

Bi-fractional Wigner functions

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Abstract. Two fractional Fourier transforms are used to define bi-fractional displacement operators, which interpolate between displacement operators and parity operators. They are used to define bi-fractional coherent states. They are also used to define the bi-fractional Wigner function, which is a two-parameter family of functions that interpolates between the Wigner function and the Weyl function. Links to the extended phase space formalism are also discussed.

1. Introduction

Wigner and Weyl functions and also P and Q functions play a central role in the general area of phase space methods [1, 2]. They are quantities which describe pseudo-probabilities and correlations associated to quantum states. There have also been other more general quantities, which interpolate among them, and provide a more general formalism. In this general context in a recent publication [3], we presented a two-parameter interpolation between displacement and parity operators (or similarly between Weyl and Wigner functions). The displacement and parity operators are related through a two-dimensional Fourier transform, which we replace with fractional Fourier transforms [4, 5, 6, 7]. This leads to a two-parameter family of operators which we call bi-fractional displacement operators, and which include as special cases the displacement and parity operators.

In the present paper we review briefly and extend this work, by discussing links with the extended phase space method in refs [8, 9, 10, 11, 12]. The latter uses the fact that the Wigner function $W(x, p)$ is related to the Weyl function $\widetilde{W}(X, P)$ through a two-dimensional Fourier transform. x, p are position and momentum entering in the Wigner function, and X, P are position and momentum increments entering in the Weyl function. One Fourier transform involves the variables x, P and the other the p, X . These Fourier transforms lead to uncertainty relations between the x, P variables (and also the X, p variables), which have been studied in [13]. An ‘extended Wigner function’ (and other ‘extended quantities’) that depend on x, p, X, P have been introduced in the extended phase space $x - p - X - P$. In this language, the two variables of the bi-fractional displacement operator studied here, are in the direction defined by the angle ϕ_1 in the $x - P$ plane, and also in the direction defined by the angle ϕ_2 in the $p - X$ plane (see fig1). Using the bi-fractional displacement operator we get a two-parameter family of functions which include as special cases the Wigner and Weyl functions.

In section 2, we discuss briefly fractional Fourier transforms, in order to define the notation. In section 3, we introduce the bifractional displacement operators. In section 4, we act with them on the vacuum to get bi-fractional coherent states. In section 5, we introduce the ‘bi-fractional Wigner functions’, which is a two parameter family of functions, that includes Wigner and Weyl



functions as special cases. In section 6, we link the present work with the extended phase space formalism. We conclude in section 7, with a discussion of our results.

2. Fractional Fourier Transform

The fractional Fourier Transform has been studied extensively for a long time. It is a generalisation of the Fourier transform given by

$$f_\phi(\alpha) = \exp(i\phi\mathcal{A}^\dagger\mathcal{A})f(\alpha); \quad \mathcal{A} = \frac{\alpha + \partial_\alpha}{\sqrt{2}}; \quad \mathcal{A}^\dagger = \frac{\alpha - \partial_\alpha}{\sqrt{2}} \quad (1)$$

This is the differential form and it is equivalent to the following integral form

$$f_\phi(\alpha) = \int \mathcal{K}_\phi(\alpha, \beta)f(\beta)d\beta, \quad (2)$$

where $\mathcal{K}_\phi(\alpha, \beta)$ is the kernel

$$\mathcal{K}_\phi(\alpha, \beta) = \begin{cases} \left[\frac{1+i \cot \phi}{2\pi} \right] \exp \left[\frac{-i(\alpha^2+\beta^2) \cot \phi}{2} + \frac{i\alpha\beta}{\sin \phi} \right], & \text{if } \phi \neq n\pi \\ \delta(\alpha - \beta), & \text{if } \phi = 2n\pi \\ \delta(\alpha + \beta), & \text{if } \phi = (2n + 1)\pi \end{cases} \quad (3)$$

It is easily seen that for $\phi = \pi/2$ we get the Fourier transform, and for $\phi = \pi$ the parity operator with respect to the origin which we denote as $\mathcal{P}(0, 0)$. We note that,

$$\int d\beta \mathcal{K}_{\phi_1}(\alpha, \beta)\mathcal{K}_{\phi_2}(\beta, \gamma) = \mathcal{K}_{\phi_1+\phi_2}(\alpha, \gamma) \quad (4)$$

The two-dimensional fractional Fourier transform is given by

$$f_{\phi_1, \phi_2}(\alpha, \beta) = \int \mathcal{K}_{\phi_1}(\beta, \alpha')\mathcal{K}_{\phi_2}(\alpha, -\beta')f(\alpha', \beta')d\alpha'd\beta' \quad (5)$$

3. Bi-fractional Displacement operator

We consider the harmonic oscillator in the Hilbert space, \mathcal{H} with \hat{x} and \hat{p} as the position and momentum operators. We define the displacement operator as

$$\mathcal{D}(X, P) = \exp(i\sqrt{2}P\hat{x} - i\sqrt{2}X\hat{p}). \quad (6)$$

The bi-fractional displacement operator is defined as

$$\mathcal{B}_{\phi_1, \phi_2}(\alpha, \beta) = [\cos(\phi_1 - \phi_2)]^{1/2} \int \mathcal{K}_{\phi_2}(\beta, X)\mathcal{K}_{\phi_1}(\alpha, -P)\mathcal{D}(X, P)dX dP. \quad (7)$$

$\mathcal{B}_{\phi_1, \phi_2}(\alpha, \beta)$ is a continuous function of ϕ_1, ϕ_2 . There is an important difference between Eqs(5),(7). In Eq.(5) α', β' are independent variables. In Eq.(7) X, P are dual quantum variables, multiplied by \hat{x}, \hat{p} which do not commute. Integrations should be performed very carefully taking into account the ordering of the operators \hat{x}, \hat{p} . A result of this is the prefactor $[\cos(\phi_1 - \phi_2)]^{1/2}$ which is important for unitarity.

In [3] we have proved that the $\mathcal{B}_{\phi_1, \phi_2}(\alpha, \beta)$ are elements of the group $\mathcal{HW} \times \mathcal{SU}(1,1)$ which is the semidirect product of the Heisenberg Weyl group \mathcal{HW} of displacements, by the $\mathcal{SU}(1,1)$ group of squeezing transformations. They are 'special elements' in the sense that not every element of this group can be written as $\mathcal{B}_{\phi_1, \phi_2}(\alpha, \beta)$.

A special case of $\mathcal{B}_{\phi_1, \phi_2}(\alpha, \beta)$ is the parity operator with respect to the point (α, β) , which is defined as

$$\mathcal{P}(\alpha, \beta) = \mathcal{D}\left(\frac{\alpha}{2}, \frac{\beta}{2}\right) \mathcal{P}(0, 0) \mathcal{D}^\dagger\left(\frac{\alpha}{2}, \frac{\beta}{2}\right) = \mathcal{D}(\alpha, \beta) \mathcal{P}(0, 0) \quad (8)$$

The parity operator is related to the displacement operator through a two-dimensional Fourier transform

$$\begin{aligned} \mathcal{P}(x, p) &= \frac{1}{2\pi} \int \mathcal{D}(X, P) \exp[i(pX - Px)] dX dP \\ &= \int \mathcal{D}(X, P) \mathcal{K}_{\frac{\pi}{2}}(p, X) \mathcal{K}_{\frac{\pi}{2}}(x, -P) dX dP \end{aligned} \quad (9)$$

For $\phi_1 = \phi_2 = \frac{\pi}{2}$, we get

$$\mathcal{B}_{\frac{\pi}{2}, \frac{\pi}{2}}(\alpha, \beta) = \mathcal{P}(\alpha, \beta) \quad (10)$$

The bi-fractional displacement operator $\mathcal{B}_{\phi_1, \phi_2}(\alpha, \beta)$ is unitary:

$$[\mathcal{B}_{\phi_1, \phi_2}(\alpha, \beta)]^\dagger = \mathcal{B}_{-\phi_1, -\phi_2}(-\alpha, -\beta) \quad (11)$$

Special cases are:

$$\begin{aligned} \mathcal{B}_{0,0}(\alpha, \beta) &= \mathcal{D}(\beta, -\alpha) \\ \mathcal{B}_{\frac{\pi}{2}, \frac{\pi}{2}}(\alpha, \beta) &= \mathcal{P}(\alpha, \beta) \\ \mathcal{B}_{\pi, \pi}(\alpha, \beta) &= \mathcal{D}(-\beta, \alpha) \end{aligned} \quad (12)$$

The relationship between two bi-fractional displacement operators with different variables is

$$\begin{aligned} \mathcal{B}_{\psi_1 + \phi_1, \psi_2 + \phi_2}(\alpha, \beta) &= \frac{|\cos(\psi_1 + \phi_1 - \psi_2 - \phi_2)|^{1/2}}{|\cos(\psi_1 - \psi_2)|^{1/2}} \\ &\times \int d\sigma d\omega \mathcal{K}_{\phi_2}(\beta, \beta') \mathcal{K}_{\phi_1}(\alpha, \alpha') \mathcal{B}_{\psi_1, \psi_2}(\alpha', \beta') \end{aligned} \quad (13)$$

Acting with the displacement operator on both sides of the bifractional displacement operator we get

$$\begin{aligned} \mathcal{D}(\nu, \xi) \mathcal{B}_{\phi_1, \phi_2}(\alpha, \beta) \mathcal{D}^\dagger(\nu, \xi) &= \mathcal{B}_{\phi_1, \phi_2}(-\alpha - 2\nu \sin \phi_1, \beta + 2\xi \sin \phi_2) \\ &\times \exp[i(2\alpha\nu \cos \phi_1 + \nu^2 \sin 2\phi_1)] \\ &\times \exp[i(2\beta\xi \cos \phi_2 + \xi^2 \sin 2\phi_2)] \end{aligned} \quad (14)$$

4. Bifractional coherent states

Acting with $\mathcal{B}_{\phi_1, \phi_2}(\alpha, \beta)$ on the vacuum $|0\rangle$ we get the generalized coherent states

$$|\alpha, \beta; \phi_1, \phi_2\rangle = \mathcal{B}_{\phi_1, \phi_2}(\alpha, \beta) |0\rangle \quad (15)$$

In the special case when $\phi_1 = \phi_2 = 0$, and $\phi_1 = \phi_2 = \frac{\pi}{2}$ we get Glauber coherent states:

$$|-\beta, \alpha; 0, 0\rangle = |\alpha, \beta; \frac{\pi}{2}, \frac{\pi}{2}\rangle = |\alpha, \beta\rangle \quad (16)$$

They are eigenstates of the creation and annihilation operators

$$\begin{aligned} b(\phi_1, \phi_2) &= \mathcal{B}_{\phi_1, \phi_2}(0, 0) a \mathcal{B}_{\phi_1, \phi_2}(0, 0) \\ b^\dagger(\phi_1, \phi_2) &= \mathcal{B}_{\phi_1, \phi_2}(0, 0) a^\dagger \mathcal{B}_{\phi_1, \phi_2}(0, 0), \end{aligned} \quad (17)$$

but they have novel non-trivial properties with respect to a, a^\dagger . They also satisfy the resolution of the identity

$$\frac{1}{2\pi} \int |\alpha, \beta; \phi_1, \phi_2\rangle \langle \alpha, \beta; \phi_1, \phi_2| d\alpha d\beta = \mathbf{1} \quad (18)$$

In [3] we have studied the Bargmann functions of these coherent states. Also the whole formalism of coherent state quantization, can be studied using the bifractional coherent states.

5. Bi-fractional Wigner functions

We consider the trace of a trace-class operator Θ with $\mathcal{B}_{\phi_1, \phi_2}(\alpha, \beta)$:

$$\begin{aligned} \mathcal{V}_{\phi_1, \phi_2}(\alpha, \beta|\Theta) &= \text{Tr}[\Theta \mathcal{B}_{\phi_1, \phi_2}(\alpha, \beta)] \\ &= |\cos(\phi_1 - \phi_2)|^{1/2} \int \mathcal{K}_{\phi_2}(\beta, X) \\ &\quad \times \mathcal{K}_{\phi_1}(\alpha, -P) \widetilde{\mathcal{W}}(X, P|\Theta) dX dP \end{aligned} \quad (19)$$

This is a generalization of the Wigner and Weyl functions. Indeed

$$\begin{aligned} \mathcal{V}_{0,0}(\alpha, \beta|\Theta) &= \widetilde{\mathcal{W}}(\beta, -\alpha|\Theta) \\ \mathcal{V}_{\frac{\pi}{2}, \frac{\pi}{2}}(\alpha, \beta|\Theta) &= \mathcal{W}(\alpha, \beta|\Theta) \\ \mathcal{V}_{\pi, \pi}(\alpha, \beta|\Theta) &= \widetilde{\mathcal{W}}(-\beta, \alpha|\Theta) \end{aligned} \quad (20)$$

In the special case that $\phi_1 = \phi_2 = 0$, the variable α becomes the variable P , the variable β becomes the variable X (see fig.1), and $\mathcal{V}_{\phi_1, \phi_2}(\alpha, \beta|\Theta)$ is the Weyl function. In the special case that $\phi_1 = \phi_2 = \frac{\pi}{2}$, the variable α becomes the variable x , the variable β becomes the variable p , and $\mathcal{V}_{\phi_1, \phi_2}(\alpha, \beta|\Theta)$ is the Wigner function. In general, the function $\mathcal{V}_{\phi_1, \phi_2}(\alpha, \beta|\Theta)$ is a two-parameter interpolation between the Wigner and Weyl functions.

When $\Theta = \rho$ is a density matrix, $\mathcal{V}_{\phi_1, \phi_2}(\alpha, \beta|\rho)$ is the bi-fractional Wigner function for the state described by ρ . As example, we consider the superposition of coherent states

$$|s\rangle = \mathcal{N}[|\alpha_0, \beta_0\rangle + |-\alpha_0, -\beta_0\rangle] \quad (21)$$

where \mathcal{N} is a normalization factor. In figs 2, 3 we plot $\Re[\mathcal{V}_{\frac{\pi}{4}, \frac{\pi}{4}}(\alpha, \beta|\Theta)]$ and $|\mathcal{V}_{\frac{\pi}{4}, \frac{\pi}{4}}(\alpha, \beta|\Theta)|$ for this state with $\alpha_0 = 1.8$ and $\beta_0 = 0$. In the Weyl function the auto-terms are in the middle and the cross-terms are in the ‘wings’, while the opposite is true in the Wigner function. Here we have both auto-terms and cross-terms in the wings.

6. Links with the extended phase space formalism

The extended phase space formalism uses the fact that the Wigner function $W(x, p)$ and the Weyl function $\widetilde{W}(X, P)$ are related through a two-dimensional Fourier transform. The Wigner function describes quantum noise, and the Weyl function quantum correlations. Starting from a function $g(u)$ with $u \in \mathbb{R}^N$, and its Fourier transform $\widetilde{g}(v)$ with $v \in \mathbb{R}^N$, we can introduce (in any context) ‘a Wigner function’ $W(u, v)$ (and other quantities) in the $u - v$ phase space, which is \mathbb{R}^{2N} . The extended phase space formalism introduces extended Wigner functions that

Figure 1. The α direction in the $x - P$ plane within the extended phase space $x - p - X - P$. Also the β direction in the $p - X$ plane. For clarity, the two planes are drawn separately, but they belong to the same four-dimensional space.

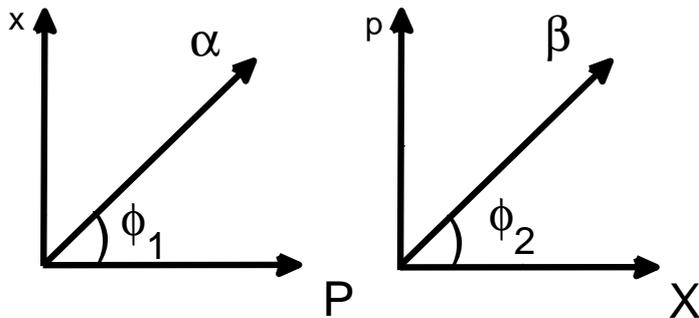
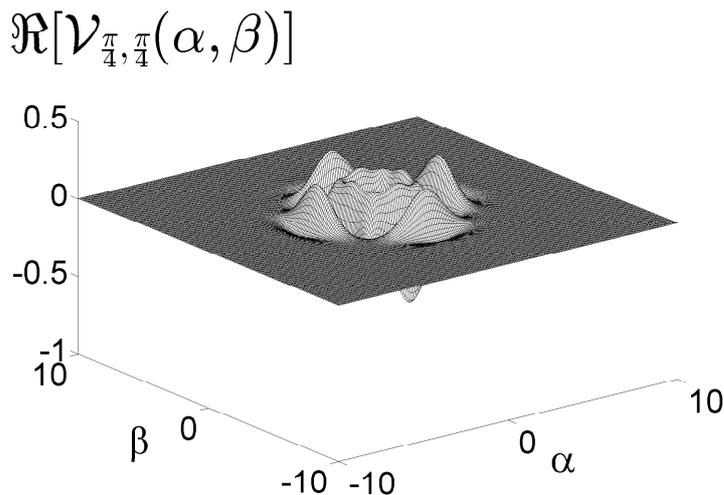


Figure 2. $\Re[\mathcal{V}_{\frac{\pi}{4}, \frac{\pi}{4}}(\alpha, \beta|\Theta)]$ for the state of Eq.(21), with $\alpha_0 = 1.8$ and $\beta_0 = 0$.

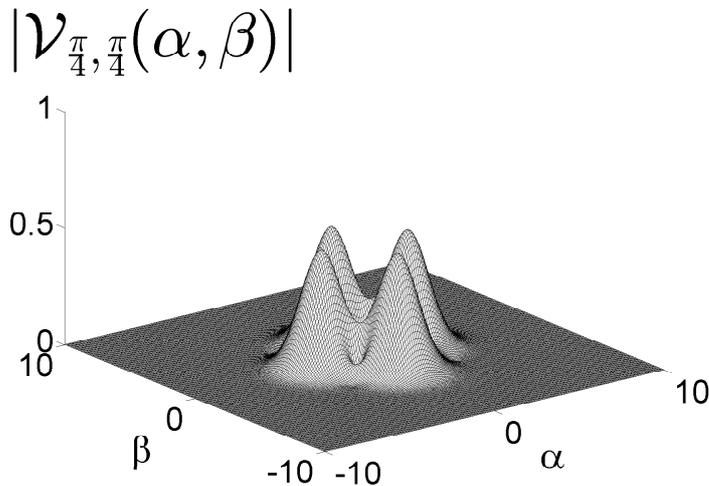


depend on four variables (x, p, X, P) and studies transformations in the extended phase space $x - p - X - P$. The $x - P$ plane links quantum noise in the x -direction, with quantum correlations in the P -direction. Uncertainty relations in the $x - P$ plane have been studied in [13]. They are related to the usual uncertainties in the case of pure states, but they are different for mixed states. Analogous comments can be made for the $p - X$ plane.

In the extended phase space $x - p - X - P$ we can introduce the following displacement operator of Weyl functions:

$$\begin{aligned} \mathfrak{D}(A_1, A_2, B_1, B_2) &= \exp[(B_2 - A_2)\partial_P] \exp\left(-i\frac{A_1 + B_1}{2}P\right) \exp[(B_1 - A_1)\partial_X] \\ &\times \exp\left(i\frac{A_2 + B_2}{2}X\right) \end{aligned} \quad (22)$$

Figure 3. $|\mathcal{V}_{\frac{\pi}{4}, \frac{\pi}{4}}(\alpha, \beta|\Theta)|$ for the state of Eq.(21), with $\alpha_0 = 1.8$ and $\beta_0 = 0$.



Acting with this on the Weyl function $\widetilde{W}(X, P|\Theta)$ we get [9]

$$\begin{aligned} \mathfrak{D}(A_1, A_2, B_1, B_2)\widetilde{W}(X, P|\Theta) &= \exp\left(-i\frac{A_1 + B_1}{2}P + i\frac{A_2 + B_2}{2}X - iA_1B_2 + iA_2B_1\right) \\ &\times \widetilde{W}(X + B_1 - A_1, P - A_2 + B_2) \end{aligned} \quad (23)$$

We call Θ' the operator with this Weyl function:

$$\widetilde{W}(X, P|\Theta') = \mathfrak{D}(A_1, A_2, B_1, B_2)\widetilde{W}(X, P|\Theta) \quad (24)$$

In this language, the present paper considers quantities in the general $\alpha - \beta$ plane (fig.1), which becomes the $X - P$ plane in the special case $\phi_1 = \phi_2 = 0$, and the $x - p$ plane in the special case $\phi_1 = \phi_2 = \frac{\pi}{2}$. It then studies relations between the various quantities. For example, we show that $\mathcal{V}_{\phi_1, \phi_2}(\alpha, \beta|\Theta')$ is related to $\widetilde{W}(X, P|\Theta)$ as follows:

$$\begin{aligned} \mathcal{V}_{\phi_1, \phi_2}(\alpha, \beta|\Theta') &= |\cos(\phi_1 - \phi_2)|^{1/2} \int \mathcal{K}_{\phi_2}(\beta, X)\mathcal{K}_{\phi_1}(\alpha, -P) \\ &\times [\mathfrak{D}(A_1, A_2, B_1, B_2)\widetilde{W}(X, P|\Theta)]dXdP \\ &= |\cos(\phi_1 - \phi_2)|^{1/2} \int \mathcal{K}_{\phi_2}(\beta, X + \mu_2) \exp(i\nu_2 X) \\ &\times \mathcal{K}_{\phi_1}(\alpha, -P - \mu_1) \exp(-i\nu_1 P) \widetilde{W}(X, P|\Theta)dXdP. \end{aligned} \quad (25)$$

Here

$$\mu_2 = A_1 - B_1; \quad \nu_2 = \frac{A_2 + B_2}{2}; \quad \mu_1 = A_2 - B_2; \quad \nu_1 = \frac{A_1 + B_1}{2}. \quad (26)$$

We next introduce the displaced kernel of the fractional Fourier transform

$$\mathfrak{K}_\phi(x, y|\kappa, \lambda) = \exp(i\kappa y) \exp(\lambda\partial_y)\mathcal{K}_\phi(x, y) = \exp(i\kappa y)\mathcal{K}_\phi(x, y + \lambda), \quad (27)$$

and rewrite Eq.(25) in the compact form

$$\begin{aligned} \mathcal{V}_{\phi_1, \phi_2}(\alpha, \beta | \Theta') &= |\cos(\phi_1 - \phi_2)|^{1/2} \int \mathfrak{K}_{\phi_2}(\beta, X | \nu_2, \mu_2) \mathfrak{K}_{\phi_1}(\alpha, -P | -\nu_1, -\mu_1) \\ &\times \widetilde{\mathcal{W}}(X, P | \Theta) dX dP \end{aligned} \quad (28)$$

Other similar relations can also be proved. In particular symplectic $Sp(4, R)$ transformations in the $x - p - X - P$ extended phase space require further study.

7. Discussion

We have studied the bi-fractional displacement operators $\mathcal{B}_{\phi_1, \phi_2}(\alpha, \beta)$. They include as special cases, both the displacement operators and the parity operators (Eq.(12)). We stress the importance of the prefactor $[\cos(\phi_1 - \phi_2)]^{1/2}$ in Eq.(7), for unitarity. This is related to the fact that in Eq.(7) X, P are dual quantum variables, multiplied by \hat{x}, \hat{p} which do not commute.

Using them we have defined the bi-fractional coherent states in Eq.(15). We have also defined the bi-fractional Wigner functions in Eq.(19), which include as special cases, both the Wigner and Weyl functions (Eq.(20)).

We have interpreted the formalism within the extended phase space, which relates quantum noise with correlations. The bi-fractional displacement operators $\mathcal{B}_{\phi_1, \phi_2}(\alpha, \beta)$, are displacement operators in the $\alpha - \beta$ plane of fig.1. The formalism provides an interpolation between various quantities in phase space.

For further work we suggest the application of these ideas to finite quantum systems with variables in \mathbb{Z}_d (the integers modulo d). We expect this generalization to be easier for prime d , in which case \mathbb{Z}_d is a field. Overall, the merit of the formalism is that different quantities appear to be special cases of a single quantity.

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