On Vacuum Polarization and Schwinger Pair Production in Intense Lasers

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Abstract. We review and elaborate the complex effective action at one-loop and at zero or finite temperature in the in-out formalism for scalar QED and the vacuum persistence in time-dependent electric fields. Using the gamma-function regularization, we find the effective action in the proper-time integral and the pair-production rate in an exponentially increasing electric field. We apply the quantum invariant theory to the scalar field in time-dependent electric fields and clarify the meaning of multi-pair states. And pair-production rates are derived when the initial state is the Minkowski vacuum or the adiabatic vacuum and are compared with two representations for the Vlasov equation. Finally, the contour integral method gives the pair-production rate in the exponentially increasing electric field.

1. Introduction

The vacuum polarization and the production of electron-positron pairs by background electric fields is one of the most interesting nonperturbative phenomena in quantum field theory. Heisenberg and Euler [1] and Schwinger [2] have shown that the vacuum under the action of a constant electromagnetic field could be polarized as a medium and the effective action could gain an imaginary part, implying the instability of the Dirac sea (Minkowski vacuum) due to the pair production. With the rapid development of intense lasers, probing the vacuum structure may be attainable by such a facility as Extreme Light Infrastructure (ELI) or International Center for Zetta-Exawatt Science and Technology (IZEST) in the near future.[1]

The coherent state of numerous low energy photons from intense lasers provides an electromagnetic field, which is spatially localized and temporally pulsed. However, computing the one-loop effective action for the localized pulse has been a theoretical challenge in quantum field theory in strong field physics (for a good review and references, see [3]). Further, electron-positron pairs can be produced in a localized, time-dependent electric field. The vacuum instability implies the one-loop effective action should be complex in electric fields. In fact, the in-out formalism by Schwinger and DeWitt [4, 5] leads to a complex effective action, the real part of which is the vacuum polarization and twice the imaginary part of which is the decay rate of the vacuum and is related to the pair production rate. The methods so far introduced for calculating the pair-production rate are the Vlasov equation [6, 7, 8], the worldline instanton [9, 10], the phase integral [11, 12], the contour integral [13, 14], to name a few. The methods for the

[1] ELI, http://www.eli-laser.eu/ and IZEST, http://www.izest.polytechnique.edu/.

vacuum polarization are the resolvent technique [15], the Feynman worldline path integral [16], the Dirac-Heisenberg-Wigner formalism [17], the Schwinger-DeWitt in-out formalism [18, 19]. There are other useful methods either for the vacuum persistence or for the vacuum polarization, which are not listed here [3].

In this paper, we review and elaborate a recent field theoretic method for the complex effective action and the quantum invariant method for quantum states for a charged field in time-dependent electric fields. In the in-out formalism, the one-loop action integrated over the four-volume is the scattering matrix $e^{i\mathcal{W}} = \langle 0, \mathrm{out}|0, \mathrm{in}\rangle$ between the out-vacuum and the in-vacuum. Further, the scattering matrix can be expressed by the Bogoliubov coefficients between two vacua in the electromagnetic field background. As quantum field theory, some renormalization scheme should be taken to regularize the UV divergences by renormalizing the vacuum energy and the charge. With the renormalization taken for granted, the task for the one-loop effective action is to find the Bogoliubov transformation between the in-vacuum and the out-vacuum. This implies that the one-loop effective action becomes complex when an electric field present regardless of the presence of a magnetic field. In other words, the complex effective action consists of the vacuum polarization and the vacuum persistence

$$\mathcal{L}^{\mathrm{C}}_{\mathrm{eff}} = \mathcal{L}_{\mathrm{vac\,pol}} + \frac{i}{2}\mathcal{L}_{\mathrm{vac\,per}}. \tag{1}$$

As will be shown in section 2, the vacuum persistence in the electric field is related to the pair-production rate $\mathcal{N}_{\mathbf{k}}$ for each momentum

$$\mathcal{L}_{\mathrm{vac\,per}} = 2\,\mathrm{Im}\mathcal{L}^{\mathrm{C}}_{\mathrm{eff}} = \sum_{\mathbf{k}} \ln(1+\mathcal{N}_{\mathbf{k}}). \tag{2}$$

We then apply the quantum invariant theory by Lewis and Riesenfeld [20] to the time-dependent Hamiltonian for a charged spinless scalar in time-dependent electric fields. The Hamiltonian is equivalent to an infinite sum of time-dependent oscillators in homogeneous, time-dependent electric fields and to coupled oscillators in the time-dependent magnetic fields with a fixed direction [21]. The time-dependent annihilation and creation operators, quantum invariant operators, construct not only the time-dependent vacuum but also excited states of multi-pairs and a thermal state. Electron-positron pairs are spontaneously produced from the vacuum and there can be stimulated production of pairs from some pairs or a thermal ensemble of pairs present in the remote past.

2. Review of In-Out Formalism for Scalar QED Action

In this section we review the complex QED action in time-dependent electric fields the in-out formalism. For the sake of simplicity, we consider a charged spinless scalar in a homogeneous, time-dependent electromagnetic field with the vector potential

$$\mathbf{A}(t) = -\int^{t} \mathbf{E}(t')dt'. \tag{3}$$

Due to the translational symmetry, the Klein-Gordon equation after the momentum-decomposition becomes a second order equation (in units of $\hbar = c = 1$)

$$\phi_{\mathbf{k}}(t) + \omega^2_{\mathbf{k}}(t)\phi_{\mathbf{k}}(t) = 0 \tag{4}$$

with a time-dependent frequency squared

$$\omega^2_{\mathbf{k}}(t) = (\mathbf{k} - q\mathbf{A}(t))^2 + m^2. \tag{5}$$

By connecting the positive and the negative frequency solutions to eq. (4) satisfying appropriate boundary conditions in the remote future and the remote past, we find the Bogoliubov transformation between the particle and antiparticle annihilation operators $\hat{a}_{\mathbf{k},\mathrm{out}}, \hat{b}_{\mathbf{k},\mathrm{out}}$ in the out-vacuum and those $\hat{a}_{\mathbf{k},\mathrm{in}}, \hat{b}_{\mathbf{k},\mathrm{in}}$ in the in-vacuum:

$$\begin{aligned} \hat{a}_{\mathbf{k},\mathrm{out}} &= \alpha_{\mathbf{k}} \hat{a}_{\mathbf{k},\mathrm{in}} + \beta^*_{\mathbf{k}} \hat{b}^\dagger_{\mathbf{k},\mathrm{in}} = \hat{U}_{\mathbf{k}} \hat{a}_{\mathbf{k},\mathrm{in}} \hat{U}^\dagger_{\mathbf{k}}, \\ \hat{b}_{\mathbf{k},\mathrm{out}} &= \alpha_{\mathbf{k}} \hat{b}_{\mathbf{k},\mathrm{in}} + \beta^*_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k},\mathrm{in}} = \hat{U}_{\mathbf{k}} \hat{B}_{\mathbf{k},\mathrm{in}} \hat{U}^\dagger_{\mathbf{k}}. \end{aligned} \tag{6}$$

Here, the evolution operator can be expressed in terms of a two-mode squeezing operator and a phase operator [18]

$$\hat{U}_{\mathbf{k}} = \exp\Big[r_{\mathbf{k}}(\hat{a}_{\mathbf{k},\mathrm{in}} \hat{b}_{\mathbf{k},\mathrm{in}} e^{-2i\vartheta_{\mathbf{k}}} - \mathrm{H.C.})\Big] \exp\Big[i\theta_{\mathbf{k}}(\hat{a}^\dagger_{\mathbf{k},\mathrm{in}} \hat{a}_{\mathbf{k},\mathrm{in}} + \hat{b}_{\mathbf{k},\mathrm{in}} \hat{b}^\dagger_{\mathbf{k},\mathrm{in}})\Big], \tag{7}$$

where the squeeze parameter $r_{\mathbf{k}}$ and angles $\theta_{\mathbf{k}}, \vartheta_{\mathbf{k}}$ are given by

$$\alpha_{\mathbf{k}} = e^{-i\theta_{\mathbf{k}}} \cosh r_{\mathbf{k}}, \quad \beta_{\mathbf{k}} = -e^{i\theta_{\mathbf{k}}}(e^{-2i\vartheta_{\mathbf{k}}} \sinh r_{\mathbf{k}}). \tag{8}$$

These operators satisfy the commutation relations

$$[\hat{a}_{\mathbf{k},\mathrm{in}}, \hat{a}^\dagger_{\mathbf{p},\mathrm{in}}] = [\hat{b}_{\mathbf{k},\mathrm{in}}, \hat{b}^\dagger_{\mathbf{p},\mathrm{in}}] = \delta(\mathbf{k} - \mathbf{p}), \tag{9}$$

and a similar expression holds for $\hat{a}_{\mathbf{k},\mathrm{out}}, \hat{b}_{\mathbf{k},\mathrm{out}}$ but all the other commutators vanish. The out-vacuum as the squeeze vacuum

$$|0, \mathrm{out}\rangle = \prod_{\mathbf{k}} e^{i\theta_{\mathbf{k}}} \exp\Big[r_{\mathbf{k}}(\hat{a}_{\mathbf{k},\mathrm{in}} \hat{b}_{\mathbf{k},\mathrm{in}} e^{-2i\vartheta_{\mathbf{k}}} - \mathrm{H.C.})\Big] |0, \mathrm{in}\rangle, \tag{10}$$

is equivalent to the summation of all one-loop diagrams with even number of external legs. Then, the out-vacuum is the multi-pair states of the in-vacuum

$$|0, \mathrm{out}\rangle = \prod_{\mathbf{k}} e^{i\theta_{\mathbf{k}}} \mathrm{sech} r_{\mathbf{k}} \Big[\sum_{n_{\mathbf{k}}=0}^{\infty} e^{i2n_{\mathbf{k}}\vartheta_{\mathbf{k}}} \tanh^{n_{\mathbf{k}}}(r_{\mathbf{k}}) |n_{\mathbf{k}}, \bar{n}_{\mathbf{k}}, \mathrm{in}\rangle\Big]. \tag{11}$$

The complex one-loop effective action at the zero temperature ($T = 0$) integrated over three-volume $\mathcal{V}$ and time is given by [18, 19]

$$\mathcal{W} = -i \ln\langle 0, \mathrm{out}|0, \mathrm{in}\rangle = i\mathcal{V} \sum_{\mathbf{k}} \alpha^*_{\mathbf{k}}, \tag{12}$$

with $\sum_{\mathbf{k}} = \int d^3\mathbf{k}/(2\pi)^3$, and the action at finite temperature T is given by [22]

$$e^{\mathcal{W}_{\mathrm{T}}} = \frac{\mathrm{Tr}\Big(\prod_{\mathbf{k}} \hat{U}^\dagger_{\mathbf{k}} \hat{\rho}_{\mathbf{k}}\Big)}{\mathrm{Tr}\Big(\prod_{\mathbf{k}} \hat{\rho}_{\mathbf{k}}\Big)}. \tag{13}$$

Here, $\hat{\rho}_{\mathbf{k}}$ denotes the density operator for an initial thermal ensemble of pairs with the energy $\omega_{\mathbf{k}}$ present in the background field. In a similar way, we may find from eq. (13) the effective action stimulated by the initial pairs in the state $\hat{\rho}_{\mathbf{k}} = |n_{\mathbf{k}}, \bar{n}_{\mathbf{k}}, \mathrm{in}\rangle\langle n_{\mathbf{k}}, \bar{n}_{\mathbf{k}}, \mathrm{in}|$, which is the QED analog of stimulated emission of photons. Then, the effective action at T per unit volume has the expression [22]

$$\mathcal{L}^{\mathrm{C}}_{\mathrm{eff}}(T) = -i \sum_{\mathbf{k}} \Big[\ln(1 + e^{-(\omega_{\mathbf{k}} - z_{\mathbf{k}})/kT}) - \frac{z_{\mathbf{k}}}{kT} - \ln(1 + e^{-\omega_{\mathbf{k}}/kT})\Big], \tag{14}$$

where $z_{\mathbf{k}}$ is the chemical potential

$$z_{\mathbf{k}} = kT(i\theta_{\mathbf{k}} - \ln\cosh r_{\mathbf{k}}). \tag{15}$$

The action (14) has the following interpretation: the first term is the effective action with the complex chemical potential $z_{\mathbf{k}}$, the second term is the zero-temperature action and the last term is the free energy for the initial ensemble. In the zero-temperature limit, eq. (14) reduces to the vacuum effective action. Remarkably, the vacuum persistence is expressed by the initial Bose-Einstein distribution n_{BE} and the complex chemical potential as [22]

$$2\mathrm{Im}\mathcal{L}^{\mathrm{C}}_{\mathrm{eff}}(T) = -\sum_{\mathbf{k}}\sum_{l=0}^{\infty}\frac{(n_{BE})^l}{l}[(e^{z_{\mathbf{k}}/kT}-1)^l + (e^{z^*_{\mathbf{k}}/kT}-1)^l]. \tag{16}$$

3. Complex QED Action in a Time-dependent Electric Field

We now illustrate the in-out formalism by working out the scalar QED in a time-dependent electric field along the z-direction

$$E_z(t) = E_0 e^{\frac{t}{\tau}}, \quad A_z(t) = -E_0\tau(e^{\frac{t}{\tau}} - 1). \tag{17}$$

The positive frequency solution to eq. (4) in the remote future is given by the Whittaker function

$$\phi^{(+)}_{\mathbf{k}}(t) = e^{-i\pi\lambda/2}\sqrt{\frac{\tau}{\xi}}W_{\lambda,-\mu}(\xi), \tag{18}$$

where

$$\xi = 2iqE_0\tau^2 e^{\frac{t}{\tau}}, \quad \mu = i\tau\bar{\omega}_{\mathbf{k}}, \quad \lambda = -i\tau\bar{k}_z, \tag{19}$$

with

$$\bar{k}_z = k_z - qE_0\tau, \quad \bar{\omega}_{\mathbf{k}} = \sqrt{m^2 + \mathbf{k}_\perp^2 + \bar{k}_z^2}. \tag{20}$$

One may be tempted to expect a catastrophic pair production since the model electric field (17) increases beyond the critical field when $t \geq \tau\ln(m^2/qE_0)$. In quantum field theory, however, the out-vacuum is defined as an asymptotic state with respect to the solution (18) in the remote future. And the solution (18) can be linearly superposed as the positive and the negative frequency solutions in the remote past, corresponding to the Minkowski vacuum. Then, the Bogoliubov coefficients are

$$\begin{aligned}
\alpha_{\mathbf{k}} &= \sqrt{\frac{2\bar{\omega}_{\mathbf{k}}\tau e^{\pi\tau\bar{\omega}_{\mathbf{k}}}}{e^{\pi\bar{k}_z}}}\frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\lambda)}, \\
\beta_{\mathbf{k}} &= \sqrt{\frac{2\bar{\omega}_{\mathbf{k}}\tau}{e^{\pi\tau\bar{\omega}_{\mathbf{k}}}e^{\pi\bar{k}_z}}}\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\lambda)}.
\end{aligned} \tag{21}$$

The pair-production rate is given by

$$\mathcal{N}_{\mathbf{k}} = |\beta_{\mathbf{k}}|^2 = \frac{e^{-2\pi\tau\bar{\omega}_{\mathbf{k}}} + e^{-2\pi\tau\bar{k}_z}}{e^{2\pi\tau\bar{\omega}_{\mathbf{k}}} - e^{-2\pi\tau\bar{\omega}_{\mathbf{k}}}}. \tag{22}$$

We find the complex effective action (12) from the in-out formalism

$$\mathcal{L}^{\mathrm{C}}_{\mathrm{eff}} = i\sum_{\mathbf{k}}[\Gamma(2\mu^*) - \Gamma(\frac{1}{2}+\mu^*-\lambda^*) - \cdots], \tag{23}$$

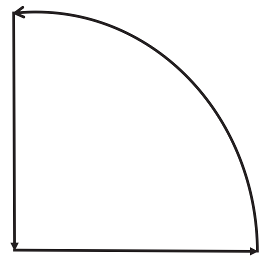

Figure 1. One of the contour integrals in the complex plane of proper time for the gamma function regularization [23].

where dots denote terms to be regulated away through renormalization. Following refs. [18, 19], the complex action can be written as the proper-time integral

$$\mathcal{L}_{\rm eff}^{\rm C} = i \sum_{\mathbf{k}} \int_0^\infty \frac{ds}{s} \Big[\frac{e^{2i\tau\bar{\omega}_{\mathbf{k}} s}}{1-e^{-s}} - \frac{e^{i\tau(\bar{\omega}_{\mathbf{k}}+\bar{k}_z)s}}{e^{\frac{s}{2}} - e^{-\frac{s}{2}}} - 1 + \frac{2}{s} \Big], \tag{24}$$

where the Schwinger subtraction scheme is employed to renormalize the vacuum energy and the charge. The Cauchy theorem for the contour in figure 1 leads to the complex effective action

$$\mathcal{L}_{\rm eff}^{\rm C} = \frac{1}{2} \sum_{\mathbf{k}} \int_0^\infty \frac{ds}{s} \Big[\frac{e^{-2\tau\bar{\omega}_{\mathbf{k}} s}}{\cos(\frac{s}{2})} - \frac{e^{-\tau(\bar{\omega}_{\mathbf{k}}+\bar{k}_z)s}}{\sin(\frac{s}{2})} - 1 + \frac{2}{s} \Big] + \frac{i}{2} \ln(1+\mathcal{N}_{\mathbf{k}}). \tag{25}$$

The imaginary part comes from the residues at simple poles $(2i\pi n)$ along the imaginary axis and thus satisfies the consistency condition from the in-out formalism. It is interesting that the first proper-time integral has the form for spinor QED while the second integral has the form for scalar QED [2]. Using the Bogoliubov coefficients (21), we may find the complex effective action (14) at finite temperature, which will not be pursued here.

4. Quantum Invariant Theory for Schrödinger Equation

In the time-dependent electric field (3), the charged field has the time-dependent Hamiltonian [21]

$$H(t) := \sum_{\mathbf{k}} H_{\mathbf{k}}(t) = \sum_{\mathbf{k}} [\pi_{\mathbf{k}}^* \pi_{\mathbf{k}} + \omega_{\mathbf{k}}^2(t) \phi_{\mathbf{k}}^* \phi_{\mathbf{k}}], \tag{26}$$

where $\pi_{\mathbf{k}} = \dot{\phi}_{\mathbf{k}}^*$, $\pi_{\mathbf{k}}^* = \dot{\phi}_{\mathbf{k}}$. In the Schrödinger picture, quantum states may be found from the time-dependent functional Schrödinger equation

$$i \frac{\partial}{\partial t} \Psi(t, \phi, \phi^*) = \hat{H}(t) \Psi(t, \phi, \phi^*). \tag{27}$$

The main advantage of the Schrödinger picture is the diversity of quantum states as in quantum mechanics and quantum optics. We further employ the quantum invariant theory for time-dependent oscillators by Lewis and Riesenfeld, which solves the Liouville-von Neumann equation for the density operator [20]

$$i \frac{\partial \hat{\rho}_{\mathbf{k}}(t)}{\partial t} + [\hat{\rho}_{\mathbf{k}}(t), \hat{H}_{\mathbf{k}}(t)] = 0. \tag{28}$$

The eigenstate of the invariant (28) is an exact solution to eq. (27) up to a time-dependent phase factor [20]. We adopt a pair of invariant operators that play the role of time-dependent annihilation and creation operators [21, 24, 25]

$$\begin{aligned}\hat{A}_{\mathbf{k}}(t) &= \frac{i}{\sqrt{2}}\Big(\varphi^*_{\mathbf{k}}(\hat{\pi}^\dagger_{\mathbf{k}} + \hat{\pi}_{\mathbf{k}}) - \dot{\varphi}^*{}_{\mathbf{k}}(\hat{\phi}^\dagger_{\mathbf{k}} + \hat{\phi}_{\mathbf{k}})\Big) = \frac{1}{\sqrt{2}}(\hat{a}_{\mathbf{k}}(t) + \hat{b}_{\mathbf{k}}(t)),\\ \hat{A}^\dagger_{\mathbf{k}}(t) &= -\frac{i}{\sqrt{2}}\Big(\varphi_{\mathbf{k}}(\hat{\pi}^\dagger_{\mathbf{k}} + \hat{\pi}_{\mathbf{k}}) - \dot{\varphi}_{\mathbf{k}}(\hat{\phi}^\dagger_{\mathbf{k}} + \hat{\phi}_{\mathbf{k}})\Big) = \frac{1}{\sqrt{2}}(\hat{a}^\dagger_{\mathbf{k}}(t) + \hat{b}^\dagger_{\mathbf{k}}(t)).\end{aligned} \tag{29}$$

Here, $\hat{a}_{\mathbf{k}}(t)$ and $\hat{b}_{\mathbf{k}}(t)$ are the annihilation operators for particle and antiparticle, respectively, and φ_α is a complex solution to the mode equation (4) and satisfies the Wronskian condition

$$\varphi_{\mathbf{k}}(t)\dot{\varphi}^*_{\mathbf{k}}(t) - \varphi^*_{\mathbf{k}}(t)\dot{\varphi}_{\mathbf{k}}(t) = i. \tag{30}$$

Then, the time-dependent annihilation and creation operators satisfy the equal-time commutator

$$[\hat{A}_{\mathbf{k}}(t), \hat{A}^\dagger_{\mathbf{p}}(t)] = \delta(\mathbf{k} - \mathbf{p}). \tag{31}$$

Note that $\hat{a}_{\mathbf{k}}(t)$ and $\hat{b}_{\mathbf{k}}(t)$ are not independently invariants and become an invariant as the pair. The fields are given by

$$\begin{aligned}\hat{\phi}(t, \mathbf{x}) &= \sum_{\mathbf{k}}[\varphi_{\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}} + \varphi^*_{\mathbf{k}}(t)\hat{b}^\dagger_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}],\\ \hat{\pi}(t, \mathbf{x}) &= \sum_{\mathbf{k}}[\dot{\varphi}_{\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}} + \dot{\varphi}^*_{\mathbf{k}}(t)\hat{b}^\dagger_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}].\end{aligned} \tag{32}$$

Note that the antiparticle carries the opposite momentum of the particle.

Using eq. (29), we can construct coherent states, squeezed states as well as multi-pair states [26, 27, 28]. The quantum invariant theory has been used to derive the nonadiabatic quantum Vlasov equation [8, 29]. The time-dependent vacuum is the product of all zero-particle states

$$|0, t\rangle = \prod_{\mathbf{k}} |0_{\mathbf{k}}, t\rangle, \quad \hat{A}_{\mathbf{k}}(t)|0_{\mathbf{k}}, t\rangle = 0. \tag{33}$$

The fields (32) have the equal-time correlation functions

$$\begin{aligned}\langle 0, t|\hat{\phi}^\dagger(t, \mathbf{y})\hat{\phi}(t, \mathbf{x})|0, t\rangle &= \sum_{\mathbf{k}} |\varphi_{\mathbf{k}}(t)|^2 e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})},\\ \langle 0, t|\hat{\pi}^\dagger(t, \mathbf{y})\hat{\pi}(t, \mathbf{x})|0, t\rangle &= \sum_{\mathbf{k}} |\dot{\varphi}_{\mathbf{k}}(t)|^2 e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}.\end{aligned} \tag{34}$$

The correlation functions (34) may be used to find the energy-momentum tensor and even the effective action, which requires independent works. For the fixed $\mathbf{k}$, acting $\hat{A}^\dagger_{\mathbf{k}}(t)$ n-times on the vacuum excites the n-pair state [21]

$$|n_{\mathbf{k}}, t\rangle = \frac{(\hat{A}^\dagger_{\mathbf{k}}(t))^{n_{\mathbf{k}}}}{\sqrt{n_{\mathbf{k}}!}}|0, t\rangle = \frac{1}{\sqrt{2^{n_{\mathbf{k}}} n_{\mathbf{k}}!}} \sum_{l_{\mathbf{k}}=0}^{n_{\mathbf{k}}} \binom{n_{\mathbf{k}}}{l_{\mathbf{k}}} |n_{\mathbf{k}} - l_{\mathbf{k}}, \bar{l}_{\mathbf{k}}, t\rangle, \tag{35}$$

where the time-dependent particle-antiparticle states are defined as

$$|k_{\mathbf{k}}, \bar{l}_{\mathbf{p}}, t\rangle = \frac{(\hat{a}^\dagger_{\mathbf{k}}(t))^{k_{\mathbf{k}}}}{\sqrt{k_{\mathbf{k}}!}} \frac{(\hat{b}^\dagger_{\mathbf{p}}(t))^{l_{\mathbf{p}}}}{\sqrt{l_{\mathbf{p}}!}} |0_{\mathbf{k}}, \bar{0}_{\mathbf{p}}, t\rangle. \tag{36}$$

The multiparticle states are orthonormal to each other. The n-pair state is completely symmetric with respect to the particle and antiparticle numbers and is an entangled state. The number of particles or antiparticles is

$$\langle n_{\mathbf{k}}, t|\hat{a}^{\dagger}_{\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(t)|n_{\mathbf{k}}, t\rangle = \langle n_{\mathbf{k}}, t|\hat{b}^{\dagger}_{\mathbf{k}}(t)\hat{b}_{\mathbf{k}}(t)|n_{\mathbf{k}}, t\rangle = \frac{n_{\mathbf{k}}}{2}. \tag{37}$$

In fact, the n-pair state (35) contains the total number $n_{\mathbf{k}}$ of particles and antiparticles. The thermal state may be given by

$$\hat{\rho}_{\mathbf{k}}(t) = \frac{1}{Z_{\mathbf{k}}} e^{-2\frac{\omega_{\mathbf{k}\mathrm{i}}}{kT}\hat{A}^{\dagger}_{\mathbf{k}}(t)\hat{A}_{\mathbf{k}}(t)}. \tag{38}$$

Here, $Z_{\mathbf{k}} = \mathrm{Tr} e^{-2(\omega_{\mathbf{k}\mathrm{i}}/kT)\hat{A}^{\dagger}_{\mathbf{k}}(t)\hat{A}_{\mathbf{k}}(t)}$ is the partition function and $\omega_{\mathbf{k}\mathrm{i}}$ is the initial energy for particle and antiparticle. The factor of two makes the density operator practically equivalent to a product of $e^{-(\omega_{\mathbf{k}\mathrm{i}}/kT)\hat{a}^{\dagger}_{\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(t)}$ and $e^{-(\omega_{\mathbf{k}\mathrm{i}}/kT)\hat{b}^{\dagger}_{\mathbf{k}}(t)\hat{b}_{\mathbf{k}}(t)}$. The thermal density (38) is also a quantum invariant and may be used to calculate the effective action at T in section 2.

5. Spontaneous and Induced Pair Production

In the quantum invariant theory for time-dependent systems, using the time-dependent annihilation and creation operators (29), we can construct various exact quantum states from time-dependent number states to the squeezed number state [27] and even to the thermal squeezed-coherent state [28]. The time-dependent vacuum (33) is an exact state for eq. (27). In the limit of $\mathbf{A} = 0$ in the remote past ($t_0 = -\infty$), the vacuum (33) becomes the Minkowski vacuum. Further, the invariant operators (29) define the number operator with the given momentum $\mathbf{k}$

$$\hat{N}_{\mathbf{k}}(t) = \hat{A}^{\dagger}_{\mathbf{k}}(t)\hat{A}_{\mathbf{k}}(t). \tag{39}$$

Now, we may ask a question how many pairs of Minkowski particle and antiparticle are contained in the vacuum (33)

$$\mathcal{N}_{\mathbf{k}}(t) = \langle 0, t|\hat{A}^{\dagger}_{\mathbf{k}}(t_0)\hat{A}_{\mathbf{k}}(t_0)|0, t\rangle, \tag{40}$$

or how many pairs defined by the number operator (39) are contained in the Minwkowski vacuum

$$\mathcal{N}_{\mathbf{k}}(t) = \langle 0, t_0|\hat{A}^{\dagger}_{\mathbf{k}}(t)\hat{A}_{\mathbf{k}}(t)|0, t_0\rangle. \tag{41}$$

The rate of spontaneous production of pairs (40) and (41) by the electric field is given by the same formula [21]

$$\mathcal{N}_{\mathbf{k}}(t) = |\dot{\varphi}_{\mathbf{k}}(t)|^2|\varphi_{\mathbf{k}}(t_0)|^2 + |\varphi_{\mathbf{k}}(t)|^2|\dot{\varphi}_{\mathbf{k}}(t_0)|^2 - \frac{1}{2}. \tag{42}$$

It is symmetric with respect to the exchange of times, t and t_0. The production of pairs induced from pairs already present in the remote past is now given by

$$\mathcal{N}_{\mathbf{k}}(t) = \langle n_{\mathbf{k}}, t_0|\hat{A}^{\dagger}_{\mathbf{k}}(t)\hat{A}_{\mathbf{k}}(t)|n_{\mathbf{k}}, t_0\rangle = (|\dot{\varphi}_{\mathbf{k}}(t)|^2|\varphi_{\mathbf{k}}(t_0)|^2 + |\varphi_{\mathbf{k}}(t)|^2|\dot{\varphi}_{\mathbf{k}}(t_0)|^2)(n_{\mathbf{k}} + 1) - \frac{1}{2}. \tag{43}$$

We may use the Hamiltonian expectation value to count the number of quanta with the given energy. The Hamiltonian expressed in terms of particle and antiparticle operators

$$\begin{aligned}\hat{H}_{\mathbf{k}}(t) &= (|\dot{\varphi}_{\mathbf{k}}(t)|^2 + \omega^2_{\mathbf{k}}(t)|\varphi_{\mathbf{k}}(t)|^2)(\hat{a}^{\dagger}_{\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(t) + \hat{b}_{\mathbf{k}}(t)\hat{b}^{\dagger}_{\mathbf{k}}(t)) \\ &\quad + (\dot{\varphi}^2_{\mathbf{k}}(t) + \omega^2_{\mathbf{k}}(t)\varphi^2_{\mathbf{k}}(t))\hat{a}_{\mathbf{k}}(t)\hat{b}_{\mathbf{k}}(t) + (\dot{\varphi}^{*2}_{\mathbf{k}}(t) + \omega^2_{\mathbf{k}}(t)\varphi^{*2}_{\mathbf{k}}(t))\hat{a}^{\dagger}_{\mathbf{k}}(t)\hat{b}^{\dagger}_{\mathbf{k}}(t).\end{aligned} \tag{44}$$

has the expectation value at the equal time

$$\langle n_{\mathbf{k}}, t|\hat{H}_{\mathbf{k}}(t)|n_{\mathbf{k}}, t\rangle = (|\dot{\varphi}_{\mathbf{k}}(t)|^2 + \omega^2_{\mathbf{k}}(t)|\varphi_{\mathbf{k}}(t)|^2)(2n_{\mathbf{k}} + 1). \tag{45}$$

In the zero-field limit $\mathbf{E}(\pm\infty) = 0$, the asymptotic solutions

$$\varphi^{\rm in}_{\mathbf{k}}(t) = \frac{e^{-i\omega_{\mathbf{k}\rm i}t}}{\sqrt{2\omega_{\mathbf{k}\rm i}}}, \quad \varphi^{\rm out}_{\mathbf{k}}(t) = \frac{e^{-i\mathbf{k}\rm o t}}{\sqrt{2\omega_{\mathbf{k}\rm o}}}, \tag{46}$$

lead to the in-vacuum and the out-vacuum

$$\begin{aligned} \hat{H}_{\mathbf{k}\rm i}|n_{\mathbf{k}}, {\rm in}\rangle &= \omega_{\mathbf{k}\rm i}\Big(n_{\mathbf{k}} + \frac{1}{2}\Big)|n_{\mathbf{k}}, {\rm in}\rangle, \\ \hat{H}_{\mathbf{k}\rm o}|n_{\mathbf{k}}, {\rm out}\rangle &= \omega_{\mathbf{k}\rm o}\Big(n_{\mathbf{k}} + \frac{1}{2}\Big)|n_{\mathbf{k}}, {\rm out}\rangle. \end{aligned} \tag{47}$$

Now, we use eq. (42) to compare with the pair-production rate in the in-in formalism and the in-out formalism [29]. Substituting eq. (46) into eq. (42), we find the pair-production rates

$$\mathcal{N}^{\rm in}_{\mathbf{k}}(t) = \frac{|\dot{\varphi}_{\mathbf{k}}(t)|^2 + \omega^2_{\mathbf{k}\rm i}|\varphi_{\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}\rm i}} - \frac{1}{2}, \tag{48}$$

and

$$\mathcal{N}^{\rm out}_{\mathbf{k}}(t) = \frac{|\dot{\varphi}_{\mathbf{k}}(t)|^2 + \omega^2_{\mathbf{k}\rm o}|\varphi_{\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}\rm o}} - \frac{1}{2}. \tag{49}$$

On the other hand, the adiabatic solution

$$\varphi^{\rm ad}_{\mathbf{k}}(t) = \frac{e^{-i\int^t \omega_{\mathbf{k}}(t')dt'}}{\sqrt{2\omega_{\mathbf{k}}(t)}} \tag{50}$$

leads to another pair-production rate

$$\mathcal{N}^{\rm ad}_{\mathbf{k}}(t) = \frac{|\dot{\varphi}_{\mathbf{k}}(t)|^2 + \omega^2_{\mathbf{k}}(t)|\varphi_{\mathbf{k}}(t)|^2}{2\omega_{\mathbf{k}}(t)} - \frac{1}{2}. \tag{51}$$

The pair-production rate (48) is the same as eq. (22) in the in-in Vlasov equation and eq. (51) is the same as eq. (12) in the in-out Vlasov equation of ref. [29]. Note that eq. (51) may be obtained by

$$\mathcal{N}^{\rm ad}_{\mathbf{k}} = \frac{1}{2\omega_{\mathbf{k}}(t)}\langle 0_{\mathbf{k}}, t|\hat{H}_{\mathbf{k}}(t)|0_{\mathbf{k}}, t\rangle - \frac{1}{2}. \tag{52}$$

The pair production (52) is the mean energy divided by the instantaneous energy, which counts the number of quanta with the energy $\omega_{\mathbf{k}}$.

6. Pair Production Via Polons

Various approximation methods have been developed to calculate the pair-production rate. The contour integral method evolves the Hamiltonian in the complex plane of time or space when the gauge potential (3) has an analytical continuation. Then, the magnitude of the in-in scattering matrix approximately gives the pair-production rate [14]

$$\mathcal{N}_{\mathbf{k}} = \Big|\sum_J \exp\Big[-i\oint_{C^{(1)}_J} \omega_{\mathbf{k}}(\zeta)d\zeta\Big]\Big|, \tag{53}$$

where $\omega_{\mathbf{k}}(\zeta)$ is the frequency in the complex plane and $C_J^{(1)}$ runs over all the possible contours of winding number one around simple poles, finite and infinite. Hence the pair production is entirely determined by simple poles, which may be dubbed "polons."

We consider a constant electric field with the vector potential $A_{\parallel}(z) = -qE_0 z$ in the complex plane of time. Then, the frequency for the spinless charge

$$\omega_{\mathbf{k}}(\zeta) = \sqrt{m^2 + \mathbf{k}_{\perp}^2 + (k_{\parallel} + qE_0 z)^2} \tag{54}$$

does not have any finite simple pole but has a simple pole at the infinity. Under a conformal transformation $z = 1/\zeta$, the contour integral gives the Schwinger formula [13]

$$e^{iqE_0 \oint \frac{d\zeta}{\zeta^3} \sqrt{(1-z_0\zeta)(1-z_0^*\zeta)}} = e^{-\frac{\pi(m^2+\mathbf{k}_{\perp}^2)}{qE_0}}, \tag{55}$$

where $z_0 = (k_{\parallel} - i\sqrt{m^2 + \mathbf{k}_{\perp}^2})/(qE_0)$ and the Cauchy theorem is applied to the contour integral after the square root is expanded. Next, we consider the electric field and vector potential (17), which has the complex frequency in the complex plane

$$\omega_{\mathbf{k}}(z) = \sqrt{m^2 + \mathbf{k}_{\perp}^2 + (\bar{k}_{\parallel} + qE_0 e^{\frac{z}{t}})^2}. \tag{56}$$

Under another conformal transformation $\zeta = e^{z/\tau}$, the pair-production rate is given by

$$e^{-iqE_0 \oint \frac{d\zeta}{\zeta} \sqrt{(\zeta-z_0)(\zeta-z_0^*)}} = e^{-2\pi\tau(\bar{\omega}_0+\bar{k}_{\parallel})}, \tag{57}$$

where the contour integral of winding number one is taken around the simple poles at $z = 0$ and $z = \infty$. The contour integral of winding number two around the simple pole at $z = 0$ yields $e^{-4\pi\tau\bar{\omega}_0}$. These two contour integrals are exclusive and thus sum to the numerator of eq. (22). It is remarkable that the exact pair-production rate

$$\mathcal{N}_{\mathbf{k}} = (e^{-(2\pi\tau\bar{\omega}_0)(2)} + e^{-2\pi\tau(\bar{k}_{\parallel}+\bar{\omega}_0)}) \sum_{l=0}^{\infty} e^{-(2\pi\tau\bar{\omega}_0)(2l)} \tag{58}$$

may be obtained by summing the possible contour integrals of all winding numbers.

7. Conclusion

We have reviewed and elaborated the QED effective action in the in-out formalism and the quantum invariant theory for the Schrödinger equation for a charged spinless scalar in time-dependent electric fields. We have also applied the contour integral method to the pair-production rate. In the in-out formalism by Schwinger and DeWitt [4, 5], the one-loop effective action is expressed by the Bogoliubov coefficients, which relate the out-vacuum to the in-vacuum. Many configurations of electric fields have the Bogoliubov coefficients in terms of the gamma functions and the gamma-function regularization leads the QED action in the proper-time integral in the constant and the Sauter-type electric fields [18, 19].

In this paper we have extended the QED action to an exponentially increasing electric field. The QED action is complex, whose real part is the vacuum polarization and whose imaginary part is related to the pair-production rate. We have also applied the quantum invariant theory to the time-dependent Hamiltonian for the charged spinless field in time-dependent electric fields. The time-dependent annihilation and creation operators construct various quantum states such as number states of pairs and a thermal state. The connection of two representations of the Vlasov equation with the formulae for pair production is also clarified. The quantum invariant theory can be applied to the Hamiltonian in the momentum space for a localized pulse of lasers. Finally, we have explained the contour integral method to calculate the pair production in the constant electric field and further elaborated the method to the exponentially increasing electric field. These recent methods seem to be not only convenient but also powerful in explicitly obtaining the QED actions and pair-production rates in intense lasers, which wait for extensive investigation.

Acknowledgments

The author would like to thank Professor Nikolay Narozhny for the invitation to LPHYS'14 and Antonino Di Piazza and Christian Schubert for useful discussions. This work was supported in part by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2012R1A1B3002852).

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