\mathbf{U}^{(2)}$ and \mathbf{V}_1 (or, equivalently, \mathbf{V}_2).

4. Difference equation for eigenvectors of the FFT

It is a remarkable fact that an appropriate difference equation for the eigenvectors $\mathbf{f}^{(n)}$ of the FFT, associated with the eigenvalues $\lambda_n = i^n$, can be constructed in complete analogy with the continuous case (that is, the Hermite functions (1.2)). To bring out this fact we now address ourselves to a study of difference analogues of the lowering \mathbf{a} and raising \mathbf{a}^\dagger differential operators (1.9), which factorize the operator \mathbf{H} in (1.11),

$$\mathbf{H} = \mathbf{a}^\dagger \mathbf{a} + \frac{1}{2} \mathbf{I} . \quad (4.1)$$

Recall also that very often it is more convenient to work directly with the number operator \mathbf{N} , defined as

$$\mathbf{N} := \mathbf{H} - \frac{1}{2} \mathbf{I} = \mathbf{a}^\dagger \mathbf{a} . \quad (4.2)$$

Then from the commutation relation (1.10) it follows at once that

$$\mathbf{N} \mathbf{a} = \mathbf{a} (\mathbf{N} - \mathbf{I}) , \quad \mathbf{N} \mathbf{a}^\dagger = \mathbf{a}^\dagger (\mathbf{N} + \mathbf{I}) , \quad (4.3)$$

and the above identities confirm algebraically that the operators \mathbf{a} and \mathbf{a}^\dagger do lower and raise, respectively, the eigenvalues of the number operator \mathbf{N} (or, equivalently, of the Hamiltonian \mathbf{H}). So in that way one naturally arrives at the conclusion that the Fourier transform (1.4) can be equivalently represented in another operator form as

$$\mathcal{F} = \exp \left(\frac{\pi i}{2} \mathbf{N} \right) . \quad (4.4)$$

Of course, from this operator form the identities (1.8) now follow at once, upon taking into account the commutation relations (4.3).

To construct difference analogues of the operators \mathbf{a} and \mathbf{a}^\dagger , which are evidently building blocks of the continuous case (4.1), we may proceed as in the previous section. So let us consider first two operators \mathcal{X} and \mathcal{D} ($N \geq 3$):

$$\mathcal{X} := \frac{1}{2i} \sqrt{\frac{N}{2\pi}} (\mathbf{Q}^{(+)} - \mathbf{Q}^{(-)}) \equiv \sqrt{\frac{N}{2\pi}} \mathbf{S} , \quad \mathcal{D} := \frac{1}{2} \sqrt{\frac{N}{2\pi}} (\mathbf{T}^{(+)} - \mathbf{T}^{(-)}) . \quad (4.5)$$

From the intertwining relations (2.7) it follows then that

$$\mathcal{D} \Phi^{(N)} = i \Phi^{(N)} \mathcal{X} , \quad \mathcal{X} \Phi^{(N)} = i \Phi^{(N)} \mathcal{D} . \quad (4.6)$$

Also, one readily verifies that the operators \mathcal{X} and \mathcal{D} are normalized in such a way that the limit relations

$$\lim_{N \rightarrow \infty} \mathcal{X} = x, \quad \lim_{N \rightarrow \infty} \mathcal{D} = \frac{d}{dx}, \quad (4.7)$$

for these two operators do hold.

Remark. It is important to observe that the action of the difference operator \mathcal{D} on the product of two functions is more complicated than the Leibniz product rule for the derivative $\frac{d}{dx}$. Indeed, it is not hard to verify that for two arbitrary functions $u(j)$ and $v(j)$ one has

$$\begin{aligned} u(j+1)v(j+1) - u(j-1)v(j-1) &= u(j+1)[v(j+1) - v(j-1)] + u(j+1)v(j-1) \\ &\quad - u(j-1)[v(j-1) - v(j+1)] - u(j-1)v(j+1) = [u(j+1) + u(j-1)][v(j+1) - v(j-1)] \\ &\quad + [u(j+1) - u(j-1)][v(j+1) + v(j-1)] - [u(j+1)v(j+1) - u(j-1)v(j-1)]. \end{aligned}$$

Consequently,

$$\begin{aligned} u(j+1)v(j+1) - u(j-1)v(j-1) &= \frac{1}{2} [u(j+1) + u(j-1)][v(j+1) - v(j-1)] \\ &\quad + \frac{1}{2} [u(j+1) - u(j-1)][v(j+1) + v(j-1)]. \end{aligned}$$

Taking into account the definition of the operator \mathcal{D} in (4.5), one concludes that

$$\mathcal{D}[u(j)v(j)] = [\mathcal{A}u(j)][\mathcal{D}v(j)] + [\mathcal{D}u(j)][\mathcal{A}v(j)], \quad (4.8)$$

where \mathcal{A} is the averaging difference operator, defined as (cf. formula (21.6.3) in [14])

$$\mathcal{A} := \frac{1}{2} \left(\mathbf{T}^{(+)} + \mathbf{T}^{(-)} \right). \quad (4.9)$$

The next natural step is to evaluate that the commutation relation between the operators \mathcal{X} and \mathcal{D} is of the form

$$[\mathcal{D}, \mathcal{X}] = \frac{N}{\pi} \sin \frac{\pi}{N} \mathbf{V}_2, \quad (4.10)$$

where \mathbf{V}_2 is defined in (3.9). This commutation relation is a discrete analogue of the Heisenberg commutation relation (1.10) for the operators x and $\frac{d}{dx}$ and on account of (3.10) the (4.10) reduces to the latter in the continuous limit as $N \rightarrow \infty$.

To imitate the steps (1.5)–(1.7) of the continuous case, one evaluates now, by employing each of the identities in (4.6) twice, that

$$\mathcal{D}^2 \Phi^{(N)} = -\Phi^{(N)} \mathcal{X}^2, \quad \mathcal{X}^2 \Phi^{(N)} = -\Phi^{(N)} \mathcal{D}^2. \quad (4.11)$$

This means that a difference operator \mathcal{H} , defined as

$$\mathcal{H} := \frac{1}{2} \left(\mathcal{X}^2 - \mathcal{D}^2 \right), \quad (4.12)$$

does commute with the FFT operator $\Phi^{(N)}$, that is, $\mathcal{H} \Phi^{(N)} = \Phi^{(N)} \mathcal{H}$. Since in the continuous limit as $N \rightarrow \infty$ the (4.12) reduces to the linear harmonic oscillator Hamiltonian $\mathbf{H}/\hbar\omega$ (because of the limit relations (4.7)), the difference operator \mathcal{H} can serve as a required discrete analogue of the continuous case (1.3).

Difference analogues of the lowering and raising operators \mathbf{a} and \mathbf{a}^\dagger in (4.1) may be now defined as

$$\mathbf{b}_N := \frac{1}{\sqrt{2}} (\mathcal{X} + \mathcal{D}) = \frac{1}{2} \sqrt{\frac{N}{\pi}} \left[\mathbf{S} + \frac{1}{2} (\mathbf{T}^{(+)} - \mathbf{T}^{(-)}) \right], \quad (4.13a)$$

$$\mathbf{b}_N^\dagger := \frac{1}{\sqrt{2}} (\mathcal{X} - \mathcal{D}) = \frac{1}{2} \sqrt{\frac{N}{\pi}} \left[\mathbf{S} - \frac{1}{2} (\mathbf{T}^{(+)} - \mathbf{T}^{(-)}) \right]. \quad (4.13b)$$

We also display the matrix form of the operators \mathbf{b}_N and \mathbf{b}_N^\dagger , respectively:

$$\begin{aligned} ((\mathbf{b}_N)_{m,m'}) &= \frac{1}{4} \sqrt{\frac{N}{\pi}} \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 \sin \frac{2\pi}{N} & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 \sin \frac{4\pi}{N} & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & -2 \sin \frac{4\pi}{N} & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 & -2 \sin \frac{2\pi}{N} \end{pmatrix}, \\ ((\mathbf{b}_N^\dagger)_{m,m'}) &= \frac{1}{4} \sqrt{\frac{N}{\pi}} \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 2 \sin \frac{2\pi}{N} & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 \sin \frac{4\pi}{N} & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 \sin \frac{4\pi}{N} & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \sin \frac{2\pi}{N} \end{pmatrix}. \end{aligned}$$

As a direct consequence of (4.10), these difference analogues of \mathbf{a} and \mathbf{a}^\dagger satisfy the following commutation relation

$$[\mathbf{b}_N, \mathbf{b}_N^\dagger] \equiv [\mathcal{D}, \mathcal{X}] = \frac{N}{\pi} \sin \frac{\pi}{N} \mathbf{V}_2 \quad (4.14)$$

and by construction they reduce to \mathbf{a} and \mathbf{a}^\dagger in the continuous limit as $N \rightarrow \infty$.

This arresting analogy with the continuous case still develops to an even greater extent since by using definitions (4.13a) and (4.13b) it is straightforward to verify that

$$\begin{aligned} \mathbf{b}_N^\dagger \mathbf{b}_N &= \frac{1}{2} (\mathcal{X} - \mathcal{D})(\mathcal{X} + \mathcal{D}) = \frac{1}{2} (\mathcal{X}^2 - \mathcal{D}^2 + \mathcal{X}\mathcal{D} - \mathcal{D}\mathcal{X}) \\ &= \mathcal{H} - \frac{1}{2} [\mathcal{D}, \mathcal{X}] = \mathcal{H} - \frac{1}{2} [\mathbf{b}_N, \mathbf{b}_N^\dagger]. \end{aligned} \quad (4.15)$$

So the difference operators \mathbf{b}_N and \mathbf{b}_N^\dagger , introduced above in (4.13), do factorize the difference operator \mathcal{H} in the form

$$\mathcal{H} = \mathbf{b}_N^\dagger \mathbf{b}_N + \frac{1}{2} [\mathbf{b}_N, \mathbf{b}_N^\dagger] = \frac{1}{2} (\mathbf{b}_N \mathbf{b}_N^\dagger + \mathbf{b}_N^\dagger \mathbf{b}_N), \quad (4.16)$$

which closely resembles the symmetrized expression (1.11) for \mathbf{H} in the continuous case. Therefore the operator $\mathcal{N} := \mathbf{b}_N^\dagger \mathbf{b}_N$ can be regarded as a difference analogue of the number operator \mathbf{N} . Observe that since the difference between the operators \mathcal{N} and \mathcal{H} is just proportional to the operator \mathbf{V}_2 ,

$$\mathcal{N} \equiv \mathcal{H} - \frac{1}{2} [\mathbf{b}_N, \mathbf{b}_N^\dagger] = \mathcal{H} - \frac{N}{2\pi} \sin \frac{\pi}{N} \mathbf{V}_2, \quad (4.16')$$

the former operator \mathcal{N} commutes with the FFT operator $\Phi^{(N)}$ as well (see (3.6) and (3.7)).

Finally, it turns out that thus introduced difference operators \mathbf{b}_N and \mathbf{b}_N^\dagger have “proper” commutation relations with the FFT operator $\Phi^{(N)}$: from intertwining relations (4.6) it follows at once that

$$\mathbf{b}_N \Phi^{(N)} = i \Phi^{(N)} \mathbf{b}_N, \quad \Phi^{(N)} \mathbf{b}_N^\dagger = i \mathbf{b}_N^\dagger \Phi^{(N)}. \quad (4.17)$$

Consequently, the difference operators \mathbf{b}_N and \mathbf{b}_N^\dagger in (4.13) actually represent n -lowering and n -raising operators, respectively, for the eigenvectors $\mathbf{f}^{(n)}$ of the FFT, associated with the eigenvalues $\lambda_n = i^n$. Thus,

$$\mathbf{b}_N \mathbf{f}^{(n)} = \mu_n^{(N)} \mathbf{f}^{(n-1)}, \quad \mathbf{b}_N^\dagger \mathbf{f}^{(n)} = \nu_{n+1}^{(N)} \mathbf{f}^{(n+1)}. \quad (4.18)$$

where $\mu_n^{(N)}$ and $\nu_n^{(N)}$ are some constant factors, which may depend on n .

Remark. For arbitrary integer N it is not difficult to construct explicitly the “ground eigenvector” $\{f_m^{(0)}\}_{m=0}^{N-1}$ of the FFT (1.12) with the eigenvalue $\lambda_0 = i^0 = 1$. Indeed, this eigenvector has the form $\{f_m^{(0)}\} := c_0^{(N)} \{\sqrt{N} + 1, 1, 1, \dots, 1\}$, which is a consequence of the fact the sum of elements in the k -th row of FFT-matrix $\left(\Phi_{m,m'}^{(N)}\right)$, $k = 1, 2, 3, \dots, N-1$, actually represents a geometric progression

$$S_N(x) := 1 + x + x^2 + \dots + x^{N-1} = \frac{1 - x^N}{1 - x}$$

in the variable $x = q^k$ and therefore all of $S_N(q^k)$ are equal to zero for any integer value of $k = 1, 2, 3, \dots, N-1$ (recall that $q^N = 1$ by definition (1.12) of q). Thus,

$$\begin{aligned} \sum_{m'=0}^{N-1} \Phi_{m,m'}^{(N)} f_{m'}^{(0)} &= \frac{c_0^{(N)}}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & q & q^2 & \dots & q^{N-2} & q^{N-1} \\ 1 & q^2 & q^4 & \dots & q^{2(N-2)} & q^{2(N-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & q^{N-1} & q^{2(N-1)} & \dots & q^{(N-2)(N-1)} & q^{(N-1)^2} \end{pmatrix} \begin{pmatrix} \sqrt{N} + 1 \\ 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} \\ &= \frac{c_0^{(N)}}{\sqrt{N}} \begin{pmatrix} \sqrt{N} + 1 + N - 1 \\ \sqrt{N} + S_n(q) \\ \sqrt{N} + S_n(q^2) \\ \dots \\ \sqrt{N} + S_n(q^{N-1}) \end{pmatrix} = c_0^{(N)} \begin{pmatrix} \sqrt{N} + 1 \\ 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} = f_m^{(0)}. \end{aligned}$$

To normalize this “ground eigenvector” $\{f_m^{(0)}\}$ to one, it is sufficient to choose the constant $c_0^{(N)} = [2\sqrt{N}(\sqrt{N} + 1)]^{-1/2}$.

It follows at once from (4.17) that the difference operator $\mathcal{B}^{(N)} := \mathbf{b}_N^\dagger \mathbf{b}_N$ commutes with the FFT operator $\Phi^{(N)}$, that is,

$$[\mathcal{B}^{(N)}, \Phi^{(N)}] = 0.$$

We thus conclude that a required *difference analogue* of the second-order differential equation $\mathbf{N} \psi_n(x) = n \psi_n(x)$, which is associated with the continuous case, has the form

$$\mathcal{B}^{(N)} \mathbf{f}^{(n)} = \lambda_n^{(N)} \mathbf{f}^{(n)}, \quad \lambda_n^{(N)} = \mu_n^{(N)} \nu_n^{(N)}, \quad (4.19)$$

where

$$\mathcal{B}^{(N)} = \frac{1}{2} (\mathcal{X} - \mathcal{D}) (\mathcal{X} + \mathcal{D}) = \frac{N}{4\pi} \left(I - \mathbf{S}_2 - 2 \sin \frac{\pi}{N} \mathbf{R}_2 \right) \quad (4.20)$$

and \mathbf{S}_2 and \mathbf{R}_2 are defined in (3.5) and (3.8), respectively. Observe that a formal solution of the difference equation (4.19) may be constructed in terms of the *ground* eigenvector $\mathbf{f}^{(0)}$, governed by, on account of (4.13), a difference equation

$$\mathbf{b}_N \mathbf{f}^{(0)} = \frac{1}{4} \sqrt{\frac{N}{\pi}} \left[\mathbf{T}_+ - \mathbf{T}_- + 2\mathbf{S} \right] \mathbf{f}^{(0)} = 0. \quad (4.21)$$

Then any other eigenvector $\mathbf{f}^{(n)}$ with an arbitrary n is found as

$$\mathbf{f}^{(n)} = \frac{1}{\prod_{k=1}^n \nu_k^{(N)}} \left(\mathbf{b}_N^\dagger \right)^n \mathbf{f}^{(0)}, \quad (4.22)$$

in conformity with the well-known formula

$$\psi_n(x) = \frac{1}{\sqrt{n!}} \left(\mathbf{a}^\dagger \right)^n \psi_0(x) \quad (4.23)$$

for the classical Hermite functions (1.2).

5. Concluding remarks

To summarize, we have constructed the explicit forms of raising and lowering difference operators that govern eigenvectors of the finite Fourier transform. The main algebraic properties of these raising and lowering difference operators have been examined in detail. We hope that this particular knowledge will help us to achieve the same degree of analytic tractability with the finite Fourier transform, as we observe in the case of its continuous counterpart, briefly outlined in Introduction.

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