Fraunhofer type diffraction of phase-modulated broad-band femtosecond pulses

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Abstract. Attosecond optical pulses with one-two cycles under the envelope diffract in non-paraxial regime on several diffraction lengths. Their intensity profile takes a form similar to Fraunhofer distribution. An analytical theory was developed, where was pointed, that such type of diffraction depends on the spectral width of the optical pulse. In this paper is shown that even for a broad-band phase-modulated femtosecond pulses the diffraction is also of Fraunhofer type.

1. Introduction

In recent years the linear regime of propagation of optical pulses with broad-band spectrum in air and transparent media attracts a considerable attention. The attosecond pulses in UV region as well as femtosecond pulses with time duration *5-15 fs* in optical and IR region admit from one to several optical cycles under their envelopes. For such pulses, as it was pointed in [1], the diffraction is not paraxial and on few diffraction lengths their shape takes parabolic form. In the theory developed in [2,3] it is shown that the non-paraxiality depends on the spectral width $\Delta k_z = \Delta \omega / v_{gr}$ of the pulse. Thus, one additional possibility appears: Is it possible to use phase-modulated broad-band femtosecond pulses with many cycles under their envelope to obtain Fraunhofer type diffraction?

In this paper we will numerically investigate Fraunhofer type diffraction of phase-modulated *25 fs* laser pulses which admit at least 10 cycles at level e^{-1} of the maximum.

This study gives us knowledge about the limits and applicability of paraxiality in the frames of femtosecond optics.

2. Basic theory

The linear scalar non-paraxial amplitude equation, governing the evolution of laser pulses in isotropic dispersive media, is written in Local type coordinates [4]:

$$-2ik_0 \frac{\partial A}{\partial z} = \Delta A_\perp + \frac{\partial^2 A}{\partial z^2} - \frac{2}{v} \frac{\partial^2 A}{\partial \tau \partial z} - \frac{\beta}{v^2} \frac{\partial^2 A}{\partial \tau^2} \quad , \tag{1}$$

where $A(\tau, z)$ is the slowly varying amplitude function of the pulse envelope, v is the group velocity, $\beta = k_0 v_{gr}^2 |k''|$ is a parameter characterizing the second order of linear dispersion of the medium, $z = z$ and $\tau = t$-z/v.

This equation can be solved by using the Fourier method. The Fourier transform is applied to the amplitude function $FFF[A(x,y,\tau,z)] = \hat{A}\left(k_x, k_y, \dfrac{\Delta\omega}{v}, z\right)$. Here $\Delta\omega = \omega - \omega_0$ and $\Delta k_z = \Delta\omega / v$ are respectively the spectral width and the wave number of the optical pulse.

After a couple of calculations the fundamental solution of equation (1) in $\left(k_x, k_y, \dfrac{\Delta\omega}{v}, z\right)$ space can be written as follow:

$$A\left(k_x, k_y, \frac{\Delta\omega}{v}, z\right) = \hat{A}(k_x, k_y, \Delta\omega, 0)\exp\left\{i[(k_0 - \frac{\Delta\omega}{v}) \pm \sqrt{(k_0 - \frac{\Delta\omega}{v})^2 + k_x^2 + k_y^2 - \frac{\beta\Delta\omega^2}{v^2}}]z\right\} \qquad (2)$$

The solution of equation (1) in *(x,y,z,t)* dimensions is simply the inverse Fourier transform of solution (2):

$$A(x, y, \tau, z) = FFF^{-1}\left\langle \hat{A}(k_x, k_y, \Delta\omega, 0)\exp\left\{i[(k_0 - \frac{\Delta\omega}{v}) \pm \sqrt{(k_0 - \frac{\Delta\omega}{v})^2 + k_x^2 + k_y^2 - \frac{\beta\Delta\omega^2}{v^2}}]z\right\}\right\rangle \quad (3)$$

Here we will investigate more precisely the case of narrow laser pulses when the condition $\Delta k_z << k_0$ is satisfied. Thus, we can assume that $k_0 - \dfrac{\Delta\omega}{v} \approx k_0$. The solution (3), in this approximation, takes the form:

$$A(x, y, \tau, z) = FFF^{-1}\left\langle \hat{A}(k_x, k_y, \Delta\omega, 0)\exp\left\{i\left[k_0 \pm k_0\sqrt{1 + \frac{k_x^2 + k_y^2 - \frac{\beta\Delta\omega^2}{v^2}}{k_0^2}}\right]z\right\}\right\rangle \qquad (4)$$

On the other hand, the expression under the square root can be presented in Taylor series, because for such type of laser pulses the condition $k_0^2 >> k_x^2 + k_y^2 - \dfrac{\beta\Delta\omega^2}{v^2}$ is satisfied. If we use the minus in front of the square root, the expression (4) takes the form:

$$A(x, y, \tau, z) = FFF^{-1}\left\langle \hat{A}(k_x, k_y, \Delta\omega, z) = \hat{A}(k_x, k_y, \Delta\omega, 0)\exp\left\{-i\left[\frac{k_x^2 + k_y^2 - \frac{\beta\Delta\omega^2}{v^2}}{2k_0^2}\right]z\right\}\right\rangle \qquad (5)$$

This is the typical spectral kernel of the spatio-temporal equation of the kind:

$$-2ik_0 \frac{\partial A}{\partial z} = \Delta A_\perp - \frac{\beta}{v^2} \frac{\partial^2 A}{\partial \tau^2} \tag{6}$$

Thus, we proved that the linear evolution of narrow-band light pulses can be described correctly by solving the paraxial spatio-temporal equation (6).

In the context of broad-band optical pulses the condition $\Delta k_z = \frac{\Delta \omega}{v} \approx k_0$ does not allow to present the spectral kernel (2) in Taylor series.

The analyses presented above lead to the next important conclusion: The non-paraxial equation (1) describes correctly the dynamics of narrow-band as well as broad-band optical pulses.

That is why the first numerical experiment with broad-band attosecond laser pulses predicts results quite different from the paraxial diffractive optics [1]. The results obtained in their paper demonstrated semi-spherical Fraunhofer type diffraction at distances of few microns. The initial laser pulse has the form of optical disc, as a superposition of plane waves. In the range of a couple of microns, before to reach the surface where are generated free electrons, the profile is changed into a semi-spherical shape, which is more close to Fraunhofer diffraction.

The problem of non-paraxial diffraction has been discussed in [2] where an analytical solution of the linear amplitude equation is obtained. The normalized non-paraxial linear amplitude equation (1) is derived from the following non-normalized equation written in Laboratory coordinates:

$$-2ik_0 \left(\frac{\partial A}{\partial z} + \frac{1}{v} \frac{\partial A}{\partial t} \right) = \Delta A - \frac{1}{v^2} \frac{\partial^2 A}{\partial t^2} \tag{7}$$

In this equation the term connected with dispersion β is neglected because in air $\beta \ll 1$ and the effects of dispersion on the evolution of laser pulses are very weak.

The method used to solve equation (7) is similar to that of equation (1). The initial condition for Gaussian pulse is: $A_x(x,y,z,t=0) = \exp\left(-\frac{x^2+y^2+z^2}{2r_0^2} \right)$. Using the Fourier method the initial condition problem can be solved and the following exact solution is obtained:

$$A_x(x,y,z,t) = \frac{i}{2\hat{r}} \exp\left(-\frac{k_0^2 r_0^2}{2} + ik_0(vt-z) \right) \times$$
$$\times \left\{ i(vt+\hat{r}) \exp\left(-\frac{1}{2r_0^2}(vt+\hat{r})^2 \right) erfc\left(\frac{i}{\sqrt{2}r_0}(vt+\hat{r}) \right) - i(vt-\hat{r}) \exp\left(-\frac{1}{2r_0^2}(vt-\hat{r})^2 \right) erfc\left(\frac{i}{\sqrt{2}r_0}(vt-\hat{r}) \right) \right\} \tag{8}$$

where $\hat{r} = \sqrt{x^2 + y^2 + (z - ir_0^2 k_0)^2}$.

This analytical result describes the diffraction of narrow-band pulses as well as that of broad-band laser pulses. For example, when $k_0 \gg r_0$ (narrow-band case) we have Fresnel diffraction on several diffraction lengths, while for $k_0 \geq r_0$ (broad-band case) the solution gives Fraunhofer type diffraction at these distances.

The analytical solution (8) as well as the numerical simulations in [1] are identical and describe very well the semi-spherical diffraction of optical pulses with few cycles under their envelope. On the other hand, the authors in [5] obtained U-type diffraction. It is important to mention that this type of

diffraction is described by solving the non-paraxial equation (UPPE equation) [5] in Local time coordinates.

One of the purposes of this work is to show that this *U*-type diffraction in Local time is simply the well-known Fraunhofer diffraction and it is due to the specific nature of Local time transformation.

To find the particularities of the diffraction of narrow and broad-band optical pulses we will investigate the evolution of initial Gaussian pulse in the frames of different types of equations.

  a) The non-paraxial equation in Galilean frame:

$$-i\frac{2k_0}{v}\frac{\partial A}{\partial t} = \Delta A_\perp - \frac{1+\beta}{v^2}\left(\frac{\partial^2 A}{\partial t^2} - 2v\frac{\partial^2 A}{\partial t \partial z}\right) - \beta\frac{\partial^2 A}{\partial z^2} \qquad (9)$$

  b) The same type of equation written in Local time frames (1).
  c) Paraxial spatio-temporal equation (6).

It is important to be pointed here that similar analytical solution as (8) is obtained for equation (9). As it can be expected, the diffraction is the same for both equations. There is only one difference: the translation of the pulse in Laboratory coordinate system and the stationarity of the pulse in Galilean.

3. Numerical results

To find differences in the diffraction picture we will solve numerically the spatio-temporal paraxial equation (6), the non-paraxial equation (1) written in Local time coordinate system and equation (9) written in Galilean frame for broad-band laser pulses with more than 10 optical cycles under the envelope.

For this purpose we studied the evolution of phase-modulated 25 *fs* laser pulses with a Gaussian spatio-temporal profile in air. The initial condition for equations (1) and (6) is:

$$A(x, y, \tau, z = 0) = A_0 \exp\left(-\frac{x^2 + y^2}{2r_0} - b\frac{\tau^2}{2\tau_0^2}\right) \quad, \qquad (10)$$

where r_0 is the transverse size (the spot of the pulse), τ_0 is the temporal size and $b=1+ia$ is a complex number where a is the linear frequency modulation (chirp of the pulse).

The initial condition for equation (9) is:

$$A(x, y, z, t = 0) = A_0 \exp\left(-\frac{x^2 + y^2}{2r_0} - b\frac{z^2}{2z_0^2}\right) \quad, \qquad (11)$$

where again r_0 is the transverse size (the spot of the pulse), $z_0 = v_{gv}\tau_0$ is the spatio-longitudinal size and a is the same linear frequency chirp.

The connection between parameter a, the real time duration of the pulse t_{FWHM} and its spectral broadening $\Delta\omega_{FWHM}$ can be obtained by using the equation [6]:

$$t_{FWHM}\Delta\omega_{FWHM} = 2\pi c_B\sqrt{1+a^2} > 2\pi c_B \qquad (12)$$

The parameter c_B is a constant. In our work the linear frequency modulation changes from 0 to 10.

In the beginning we will investigate the evolution of *25 fs* spectrally-limited optical pulse (chirp parameter *a=0*) governed in the frames of paraxial equation (6) under the initial condition (10). The

numerical results are shown in Figure 1. Here, as well as on the next fiqures, the side projection of the intensity profile is presented trought a distances of one diffractional length.

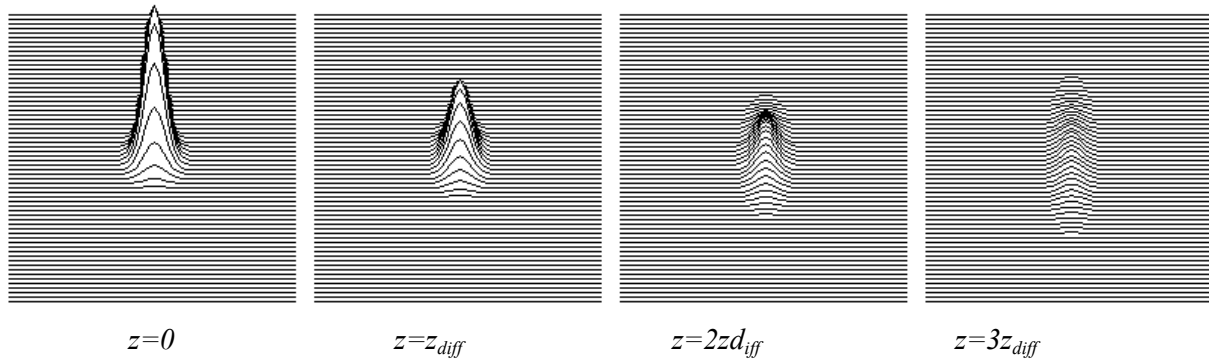$$z=0 \qquad z=z_{diff} \qquad z=2zd_{iff} \qquad z=3z_{diff}$$

Figure 1. Plot of the side *(x-τ)* projection of the intensity profile of *25 fs* spectrally-limited optical pulse. The numerical result is obtained by solving the paraxial Eq. (6) and initial condition (10). The parameter *a=0*. The typical Fresnel diffraction is clearly seen. At one diffraction length the transverse size of the intensity enlarges tice and the initial amplitude decreases by factor $\sqrt{2}$.

The form of the pulse corresponds to the typical diffractive profile on a several diffractive lengths in the context of paraxial optics.
In Figure 2 are presented the results obtained by using the initial condition (10) and the equation (6). The chirp parameter is *a=10*.
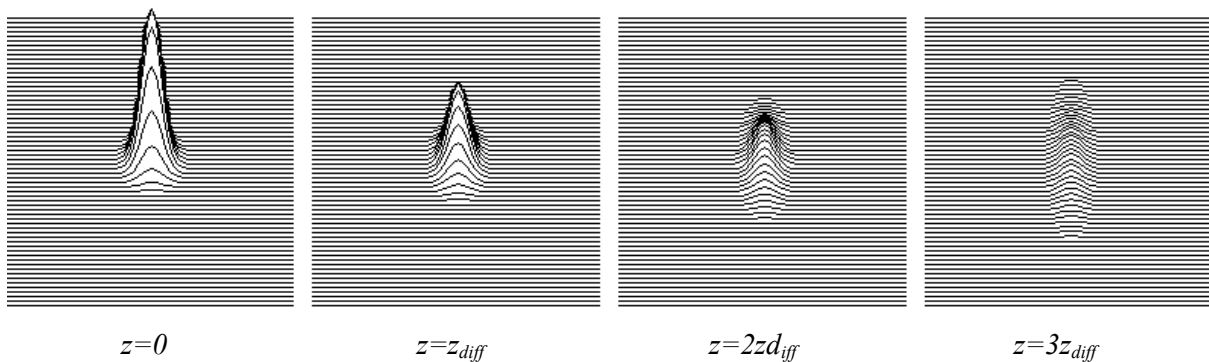

$$z=0 \qquad z=z_{diff} \qquad z=2zd_{iff} \qquad z=3z_{diff}$$

Figure 2. Plot of the side *(x-τ)* projection of the intensity profile of *25 fs* phase-modulated optical pulse. The results are obtained by solving the paraxial Eq. (6) and the same initial condition (10). The pulse is chirped: *a=10*. The phase-modulated pulse governed by the paraxial Eq.(6) again gives Fresnel diffraction. The result correspond to the simple fact that in the frames of paraxial optics the chirp and the spectral band of the pulse do not influence on the diffraction.

The form of the pulse corresponds to the typical diffractive profile on a several diffractive lengths in the context of paraxial optics. The figure shows that in the frames of paraxial optics the chirp and the spectral band of the pulse do not influence on the diffraction.
The same result is obtained by using the non-paraxial spatio-temporal equation (1) for unchirped pulse under the same initial condition (10).
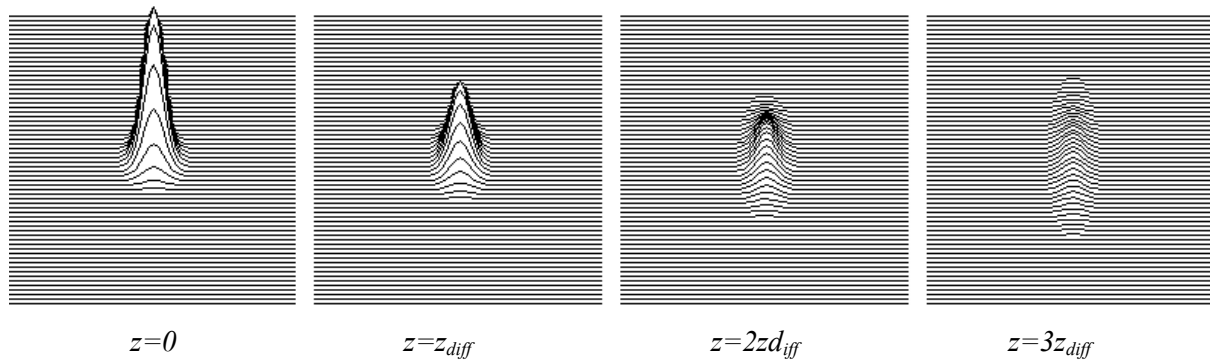
$z=0$ $z=z_{diff}$ $z=2zd_{iff}$ $z=3z_{diff}$

Figure 3. Plot of the side *(x-τ)* projection of the intensity profile of *25 fs* spectrally-limited optical pulse with parameter *a=0*. This result is obtained by solving the spatio-temporal non-paraxial Eq. (1) under the same initial condition (10). In its nature the non-modulated *25 fs* pulse is narrow-band, we see again Fresnel diffraction.

The form of the pulse is not different from the typical diffractive profile obtained in the frames of paraxial optics.
In Figure 4 are presented the results obtained under the same initial condition (10), equation (1) and chirp parameter *a=10*.
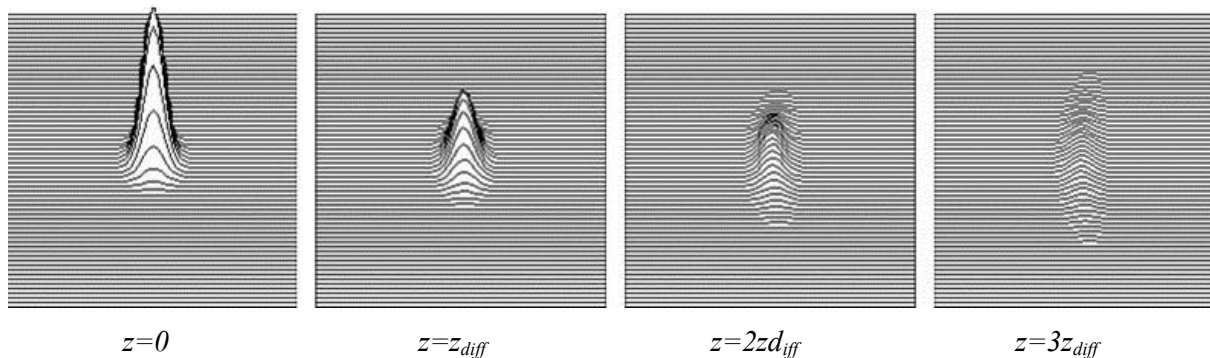

$z=0$ $z=z_{diff}$ $z=2zd_{iff}$ $z=3z_{diff}$

Figure 4. Plot of the side *(x-τ)* projection of the intensity profile of *25 fs* phase-modulated optical pulse with parameter *a=10* under the same initial condition (10) and the spatio-temporal non-paraxial Eq. (1). The phase-modulated pulse is broad-band and its diffraction is of *U*-type (inverted semi-sphere) as in the case of attosecond pulses.

The form of the pulse significantly differs from the typical diffractive profile obtained in the frames of paraxial optics (Figure 2). The semi-spherical shape of the pulse is clearly seen. The parabolic form of the phase-modulated femtosecond laser pulse is observed on a several diffractive lengths. The Local coordinate system gives an inverted image of the real pulse deformation with respect to the axis *Oz*.
The question is: Is it possible to observe the real Fraunhofer type deformation obtained in the experiments [1]?
The real parabolic shape of the pulse is observed when we use equation (9) in Galilean frame under the initial condition of the kind (11) and chirp parameter *a=10*. The results are shown in Figure 5.
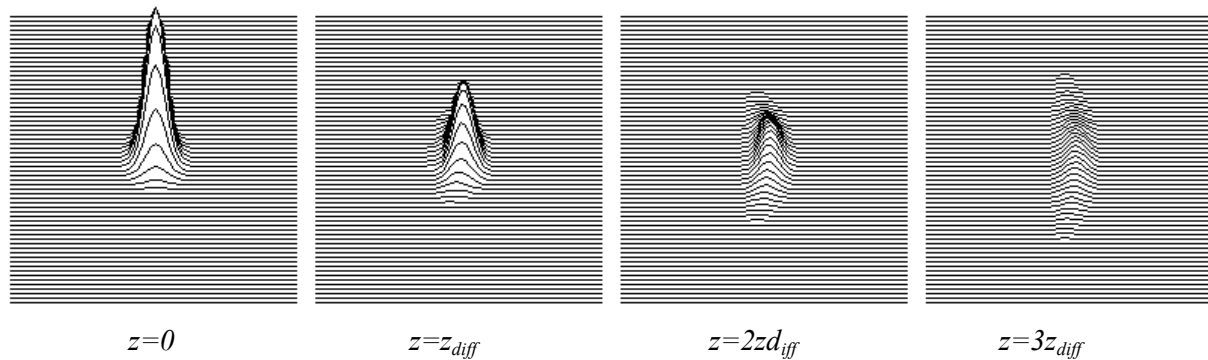
$z=0$ \qquad $z=z_{diff}$ \qquad $z=2zd_{iff}$ \qquad $z=3z_{diff}$

Figure 5. Plot of the side *(x-z)* projection of the intensity profile of *25 fs* phase-modulated optical pulse. The real spatial Fraunhofer type semi-spherical deformation is clearly seen. The initial condition (11), the non-paraxial Eq. (9) in Galilean frame and chirp parameter *a=10* are used. It is obvious that the Local coordinate system gives an inverted image (with respect to axis *Oz*) of the real pulse deformation.

The intensity profile of the laser pulses in Figure 4 and 5 is a typical example of Fraunhofer diffraction for broad-band optical pulses which is the same as the diffraction of laser pulses with one-two cycles under the envelope (attosecond optical pulses). The solutions in Galilean fames give a real diffractive deformation of the pulses in the space. This is the reason why it is more appropriate to use this coordinate system instead of the Local time.

4. Conclusion
In the present work the propagation of optical pulses with narrow and broad-band spectrum in linear regime is investigated in the frames of three different evolution equations: paraxial spatio-temporal equation (6), non-paraxial amplitude equation in Local tame coordinate system (1) and non-paraxial amplitude equation in Galilean frames (9). The Fraunhofer type diffraction of phase-modulated *25 fs* laser pulses can be observed only in the frames of non-paraxial optics: equations (1) and (9). The paraxial optics is not sensitive to the spectral band of the pulses and cannot present this effect.
Our numerical results have shown that:
- The non-paraxial equation in Local coordinate system (1) gives the correct profile deformation of optical pulses but inverted with respect to the axis *Oz*;
- The real spatial Fraunhofer diffraction of femtosecond and attosecond laser pulses is obtained when the non-paraxial equation is solved in Galilean (9) or Laboratory coordinate system (1).

In this paper we have shown that the Fraunhofer type diffraction is not a characteristic only to attosecond or femtosecond pulses with one or few optical cycles under the envelope. There are different methods to obtain broad-band laser pulses with many cycles under the envelope. For examples: nonlinear self-phase modulation, optical grating etc.
We demonstrated that even in the case of phase-modulated femtosecond pulses with many cycles under the envelope Fraunhofer type diffraction can be observed.

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