

Quantum theory of measurements as quantum decision theory

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Abstract. Theory of quantum measurements is often classified as decision theory. An event in decision theory corresponds to the measurement of an observable. This analogy looks clear for operationally testable simple events. However, the situation is essentially more complicated in the case of composite events. The most difficult point is the relation between decisions under uncertainty and measurements under uncertainty. We suggest a unified language for describing the processes of quantum decision making and quantum measurements. The notion of quantum measurements under uncertainty is introduced. We show that the correct mathematical foundation for the theory of measurements under uncertainty, as well as for quantum decision theory dealing with uncertain events, requires the use of positive operator-valued measure that is a generalization of projection-valued measure. The latter is appropriate for operationally testable events, while the former is necessary for characterizing operationally uncertain events. In both decision making and quantum measurements, one has to distinguish composite non-entangled events from composite entangled events. Quantum probability can be essentially different from classical probability only for entangled events. The necessary condition for the appearance of an interference term in the quantum probability is the occurrence of entangled prospects and the existence of an entangled strategic state of a decision maker or of an entangled statistical state of a measuring device.

1. Introduction

Developing the theory of quantum measurements, von Neumann [1] mentioned that the process of quantum measurements is analogous to decision making. The formal analogy between these processes has been described in several mathematical works [2–5]. However, this analogy remained rather formal, without comparing quantum measurements with real decision making, as done by humans. Several important questions have not been answered:

- (i) First of all, in what sense human decision making could be characterized by quantum measurements?
- (ii) What would be a general scheme for describing measurements under uncertainty and decisions under uncertainty?
- (iii) How to correctly define a quantum probability of non-commuting events for both, quantum measurements and quantum decisions?
- (iv) What is the role of entanglement in quantum measurements, and in quantum decision making?



(v) When should decision making be treated by quantum rules and when is it sufficient to use classical theory?

In this report, we present a general approach and mathematical techniques common to quantum measurements and decision making that provides natural answers to these questions. The reason for developing a common quantum approach to measurements and to decision making is twofold. First, quantum theory provides tools for taking into account behavioral biases in human decision making [6–8]. Note that the possibility of describing cognition effects by means of quantum theory was suggested by Bohr [9, 10]. Second, formulating a quantum theory of decision making defines the main directions for creating artificial quantum intelligence [11, 12].

2. Operationally testable events

Let us first briefly recall the definition of quantum probabilities for measurements corresponding to operationally testable events and their connection to simple events in decision making [6–8].

The operator of an observable \hat{A} is a self-adjoint operator, whose eigenvectors are given by the eigenproblem

$$\hat{A}|n\rangle = A_n|n\rangle, \quad (1)$$

forming a complete basis. The closed linear envelope of this basis composes a Hilbert space $\mathcal{H}_A \equiv \text{span}\{|n\rangle\}$. The measurement of an eigenvalue A_n is an event that we denote by the same letter. This event is represented by a projector \hat{P}_n according to the correspondence

$$A_n \rightarrow \hat{P}_n \equiv |n\rangle\langle n|. \quad (2)$$

The system statistical state, or the decision maker strategic state, is given by a statistical operator $\hat{\rho}_A$. The probability of an event A_n is

$$p(A_n) \equiv \text{Tr}_A \hat{\rho}_A \hat{P}_n \equiv \langle \hat{P}_n \rangle. \quad (3)$$

In the theory of measurements or decision theory, the set $\mathcal{A} \equiv \{\hat{P}_n\}$ of projectors plays the role of the algebra of observables, with the expected value (3) being the event probability.

In eigenproblem (1), a nondegenerate spectrum is tacitly assumed. If the spectrum is degenerate, then in the eigenproblem

$$\hat{A}|n_j\rangle = A_n|n_j\rangle \quad (4)$$

an eigenvalue A_n corresponds to several eigenvectors $|n_j\rangle$, where $j = 1, 2, \dots$. The Hilbert space can again be composed spanning all these eigenvectors, $\mathcal{H}_A \equiv \{|n_j\rangle\}$. Now an event of measuring A_n is associated with the projector

$$\hat{P}_n \equiv \sum_j \hat{P}_{n_j} \quad (\hat{P}_{n_j} \equiv |n_j\rangle\langle n_j|). \quad (5)$$

And the event probability is defined as in Eq. (3).

A basis, formed by the eigenvectors of a self-adjoint operator, is complete, because a self-adjoint operator is normal. The set of eigenvectors of any normal operator, such that $\hat{A}^+ \hat{A} = \hat{A} \hat{A}^+$, forms a complete basis. Normal operators include self-adjoint and unitary operators. In the case of degenerate spectra, the basis may be not uniquely defined. In quantum measurements, this does not lead to any principal problem, since one can use the summary projector (5). But in decision theory, the question arises: in what sense a single event A_n can correspond to several projectors \hat{P}_{n_j} , since an operationally testable event either happens or not?

Fortunately, this problem of nonuniqueness is easily avoidable, both in the theory of quantum measurements and in decision theory. This is done by invoking the von Neumann suggestion of degeneracy lifting [1]. For this purpose, the operator of the observable \hat{A} is slightly shifted by an infinitesimal term,

$$\hat{A} \rightarrow \hat{A} + \nu \hat{\Gamma} \quad (\nu \rightarrow 0) , \quad (6)$$

where $\hat{\Gamma}$ is an operator lifting the symmetry responsible for the spectrum degeneracy. Then the operator spectrum splits into the set

$$A_n \rightarrow A_n + \nu \Gamma_{n_j} , \quad (7)$$

thus, removing the degeneracy. Finally, the event probability is defined as

$$p(A_n) = \lim_{\nu \rightarrow 0} p(A_n + \nu \Gamma_{n_j}) . \quad (8)$$

In that way, the degeneracy is avoided, and one always deals with a unique correspondence between eigenvalues, representing events, and eigenvectors. This procedure is also analogous to the Bogolubov method of quasi-averages, breaking the symmetry of statistical systems by introducing infinitesimal sources [13, 14].

The union of mutually orthogonal (incompatible) events is represented by the projector sum:

$$\bigcup_n A_n \rightarrow \sum_n \hat{P}_n \quad (\hat{P}_m \hat{P}_n = \delta_{mn} \hat{P}_n) . \quad (9)$$

The probability of such a union is additive:

$$p\left(\bigcup_n A_n\right) = \sum_n p(A_n) . \quad (10)$$

In the definition of the quantum probability (3), the averaging is done with a statistical operator $\hat{\rho}_A$, generally, implying a mixed state. In some special cases, a quantum system can be prepared in a pure state described by a wave function $|\psi\rangle$, which corresponds to the statistical operator $|\psi\rangle\langle\psi|$. The setups of experiments with physical systems and with decision makers are quite different. Accomplishing quantum measurements, we may meet two types of systems, open and quasi-isolated.

An *open* system interacts with its surrounding, keeps information on the preparation of its initial state and, generally, can feel past interactions through retardation memory effects. Even if, at the given time, the interactions with its surrounding can be neglected, the system remains open, if it possesses the memory of its preparation and of past interactions. In addition, there can exist quantum statistical correlations making the system entangled with its surrounding.

A physical system is *quasi-isolated* when it is isolated from its surrounding interactions, does not have retardation memory effects, and is not entangled with surrounding through quantum statistical correlations. Such a system can be conditionally treated as isolated for a short instance of time. However, to confirm its isolation, one needs to realize measurements, at least nondestructive, which perturbs the system making it not absolutely isolated, but quasi-isolated [15–17].

Contrary to physical systems, a decision maker, even being isolated from a surrounding society, always keeps memory and information of many previous interactions. Therefore a decision maker has always to be considered as an open system.

3. Composite separable events

When one deals with not just a single event, but with several events, the situation becomes more involved. This especially concerns the measurements of noncommuting observables, whose operators do not commute with each other [18]. There has been an old problem of correctly defining the quantum probability of such events. It has been shown [19] that the Lüders probability [20] of consecutive measurements is a transition probability between two quantum states and cannot be accepted as a quantum extension of the classical conditional probability. Also, the Wigner distribution [21] is just a weighted Lüders probability and it cannot be treated as a quantum extension of the classical joint probability. The correct and most general definition of a quantum joint probability can be done [19] by employing the Choi-Jamiołkowski isomorphism [22, 23] expressing the channel-state duality.

Composite events, that are composed of two or more events, can be separable or entangled. First, we consider separable events.

Let us be interested in the measurements involving two observables associated with two operators, \hat{A} and \hat{B} , with the related eigenproblems

$$\hat{A}|n\rangle = A_n|n\rangle, \quad \hat{B}|\alpha\rangle = B_\alpha|\alpha\rangle. \quad (11)$$

As above, we shall denote the events of measuring the eigenvalues A_n and B_α by the same letters, respectively. To each event, we put into correspondence the appropriate projectors,

$$A_n \rightarrow \hat{P}_n \equiv |n\rangle\langle n|, \quad B_\alpha \rightarrow \hat{P}_\alpha \equiv |\alpha\rangle\langle\alpha|. \quad (12)$$

Constructing two Hilbert space copies

$$\mathcal{H}_A \equiv \text{span}\{|n\rangle\}, \quad \mathcal{H}_B \equiv \text{span}\{|\alpha\rangle\}, \quad (13)$$

we define the algebras of observables

$$\mathcal{A} \equiv \{\hat{P}_n\}, \quad \mathcal{B} \equiv \{\hat{P}_\alpha\}, \quad (14)$$

acting on the corresponding Hilbert spaces. The composite algebra $\mathcal{A} \otimes \mathcal{B}$ acts on the tensor-product space $\mathcal{H}_A \otimes \mathcal{H}_B$. The system statistical state (decision-maker strategic state) also acts on the space $\mathcal{H}_A \otimes \mathcal{H}_B$. The joint probability of events, corresponding to the observables from the algebra $\mathcal{A} \otimes \mathcal{B}$, is defined as

$$p(\mathcal{A} \otimes \mathcal{B}) \equiv \text{Tr}_{AB} \hat{\rho} \mathcal{A} \otimes \mathcal{B}, \quad (15)$$

with the trace over $\mathcal{H}_A \otimes \mathcal{H}_B$.

For any two operators from the algebra \mathcal{A} , it is possible to introduce the Hilbert-Schmidt scalar product that is a map

$$\sigma_A : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C}. \quad (16)$$

Thus, for the operators \hat{A}_1 and \hat{A}_2 from the algebra \mathcal{A} , the scalar product reads as

$$(\hat{A}_1, \hat{A}_2) \equiv \text{Tr}_A \hat{A}_1^\dagger \hat{A}_2, \quad (17)$$

with the trace over \mathcal{H}_A . The operators from the algebra \mathcal{A} , acting on the Hilbert space \mathcal{H}_A , and complemented with the scalar product σ_A , form the Hilbert-Schmidt operator space

$$\tilde{\mathcal{A}} \equiv \{\mathcal{A}, \mathcal{H}_A, \sigma_A\}. \quad (18)$$

Similarly, one defines the Hilbert-Schmidt space

$$\tilde{\mathcal{B}} \equiv \{\mathcal{B}, \mathcal{H}_B, \sigma_B\} . \quad (19)$$

The tensor-product of the above Hilbert-Schmidt spaces forms the space

$$\tilde{\mathcal{C}} \equiv \tilde{\mathcal{A}} \otimes \tilde{\mathcal{B}} . \quad (20)$$

An operator \hat{C} acting on $\tilde{\mathcal{C}}$ is called *separable* if and only if it can be represented as a sum

$$\hat{C} = \sum_i \hat{A}_i \otimes \hat{B}_i \quad (\hat{A}_i \in \tilde{\mathcal{A}}, \hat{B}_i \in \tilde{\mathcal{B}}) . \quad (21)$$

A composite event is named a prospect. The prospect $A_n \otimes B_\alpha$ corresponds to the prospect operator

$$\hat{P}(A_n \otimes B_\alpha) = \hat{P}_n \otimes \hat{P}_\alpha . \quad (22)$$

The prospect operator (22) is evidently separable. Therefore the corresponding prospect $A_n \otimes B_\alpha$ is also called separable. The related prospect probability writes as

$$p(A_n \otimes B_\alpha) = \text{Tr}_{AB} \hat{\rho} \hat{P}_n \otimes \hat{P}_\alpha , \quad (23)$$

with the trace over $\mathcal{H}_A \otimes \mathcal{H}_B$.

More generally, the prospect $A_n \otimes \bigcup_\alpha B_\alpha$, where $\bigcup_\alpha B_\alpha$ is a union of mutually orthogonal events, corresponds to the prospect operator

$$\hat{P}\left(A_n \otimes \bigcup_\alpha B_\alpha\right) = \sum_\alpha \hat{P}_n \otimes \hat{P}_\alpha . \quad (24)$$

This operator is separable, and the related prospect probability

$$p\left(A_n \otimes \bigcup_\alpha B_\alpha\right) = \sum_\alpha p(A_n \otimes B_\alpha) \quad (25)$$

is additive with respect to the events $A_n \otimes B_\alpha$.

4. Composite entangled events

An operator \hat{C} from the Hilbert-Schmidt space (20) is termed *entangled*, or non-separable, if it cannot be represented as sum (21), so that

$$\hat{C} \neq \sum_i \hat{A}_i \otimes \hat{B}_i \quad (\hat{A}_i \in \tilde{\mathcal{A}}, \hat{B}_i \in \tilde{\mathcal{B}}) . \quad (26)$$

The appearance of entangled events, corresponding to entangled operators, is connected with the existence of not operationally testable measurements that also are called uncertain measurements, or incomplete measurements, or indecisive measurements, or inconclusive measurements. Respectively, one can keep in mind uncertain events in decision making.

Let us define an uncertain event B as a set of possible events B_α , characterized by weights $|b_\alpha|^2$,

$$B = \{B_\alpha : \alpha = 1, 2, \dots\} . \quad (27)$$

Since the uncertain event is not operationally testable, it is not required that the weights $|b_\alpha|^2$ be summed to one. The uncertain-event operator is

$$\hat{P}(B) = |B\rangle\langle B| , \quad (28)$$

with the state

$$|B\rangle = \sum_{\alpha} b_{\alpha} |\alpha\rangle \quad (29)$$

that is not necessarily normalized to one. The uncertain-event operator (28), which can be written as

$$\hat{P}(B) = \sum_{\alpha\beta} b_{\alpha} b_{\beta}^* |\alpha\rangle\langle\beta| , \quad (30)$$

is not a projector onto a subspace that would correspond to a degenerate spectrum, because it does not have form (5). Moreover, it is not a projector at all, as far as it is not necessarily idempotent,

$$\hat{P}^2(B) = \langle B|B\rangle \hat{P}(B) \neq \hat{P}(B) ,$$

since state (29) is not, generally, normalized to one.

A composite event, formed of an operationally testable event A_n and an uncertain event (27), is the uncertain prospect

$$\pi_n = A_n \otimes B , \quad (31)$$

whose prospect operator is

$$\hat{P}(\pi_n) \equiv |\pi_n\rangle\langle\pi_n| = \sum_{\alpha} |b_{\alpha}|^2 \hat{P}_n \otimes \hat{P}_{\alpha} + \sum_{\alpha \neq \beta} b_{\alpha} b_{\beta}^* \hat{P}_n \otimes |\alpha\rangle\langle\beta| . \quad (32)$$

This operator is not separable in the Hilbert-Schmidt space (20), because the operator $|\alpha\rangle\langle\beta|$ does not pertain to space (19) composed of projectors \hat{P}_{α} . Hence, operator (32) is entangled. The corresponding prospect (31) is termed entangled.

The prospect operators are assumed to satisfy the resolution of unity

$$\sum_n \hat{P}(\pi_n) = \hat{1} , \quad (33)$$

where $\hat{1}$ is the identity operator in the space $\mathcal{H}_A \otimes \mathcal{H}_B$. But these prospect operators are not necessarily orthogonal, since

$$\hat{P}(\pi_m) \hat{P}(\pi_n) = \langle \pi_m | \pi_n \rangle |\pi_m\rangle\langle\pi_n| ,$$

and they are not idempotent, as far as

$$\hat{P}^2(\pi_n) = \langle \pi_n | \pi_n \rangle \hat{P}_n .$$

That is, they are not projectors. The family $\{\hat{P}(\pi_n)\}$ of such positive operators, obeying the resolution of unity (33) is named *positive operator-valued measure* [2, 4, 5].

For a given lattice $\{\pi_n\}$ of prospects, the prospect probabilities

$$p(\pi_n) = \text{Tr} \hat{P}(\pi_n) , \quad (34)$$

where the trace is over $\mathcal{H}_A \otimes \mathcal{H}_B$, satisfy the conditions

$$\sum_n p(\pi_n) = 1 , \quad 0 \leq p(\pi_n) \leq 1 , \quad (35)$$

making the set $\{p(\pi_n)\}$ a probability measure.

The analysis of the prospect probability (34) results in the following properties [24–26]. The probability can be written in the form

$$p(\pi_n) = f(\pi_n) + q(\pi_n) . \quad (36)$$

The first term,

$$f(\pi_n) = \sum_{\alpha} |b_{\alpha}|^2 p(A_n \otimes B_{\alpha}) , \quad (37)$$

corresponds to classical probability, possessing the features

$$\sum_n f(\pi_n) = 1 , \quad 0 \leq f(\pi_n) \leq 1 . \quad (38)$$

The classical term (37) is an objective quantity reflecting the given properties of the prospect. The second term,

$$q(\pi_n) = \sum_{\alpha \neq \beta} b_{\alpha} b_{\beta}^* \langle n\alpha | \hat{\rho} | n\beta \rangle , \quad (39)$$

is purely quantum, caused by interference and coherence effects. This quantum term in the theory of measurements is called interference factor, or coherence factor, and in decision theory, attraction factor, since it characterizes the subjective attractiveness of different prospects to the decision maker.

According to the *quantum-classical correspondence principle* [27], when quantum effects disappear, quantum theory should reduce to classical theory, which implies

$$p(\pi_n) \rightarrow f(\pi_n) , \quad q(\pi_n) \rightarrow 0 . \quad (40)$$

In general, the reduction of quantum measurements to their classical counterparts is called decoherence [28].

The quantum factor (39) varies in the range

$$-1 \leq q(\pi_n) \leq 1 \quad (41)$$

and satisfies the *alternation law*

$$\sum_n q(\pi_n) = 0 . \quad (42)$$

For a large number of considered prospects N , we get the *quarter law*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |q(\pi_n)| = \frac{1}{4} . \quad (43)$$

A known example of the arising interference under measurements is provided by the double-slit experiment [1]. The passage of a particle through one of two slits corresponds to prospect (31). The operationally testable event A_n is the registration of the considered particle by an n -th detector, while the passage through one of the two slits, either B_1 or B_2 is described by the uncertain event $B = \{B_1, B_2\}$.

The quantum term is not always nontrivial [29]. To be nonzero, it requires the validity of two necessary conditions. The first condition is that the considered prospect be entangled, as described in Sec. 4. The second necessary requirement is the entanglement of the statistical operator. The latter is entangled when, e.g.,

$$\hat{\rho} \neq \hat{\rho}_A \otimes \hat{\rho}_B , \quad (44)$$

where

$$\hat{\rho}_A \equiv \text{Tr}_B \hat{\rho}, \quad \hat{\rho}_B \equiv \text{Tr}_A \hat{\rho},$$

that is, when the statistical state of a composite system is not a product of its partial subsystem states. More generally, the system state is entangled if it cannot be represented in the form

$$\hat{\rho} \neq \sum_{\alpha} \lambda_{\alpha} \hat{\rho}_{\alpha A} \otimes \hat{\rho}_{\alpha B}, \quad (45)$$

where

$$\sum_{\alpha} \lambda_{\alpha} = 1, \quad 0 \leq \lambda_{\alpha} \leq 1.$$

Condition (45) is necessary, but not sufficient for the occurrence of a nonzero term (39).

In quantum theory, entanglement is a well known notion. In decision theory, it corresponds to the correlations between different possible events that are perceived by the decision maker.

5. Conclusion

We have shown that quantum measurements and quantum decision making can be described by the same mathematical tools. In both these cases, the problem of degeneracy can be avoided by employing the von Neumann method of degeneracy lifting, which is analogous to the Bogolubov quasi-averaging procedure. The correct definition of quantum probabilities of composite events, called prospects, is done by using the Choi-Jamiolkowsky isomorphism. This allows us to describe any type of composite events, including those corresponding to noncommutative observables.

The notion of uncertain events and measurements makes it feasible to give a general scheme for describing measurements under uncertainty and decisions under uncertainty. This is done by means of positive operator-valued measures. Prospects are classified into two principally different types, separable and entangled, depending on the structure of the related prospect operators in the Hilbert-Schmidt space.

The appearance of a quantum interference term in the quantum probability of composite events is shown to require two necessary conditions, prospect entanglement and statistical state entanglement.

Classical measurements and classical decision making are particular cases of the corresponding quantum counterparts. The reduction of quantum measurements and decision making to the classical limit occurs when the interference term becomes zero.

The investigation of the analogies between quantum measurements and quantum decision making suggests the ways of creating artificial quantum intelligence [11, 12] and gives keys for better understanding of quantum effects in self-organization of complex systems [30].

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