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Comments on supersymmetric quantum field theories

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Abstract. New mathematical conjectures have often been extracted from the analysis of supersymmetric quantum field theories. Here we would like to illustrate how this extraction process is performed.

1. Mathematical conjectures from QFTs: the classic example

The prime example where a big mathematical conjecture was obtained from the analysis of supersymmetric quantum field theories (QFTs) was the discovery of the mirror symmetry in the early 1990s. Let us quickly recall what this symmetry is.

Given a Calabi-Yau manifold X, there is an associated 2d supersymmetric QFT $Q(X)$:

$$X \mapsto Q(X) \tag{1}$$

In the space of 2d supersymmetric QFTs, there is an involutive automorphism σ

$$Q \mapsto \sigma(Q) \mapsto \sigma^2(Q) = Q. \tag{2}$$

Note that there is not yet a precise mathematical definition of what a 2d supersymmetric QFT is, or what this automorphism σ is. However, the quantum field theorists knew these concepts well enough, to find the following. Namely, given a Calabi-Yau X, it often happens that there is another Calabi-Yau Y such that

$$\sigma(Q(X)) = Q(Y). \tag{3}$$

Properties of X and Y related in a way very mysterious to those who do not know the background QFT material. For example, the complex structure of X is related to the symplectic structure of Y.

By now there is a big mathematical industry studying the relation between X and Y. However, mathematicians still do not talk directly about $Q(X)$, as if it is the one-who-must-not-be-named.

Journal of Physics: Conference Series **597** (2015) 012005 doi:10.1088/1742-6596/597/1/012005

2. Mathematical conjectures from QFTs: a recent one

There are many other minor such events where QFT led to new mathematical relations. For example, consider the following expression

$$I(u,v;x,y)=\oint \frac{\prod_{\pm\pm\pm}\Gamma_{p,q}(tu^{\pm1}v^{\pm1}z^{\pm1})\prod_{\pm\pm\pm}\Gamma_{p,q}(tx^{\pm1}y^{\pm1}z^{\pm1})}{\prod_{\pm}\Gamma_{p,q}(t^2z^{\pm2})\prod_{\pm}\Gamma_{p,q}(z^{\pm2})}\frac{dz}{2\pi iz} \tag{4}$$

where

$$\Gamma_{p,q}(z)=\prod_{j,k\geq 0}\frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k} \tag{5}$$

is the elliptic gamma function.

This integral $I(u,v;x,y)$ is obviously symmetric under the exchanges $u\leftrightarrow v$, $x\leftrightarrow y$. However, it is also symmetric under the exchange $u\leftrightarrow x$. On the physics side, this symmetry was conjectured in [1] in 2009. On the math side, this symmetry was proved completely independently in [2] in 2009.

3. Previous formalisms of QFTs

What is this thing called QFT, that touches many parts of mathematics? Those who practice QFT, including the author, feel they know what that is. However, those feelings are not enough to convey it to the diverse audience of this conference series. We clearly need some formalism, and we want some solid starting point. At this point, a reader might wonder that in fact there have been many such formalisms. Examples include Wightman axioms, algebraic quantum field theories, topological quantum field theories, and vertex operator algebras. They indeed capture some of the aspects of QFTs, but none of them is comprehensive enough to even state what the mirror symmetry is, or what the Seiberg-Witten theory is.

Such a formalism has not been written down anywhere yet, but the author thinks it should not be impossible to do so. After all, the axioms of a group, that look so straightforward today, took many years to be straightened out.

There is one other thing to be pointed out. In the study of QFT, in whatever formalization mentioned so far, or in particle physics community in general, people tend to study each individual QFT, Q_1, Q_2, ...one by one. For example, the whole experimental high energy particle physics can be said to be the quest to find exactly which QFT Q_{SM} describes elementary particles, and thus, the universe.

But this is like studying groups one by one. There is nothing wrong with that, particularly when there is a few particularly interesting groups / QFTs. We should, however, also study group homomorphism $G\to H$, representations of groups, action of groups on spaces, the quotient space G/H, etc. Similarly, we should study the interrelation among QFTs, the relation of QFTs with other mathematical objects, etc.

4. A formalism of QFTs

From now on, a QFT is always assumed to be four-dimensional and $\mathcal{N}=2$ supersymmetric, unless otherwise specified. A few basic formal axioms of QFTs are the following:

- Given a QFT Q and a 4d manifold X, we can form the partition function $Z_Q(X)\in\mathbb{C}$.
- Given two QFTs Q_1 and Q_2, we can form a new QFT $Q_1\times Q_2$.
- This product is commutative, associative, has a unit $\bullet$, i.e. $Q\times\bullet=Q$.
- The product is compatible with taking the partition function $Z_{Q_1\times Q_2}(X)=Z_{Q_1}(X)Z_{Q_2}(Y)$.

We also need a concept of G-symmetric QFTs:

Journal of Physics: Conference Series **597** (2015) 012005 doi:10.1088/1742-6596/597/1/012005

- Given a G-symmetric QFT Q and a 4d manifold X with G connection, we can form the partition function $Z_Q(X) \in \mathbb{C}$.
- Given a G-symmetric QFT Q and a homomorphism $\varphi : H \to G$, we can regard Q as an H-symmetric QFT.
- Given a G_1-symmetric QFT Q_1 and a G_2-symmetric QFT G_2, their product $Q_1 \times Q_2$ is $G_1 \times G_2$-symmetric, and $Z_{Q_1 \times Q_2}(X) = Z_{Q_1}(X) Z_{Q_2}(X)$.
- Finally, the unit $\bullet$ is G-symmetric for any G.

From this extended viewpoint we can think of an unqualified QFT as an $\{id\}$-symmetric QFT. Note that these formal properties are very much like spaces with G action:

- Given a space X with G action and a homomorphism $\varphi : H \to G$, we can regard that X has H action.
- Given a space X_1 with G_1 action and a space X_2 with G_2 action, their product $X \times Y$ has $G_1 \times G_2$ action.
- A point $\bullet$ has trivial G action for any G.

Now, recall that given a space X with $G \times H$ action, X/G is a space with H action. Similarly, given a QFT Q that is $G \times H$-symmetric, we can form a new QFT $Q /\!\!/\!\!/ G$, which is H-symmetric. Note that the G symmetry is gone. Usually this operation is called coupling to the gauge group G. With the formal operations so far, we already have interesting QFTs:

$$\bullet /\!\!/\!\!/ G. \tag{6}$$

They are called pure $\mathcal{N} = 2$ gauge theories with gauge group G in the standard physics literature. For a four-manifold M, a slightly modified version of

$$Z_{\bullet /\!\!/\!\!/ \mathrm{SU}(2)}(M) \tag{7}$$

is the Donaldson invariant, as shown in [3].

Another basic construction is the following. Given a symplectic representation R of G, there is a G-symmetric QFT we denote by $\mathrm{Hyp}(R)$:

$$R \mapsto \mathrm{Hyp}(R). \tag{8}$$

This is usually called the free hypermultiplet in R, and has the following formal properties:

$$\mathrm{Hyp}(R \oplus R') = \mathrm{Hyp}(R) \times \mathrm{Hyp}(R'), \qquad \mathrm{Hyp}(0) = \bullet \tag{9}$$

where 0 stands for a trivial, zero-dimensional representation.

Now we know there are QFTs of the form $\mathrm{Hyp}(R) /\!\!/\!\!/ G$, which are usually called $\mathcal{N} = 2$ supersymmetric gauge theories. For example, take $G = \mathrm{U}(1)$, and $R = V \oplus V^*$, where $V \simeq \mathbb{C}$ is a standard 1-dimensional representation of $\mathrm{U}(1)$. Let

$$Q' = \mathrm{Hyp}(R) /\!\!/\!\!/ \mathrm{U}(1). \tag{10}$$

A version of its partition function, $Z_{Q'}(M)$, is the Seiberg-Witten invariant. These invariants are concrete objects but rather deep, and will not be discussed further in this note.

Journal of Physics: Conference Series **597** (2015) 012005 doi:10.1088/1742-6596/597/1/012005

5. Mathematical objects associated to QFTs

Let us discuss two easier objects to discuss, associated to a given G-symmetric QFT Q. These are the Higgs branch $\mathcal{M}_{\text{Higgs}}(Q)$ and the superconformal index $\text{SCI}(Q)$.

The Higgs branch $\mathcal{M}_{\text{Higgs}}(Q)$ is a Hyperkähler space with G action, and the superconformal index $\text{SCI}(Q)$ is a class function on G that is a formal power series in p, q, t. They preserve formal properties under the multiplication of the QFTs:

$$\mathcal{M}_{\text{Higgs}}(Q_1 \times Q_2) = \mathcal{M}_{\text{Higgs}}(Q_1) \times \mathcal{M}_{\text{Higgs}}(Q_2), \qquad \text{SCI}(Q_1 \times Q_2) = \text{SCI}(Q_1) \times \text{SCI}(Q_2). \tag{11}$$

For the gauging $Q /\!\!/\!\!/\!\!/ G$, we have

$$\mathcal{M}_{\text{Higgs}}(Q /\!\!/\!\!/\!\!/ G) = \mathcal{M}_{\text{Higgs}}(Q) /\!/\!/ G \tag{12}$$

where $/\!/\!/ G$ is the hyperkähler quotient construction and

$$\text{SCI}(Q /\!\!/\!\!/\!\!/ G) = \left(\frac{1}{\Gamma_{p,q}(t)\Gamma'_{p,q}(1)}\right)^r \frac{1}{|W_G|} \oint \frac{\text{SCI}(Q)(z)}{\prod_\alpha \Gamma_{p,q}(z^\alpha)\Gamma_{p,q}(t^2 z^\alpha)} \prod_{i=1}^r \frac{dz_i}{2\pi\sqrt{-1}z_i} \tag{13}$$

where α runs over the roots of G. Here we restricted $z \in G$ to lie in the Cartan torus $z \in \text{U}(1)^r \subset G$.

For the free hypermultiplets $Q = \text{Hyp}(R)$, we have

$$\mathcal{M}_{\text{Higgs}}(\text{Hyp}(R)) = R, \qquad \text{SCI}(\text{Hyp}(R))(z) = \prod_w \Gamma_{p,q}(tz^w) \tag{14}$$

where z is in the Cartan of G and w runs over the weights of R.

Now, given a symplectic representation R of $G \times H$, we can form $Q = \text{Hyp}(R) /\!\!/\!\!/\!\!/ G$ that is H-symmetric. For this QFT, we have

$$\mathcal{M}_{\text{Higgs}}(\text{Hyp}(R) /\!\!/\!\!/\!\!/ G) = R /\!/\!/ G, \tag{15}$$

which is a hyperkähler space with H action. Similarly, we have

$$\text{SCI}(\text{Hyp}(R) /\!\!/\!\!/\!\!/ G)(y) = \oint \frac{\prod_{w \oplus v} \Gamma_{p,q,t}(z^w y^v)}{\prod_\alpha \Gamma_{p,q}(z^\alpha)\Gamma_{p,q}(t^2 z^\alpha)} \prod_{i=1}^r \frac{dz_i}{2\pi z_i}, \tag{16}$$

which is the so-called elliptic beta integral. They are both well-studied in mathematics.

Note that we start from a group G and its representation R, that are both well established mathematical objects. We then pass to $\text{Hyp}(R) /\!\!/\!\!/\!\!/ G$, which is not well formulated yet. Then we pass back to $\mathcal{M}_{\text{Higgs}}(\text{Hyp}(R) /\!\!/\!\!/\!\!/ G)$ or $\text{SCI}(\text{Hyp}(R) /\!\!/\!\!/\!\!/ G)$, both of which are again well-established mathematical-physical objects. How do we get something new?

6. Gaiotto's construction

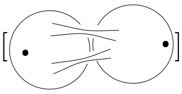
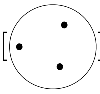
The key is that there are a more transcendental construction coming from six dimensions. Let G be a simply-laced group. Then it is believed that there is a 6-dimensional supersymmetric QFT $\mathcal{S}_G$ with very good properties, called the 6d $\mathcal{N}{=}(2,0)$ theory. Given a k-punctured Riemann surface C and a 4d manifold X, define

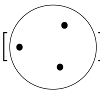
$$Z_G[C](X) = \mathcal{S}_G(X \times C). \tag{17}$$

This gives a 4d QFT $\mathcal{S}_G[C]$ depending on C.

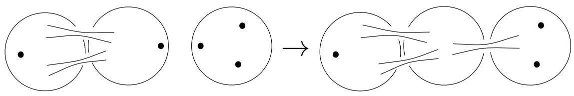
Journal of Physics: Conference Series **597** (2015) 012005 doi:10.1088/1742-6596/597/1/012005

With k-points, $\mathcal{S}_G[C]$ is G^k symmetric. For example,

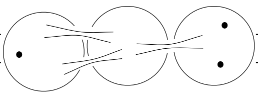
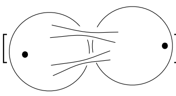
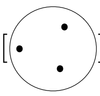
$\mathcal{S}_G[$  $],\qquad \mathcal{S}_G[$  $]$ (18)

are G^2, G^3 symmetric, respectively. Not only that, $\mathcal{S}_G[C]$ depends only on the topology of C: you can exchange points without changing the theory. More mathematically, there is an S_k action on k marked points on C. Therefore, when C has k marked points, $\mathcal{S}_G[C]$ is $S_k \ltimes G^k$ symmetric. For example, $\mathcal{S}_G[$  $]$ is $S_3 \ltimes G^3$ symmetric.

Now, we can connect two punctures of two Riemann surfaces:

 . (19)

Correspondingly, we have the crucial property

$\mathcal{S}_G[$  $] = (\mathcal{S}_G[$  $] \times \mathcal{S}_G[$  $]) /\!\!/\!\!/ G$ (20)

where the gauging G is performed with respect to $G_{\text{diag}} \to G \times G$ associated to the two punctures connected. This was first formulated by Gaiotto in [4] and opened up a new direction in the study of supersymmetric quantum field theories.

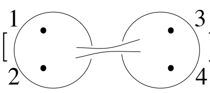
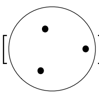
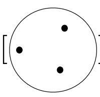
We can say that the assignment

$$C \mapsto \mathcal{S}_G[C] \tag{21}$$

maps the operations among Riemann surfaces to the operations among QFTs. For those who know the axioms of 2d topological QFT, this can be phrased as follows: a usual 2d topological QFT takes values in the monoidal category of vector spaces, whereas this topological QFT $\mathcal{S}_G$ takes values in (a higher version of) the monoidal category of 4d supersymmetric QFTs.

7. Gaiotto's construction and mathematical conjectures

Now, consider the equation

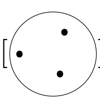
$\mathcal{S}_G[$  $] = (\mathcal{S}_G[$  $] \times \mathcal{S}_G[$  $]) /\!\!/\!\!/ G.$ (22)

Note that on the left hand side, G^4 are manifestly interchangeable, by the formal property. There is an action of $S_4 \ltimes G^4$. On the right hand side, however, we started from two objects with $S_3 \ltimes G^3$ symmetry. But by connecting two G, we only have the symmetry

$$S_2 \ltimes ((S_2 \ltimes G^2) \times (S_2 \ltimes G^2)). \tag{23}$$

This means that something nontrivial is going on. This non-triviality, however, happens within the category of QFTs.

We can get something nontrivial happening in something well defined, by applying $\mathcal{M}_{\text{Higgs}}$ or SCI. For this, let us take $G = \mathrm{SU}(2)$. Then it is known that

$\mathrm{Hyp}(V_1 \otimes V_2 \otimes V_3) = \mathcal{S}_{\mathrm{SU}(2)}[$  $]$ (24)

Journal of Physics: Conference Series **597** (2015) 012005 doi:10.1088/1742-6596/597/1/012005

where $V_i \simeq \mathbb{C}^2$ is the defining representation of SU(2). The left hand side correctly has $\mathrm{SU}(2)^3$ action, with S_3 permuting them. We now have

$$
\begin{aligned}
&\mathcal{M}_{\mathrm{Higgs}}(\mathcal{S}_{\mathrm{SU}(2)}[{}^{1}_{2}\qquad{}^{3}_{4}]) \\
&= \mathcal{M}_{\mathrm{Higgs}}((\mathcal{S}_{\mathrm{SU}(2)}[\quad] \times \mathcal{S}_{\mathrm{SU}(2)}[\quad]) /\!\!/\!\!/ \mathrm{SU}(2)) \\
&= (V_1 \otimes V_2 \otimes V \oplus V \otimes V_3 \otimes V_4) /\!\!/\!\!/ \mathrm{SU}(2)
\end{aligned}
\tag{25}
$$

From the left hand side, it should have S_4 permuting four SU(2)s. The right hand side does not obviously have this symmetry. In fact, the right hand side is the ADHM construction of the minimal nilpotent orbit of $\mathrm{SO}(8)_{\mathbb{C}}$. There is an outer automorphism S_3 acting on SO(8), that provides S_4 permuting SU(2)s.

We also have

$$
\begin{aligned}
&\mathrm{SCI}(\mathcal{S}_{\mathrm{SU}(2)}[{}^{1}_{2}\qquad{}^{3}_{4}])(u,v,x,y) \\
&= \mathrm{SCI}((\mathcal{S}_{\mathrm{SU}(2)}[\quad] \times \mathcal{S}_{\mathrm{SU}(2)}[\quad]) /\!\!/\!\!/ \mathrm{SU}(2))(u,v,x,y) \\
&= \mathrm{SCI}((V_1 \otimes V_2 \otimes V \oplus V \otimes V_3 \otimes V_4) /\!\!/\!\!/ \mathrm{SU}(2))(u,v,x,y) \\
&= \frac{1}{2\Gamma_{p,q}(t)\Gamma'_{p,q}(1)} I(u,v;x,y)
\end{aligned}
\tag{26}
$$

where $I(u,v;x,y)$ is the hyperelliptic beta integral defined in (4) From the left hand side, we see that the right hand side should have the S_4 symmetry permuting u,v,x,y. This was how it was conjectured on the physics side in [1]. On the math side, this symmetry was proved completely independently in [2], as already mentioned.

There are more operations, extracting concrete, well-defined mathematical physical objects from QFTs. For example, given a G-symmetric QFT Q, there is a vertex operator algebra $W(Q)$ that has $\hat{g}$ affine Lie algebra as a vertex subalgebra. Further, $W(Q /\!\!/\!\!/ G)$ is given by something like the Drinfeld-Sokolov reduction of $W(Q)$ with respect to $\hat{g}$, and $W(\mathrm{Hyp}(R))$ is the standard symplectic boson VOA. This operation was introduced in [5]. Then we have

$$
\begin{aligned}
&W(\mathcal{S}_{\mathrm{SU}(2)}[{}^{1}_{2}\qquad{}^{3}_{4}]) \\
&= W((\mathcal{S}_{\mathrm{SU}(2)}[\quad] \times \mathcal{S}_{\mathrm{SU}(2)}[\quad]) /\!\!/\!\!/ \mathrm{SU}(2)) . \\
&= W((V_1 \otimes V_2 \otimes V \oplus V \otimes V_3 \otimes V_4) /\!\!/\!\!/ \mathrm{SU}(2))
\end{aligned}
\tag{27}
$$

This is a VOA that can explicitly be written down, and has four $\widehat{\mathfrak{su}(2)}$ affine Lie subalgebra. But the existence of S_4 action permuting four $\widehat{\mathfrak{su}(2)}$ has not been proved. In fact the final VOA is conjectured to be just $\widehat{\mathfrak{so}(8)}_{-2}$, and any mathematician reading this note should consult [5] and prove it.

Yet another object one can associate to a G-symmetric Q is the Nekrasov partition function $Z_{\mathrm{Nekrasov}}(Q)$, which is an element in the equivariant cohomology of the moduli space of

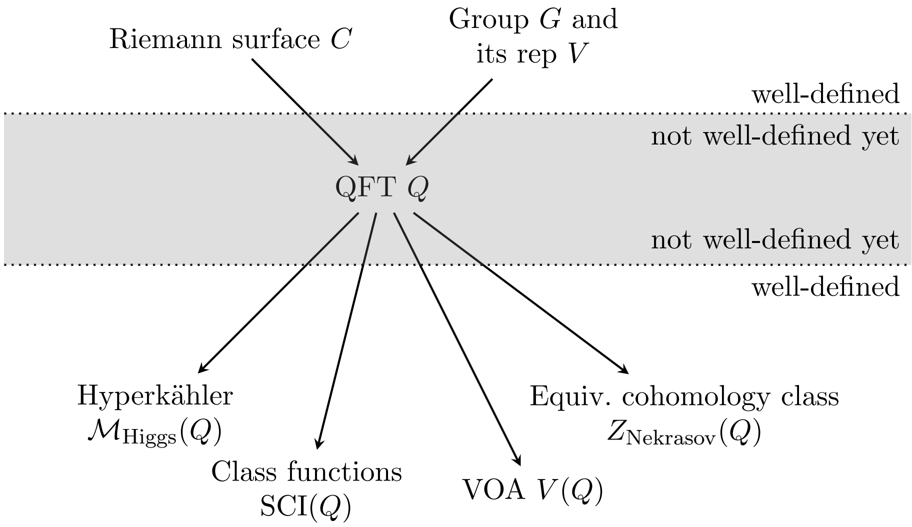


Figure 1. Relations of various objects discussed in the note.

G instantons. $Z_{\text{Nekrasov}}(\text{Hyp}(R))$ is determined by the index bundle of the Dirac operator associated to the representation R of G in the instanton background, and $Z_{\text{Nekrasov}}(Q /\!/\!/ G)$ is also computable, given $Z_{\text{Nekrasov}}(Q)$.

There are more formal properties satisfied by $\mathcal{S}_G[C]$ that depends on the complex structure of C, which was not explained in this note. The ability to permute four points on

$$Z_{\text{Nekrasov}}(\mathcal{S}_G[\quad]), \tag{28}$$

when carefully analyzed, turns out to imply that there should be a $W(G)$-algebra action on the equivariant cohomology of the moduli space of G instantons.

This was how L. Fernando Alday, Davide Gaiotto and the author conjectured the relation between 4d gauge theory and 2d conformal field theory in [6], although the argument presented here was streamlined with lots of hindsight today. That conjecture, which has recently proved by mathematicians, was the main reason the author was awarded a prize in this conference.

8. Conclusions

As a summary, in Fig. 1, the relations of various objects discussed in this note are summarized in a picture. The steps are simple to understand: we pass from a well-defined realm to a not-well-defined realm, where some properties are 'known'. We then pass back to a well-defined realm again, producing precisely-formulated mathematical conjectures.

The author believes that many properties of QFT that are used to derive new mathematical conjectures are formalizable. Of course, there will remain some 'deus ex machina' in the process. For example, the existence of the operation

$$Q \mapsto Q /\!/\!/ G \tag{29}$$

almost contains the solution to the mass-gap problem. This is because of the following. Seiberg and Witten have already shown in [7] that, given some basic properties of $\bullet /\!/\!/ G$, we can easily deduce that the pure non-supersymmetric G gauge theory has a mass gap, which is one of the Clay Millenium Problems.

Similarly, the existence of the 6d $\mathcal{N} = (2,0)$ theory $\mathcal{S}_G$ is another 'deus ex machina'. But, assuming that, the map $C \mapsto \mathcal{S}_G[C]$ can be constructed rather formally, and many of the properties follow straightforwardly, in a way understandable to mathematicians and mathematical physicists. So, we will be able to rigorously show that the symmetry $u \leftrightarrow x$ of the hyperelliptic beta integral $I(u,v;x,y)$ in (4) follows from the existence of $\mathcal{S}_{\mathrm{SU}(2)}$.

The author hopes that in this way, mathematicians will be able to see which part of the analysis done by quantum field theorists is truly deep and which part is more mundane.

Acknowledgments

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