Analytical formula for residual current density excited in the process of gas ionization by a few-cycle laser pulse in the low-intensity limit

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Abstract. This work is devoted to analytical study of excitation of the residual current density (RCD) in the process of gas ionization by a few-cycle laser pulse. The RCD remains in the laser-produced plasma after the passage of the laser pulse and is as an initial push leading to excitation of the plasma oscillations which can radiate terahertz waves. We derive simple closed-form analytical formula for RCD for relatively small peak intensity of few-cycle laser pulse, which corresponds to small final degree of ionization. The dependences of the RCD on laser pulse parameters are discussed.

1. Introduction

Recently, great attention has been given to the phenomenon of excitation of the residual current density (RCD) during gas ionization by few-cycle laser pulses [1–4]. The essence of this phenomenon can be explained as follows. In the process of ionization of atoms and molecules by an intense laser field, the newly born free electrons acquire, along with oscillatory velocity, the constant drift velocity. The value and direction of this velocity are determined by the phase of the electric field at the moment of ionization. As a result, after the passage of the laser pulse, quasi-constant RCD can arise in the produced plasma. The value of this RCD depends on the carrier-envelope phase (CEP) of few-cycle laser pulse and can reach high values at sufficiently short durations of the laser pulse [1–5]. The interest to this phenomenon is connected with the possibility of its application for generation of radiation in the terahertz (THz) range, which was demonstrated experimentally in several works [6–9]. The residual current serves as the initial push for plasma polarization and excitation of radiating plasma oscillations which have THz frequencies in a wide range of gas pressures and laser pulse intensities [1, 10].

Earlier studies of RCD excitation during gas ionization by few-cycle laser pulse were carried out by using numerical calculations. These calculations were based on semiclassical [1] and quantum-mechanical [3] approaches. The semiclassical approach is based on solution the hydrodynamic equation for the electron current density and

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the equation for free-electron density, which uses static-field probability of tunneling ionization per time unit. The quantum-mechanical approach is based on the *ab initio* numerical solution of the three-dimensional (3D) time-dependent Schrödinger equation. These approaches were used to calculate RCD dependences on the carrier-envelope phase (CEP), intensity, and duration of the laser pulse [2–5, 11–13]. In particular, comparison of the results obtained on the basis of semiclassical and quantum-mechanical approaches showed that the range of applicability of the semiclassical approach is limited by the parameters of the laser pulse, at which the conditions for tunneling ionization of the gas are fulfilled at the maximum of the pulse envelope [3, **?**]. These conditions correspond to a low value of the Keldysh parameter γ [14].

In this work we derive analytical formula for RCD excited in the process of gas ionization by few-cycle laser pulse. To develop the analytical model, we use the semiclassical approach. We assume for simplicity that the intensity of the laser pulse is sufficiently small so that the final gas ionization dergee is much smaller than unity. Basing on the derived analytical formula, the dependences of RCD on CEP, intensity, duration and wavelength of linearly-polarized pulses are determined.

This paper is structured as follows. Section 2 contains the statement of the problem. In Section 3 the analytical formulas for RCD are derived. Then, in Section 4 we discuss the found analytical dependences of RCD on the main parameters of the laser pulses. Section 5 contains the summary and conclusions of the research.

2. Statement of the problem

We assume that the electric field $\mathbf{E}(t)$ of the laser pulse is polarized linearly along the z axis ($\mathbf{E}(t) = \hat{\mathbf{z}}E(t)$) and is given by the formula

$$\begin{cases} E(t) = \dfrac{E_0}{\omega_0}\dfrac{da}{dt}, & (1) \\ a(t) = f(t)\sin(\omega_0 t + \varphi_{\text{CEP}}). & (2) \end{cases}$$

Here, E_0 is the maximum of the laser pulse envelope, ω_0 is the carrier frequency, φ_{CEP} is CEP, $f(t)$ is the slowly varying envelope of the pulse.

To derive analytical formula for RCD, we use the so-called semiclassical approach, which is based on solving the equation for the free-electron density $N(t)$ and the classical equation for the electron density $\mathbf{j}(t) = \hat{\mathbf{z}}j(t)$ in plasma with a variable number of particles

$$\begin{cases} \dfrac{\partial N}{\partial t} = (N_g - N)w(|E|), & (3) \\ j_{\text{RCD}} = \dfrac{e^2}{m}\displaystyle\int_{-\infty}^{\infty} N(t)E(t)dt. & (4) \end{cases}$$

Here, $w(|E|)$ is the static-field probability of tunneling ionization per time unit, N_g is the unperturbed gas density, and e and m are the charge and mass of the electron,

respectively. The probability of atom ionization per unit of time is assumed to be determined by the function of the electric-field strength. This function is taken from the solution of the quantum-mechanical problem of the ionization of a hydrogen atom from the ground state in a static electric field. We assume further that final ionization degree is sufficiently small and neglect the term responsible for atom depletion in Eq. 3.

We will normalize RCD to the maximum possible oscillatory current density in the pulse $j_{\rm osc} = e^2 E_0 N_g / m\omega_0$. The normalized current density obtained by this method,

$$j_{\rm norm} = j_{\rm RCD}/j_{\rm osc}, \tag{5}$$

does not depend on N_g and its squared value $j^2_{\rm norm}$ characterizes the efficiency of conversion of the laser pulse energy to the energy of low-frequency THz waves [1, 3]. Using integrating by parts in Eq. 4, we obtain equation for $j_{\rm norm}$:

$$j_{\rm norm} = -\int_{-\infty}^{\infty} a(t) w(|E(t)|) dt. \tag{6}$$

3. Analytical model

An important fact for the further study is that in the range of the values of E being of interest, the probability of ionization per time unit is a sharp function of the electric-field strength. The parameter that characterizes the sharpness of $w(E)$ is [15]

$$\eta(E) = \frac{w(E)}{E w'(E)}. \tag{7}$$

The lower is the quantity of η, the sharper is the behavior of $w(E)$ at the point E. At sufficiently low strength $|E|$ of the electric field, the value η is a small parameter, $\eta \ll 1$. Therefore the function $w(|E(t)|)$ consists of narrow peaks localized near the moments of maximums $|E(t)|$. In the case of sufficiently low peak intensity of the laser pulse these peaks can be approximated by the Gaussian functions

$$w(|E(t)|) \approx \sum_{k=-\infty}^{\infty} w(|F(t_{e,k})|) \exp\left(-\frac{2(t-t_{e,k})^2}{\tau_{e,k}^2}\right). \tag{8}$$

Here, $F(t) = E_0 f(t)$, k is an integer number, and $t_{e,k}$ are the moments of maximums $|E(t)|$, which are approximately equal to

$$t_{e,k} \approx \left(1 - 4(\omega_0 \tau)^{-2}\right) t_{a,k}, \tag{9}$$

where

$$t_{a,k} = (\pi k - \varphi_{\rm CEP})\omega_0^{-1}. \tag{10}$$

The value $\tau_{e,k} = 2\sqrt{\eta(F(t_{e,k}))}\omega_0^{-1}$ is the characteristic width of the kth peak, $\tau = \sqrt{2/|\ddot{f}(0)|}$ (the dots means the second derivative with respect to time) is characteristic width of the pulse envelope. In what follows τ will be called the pulse duration. Since $\eta \ll 1$, $\tau_{e,k}$ is much shorter than the laser pulse period $T_0 = 2\pi/\omega_0$. Therefore, the spectrum of the function $w(|E(t)|)$ is concentrated near even harmonics of the laser pulse frequency. The function $w(|F(t_{e,k})|)$ from Eq. 8 is approximately equal to

$$w(F(t_{e,k})) \approx w(E_0) e^{-2t_{e,k}^2/\tau_i^2}. \tag{11}$$

$$\tau_i \approx \sqrt{2\eta_0}\tau, \quad \eta_0 = \eta(E_0) \tag{12}$$

is the characteristic time of the concentration rise, which will be called the *ionization time* in what follows [15]. Since $\eta_0 \ll 1$, τ_i is much shorter than the pulse duration.

Let us now obtain approximate analytical formula for the normalized RCD. The integral in formula 6 can be subdivided into a sum of integrals in the small vicinity of maximums $|E(t)|$. Thus, the normalized RCD is subdivided to the contributions Δj_k made by the electrons born on separate humps of $|E(t)|$,

$$j_{\text{norm}} = \sum_{k=-\infty}^{\infty} \Delta j_k, \quad \Delta j_k = -\frac{\pi}{\omega_0} a(t_{e,k}) w(F(t_{e,k})). \tag{13}$$

The value of $a(t)$ at the time moment $t_{e,k}$ is

$$a(t_{e,k}) \approx -4(-1)^k f(t_{e,k})(\omega_0\tau)^{-2}\omega_0 t_{e,k}. \tag{14}$$

Substituting 14 to Eq. 13 and taking into account that $\tau \gg \tau_i$, we obtain

$$j_{\text{norm}} \approx 4\pi \frac{w(E_0)}{(\omega_0\tau)^2} \sum_{k=-\infty}^{\infty} (-1)^k t_{e,k} e^{-2t_{e,k}^2/\tau_i^2}. \tag{15}$$

To calculate the sum in this formula, we will use approximate relationship

$$\sum_{k=-\infty}^{\infty} (-1)^k e^{-b(k-k_0)^2} \approx 2\sqrt{\pi/b} e^{-\pi^2/4b} \cos(\pi k_0), \tag{16}$$

which is valid at $\exp(-3\pi^2/b) \ll 1$. In order to apply Eq. 16 for calculation of the sum in Eq. 15, it is necessary to ensure fulfillment of inequality $\exp(-3\omega_0^2\tau_i^2/2) \ll 1$, which is knowingly fulfilled at the values of pulse intensity and duration being of interest. Using Eq.16 in formula 15, we obtain

$$j_{\text{norm}} = -j_{\max} \sin(\varphi_{\text{CEP}}). \tag{17}$$

Thus, j_{norm} is subdivided into the product of two factors, one of which depends only on CEP, and the other, on the other parameters of the laser pulse. The maximum normalized RCD, which corresponds to the optimal CEP, is equal to

$$j_{\max} = 2\sqrt{\eta_0} w(E_0) \frac{\tau_i^3}{\tau^2} \exp\left(-\frac{\omega_0^2\tau_i^2}{8}\right) = 4\sqrt{2}\eta_0^2 w(E_0)\tau \exp\left(-\eta_0 \frac{\omega_0^2\tau^2}{4}\right). \tag{18}$$

Let us assume that $w(E)$ is approximated by the empirical formula proposed by Tong and Lin [16], which yields well approximation of the precise value w in both the regimes of tunneling and above-barrier ionization. In this regime, the tunneling formula $w_{TI}(E)$ [17, 18] yields strong overrating of the ionization probability. The adjusted formula proposed in [16] differs from $w_{TI}(E)$ in an additional adjustment factor:

$$w(E) = w_{TI}(E) e^{-\nu E/E_a}, \tag{19}$$

Here, $E_a = 5.14 \times 10^9$ V/cm is the atomic field strength, ν is the coefficient introduced to make formula 19 coincide with the dependence $w(|E|)$ found by means of direct numerical solution of the Schrödinger equation. The values of ν calculated for the

hydrogen atoms and different noble gases can be found in [16]. For example, for hydrogen atoms $\nu = 12$. Since ν is sufficiently great, the adjustment factor in Eq. 19 is significant even at moderate electric field strength being a tenth of E_a. The tunneling formula $w_{TI}(E)$ entering into Eq. 19 has the following form [17, 18]:

$$w_{\mathrm{TI}}(E) = \alpha\Omega_a \left(\frac{E_a}{E}\right)^s e^{-\beta E_a/E}, \tag{20}$$

where $\Omega_a = 4.13 \times 10^{16}$ s^{-1} is the atomic unit of frequency. For hydrogen atoms, $\alpha = 4$, $\beta = 2/3$, and $s = 1$. In the case when w is specified by adjusted tunneling formula 19 the formula for $j_{\max}$ is

$$j_{\max} = \frac{4\sqrt{2}\alpha}{\beta^2}\Omega_a\tau \left(\frac{E_0}{E_a}\right)^{2-s} \exp\left(-\frac{\beta E_a}{E_0} - \frac{\nu E_0}{E_a} - \frac{E_0}{\beta E_a}\frac{\omega_0^2\tau^2}{4}\right). \tag{21}$$

4. Discussion

It follow from general formula 17 that a the case of low laser pulse intensities the dependence of RCD on CEP is sinusoidal,

$$j_{\mathrm{RCD}} \propto \sin(\varphi_{\mathrm{CEP}}), \tag{22}$$

independently of the shape of the function $w(E)$. Thus, the analytical formula confirm the results of numerical calculations of the RCD dependence on CEP in the tunneling regime of ionization [1–5, 11]. The optimal CEP is equal to $\varphi_{\mathrm{opt}} = \pi/2$, which corresponds to the asymmetric (sine-shaped) electric field of the laser pulse. This can be explained as follows. In the case of a symmetric pulse, the drift velocities of the electrons born at certain time moments $t = t_0$ and $t = -t_0$ are equal in their absolute values, but opposite in direction. As a result, since the ionization probability per time unit is the even function t, the currents of the electrons born at $t < 0$ and $t > 0$ compensate each other totally, and RCD is equal to zero. If $\varphi_{\mathrm{CEP}} \neq 0$, then the currents of the electrons moving in different directions do not coincide in the absolute values, which results in appearance of non-zero RCD. Since the dependence of RCD on CEP is sinusoidal, the optimal CEP is shifted from the nonoptimal one to $\pi/2$, i.e., it is equal to $\varphi_{\mathrm{opt}} = \pi/2$, which corresponds to the antisymmetric (sine-shaped) electric field of the laser pulse.

It is important to focuse on the following issue. It was noted in the introduction that the region of applicability of the semiclassical approach to RCD calculation is limited by small values of the Keldysh parameter, $\gamma = \sqrt{I_p/2U_p} \ll 1$ [14]. Here, $U_p = e^2E_0^2/4m\omega_0^2$ is the averaged ponderomotive energy of the electron in the pulse. Since $\gamma \propto 1/E_0$, γ increases as the intensity decreases. At low values of intensity, when final ionization degree is small, the parameter γ can be comparable with or exceed unity. In this case, the semiclassical approach goes beyond the region of applicability. However, when the laser pulse wavelength increases, the Keldysh parameter decreases in proportion to λ^{-1}, and the region of applicability of the semiclassical approach becomes wider. As a result, the semiclassical approach (and, in particular, analytical formula 17) is valid in the

mid-infrared band including the case of low intensities corresponding to low final dergee of ionization.

Let us now consider the dependence of the maximum normalized RCD $j_{\max} = |j_{\text{norm}}(\varphi_{\text{opt}})|$, which corresponds to the optimal CEP, on the duration of the laser pulse. At low intensities $j_{\max} \propto \tau \exp(-\omega_0^2 \tau_i^2/8)$. However, the pre-exponential factor affects the dependence $j_{\max}(\tau)$ only at very short durations of $\tau << T_0$ and is not important for the values of τ being of interest. Since the ionization time τ_i is proportional to τ, $j_{\max}$ decreases with an increasing duration obeying the Gaussian law:

$$j_{\max}(\tau) \propto \exp\left(-\eta_0 \frac{\omega_0^2 \tau^2}{4}\right). \tag{23}$$

Since η_0 is a small parameter, the characteristic scale of the decrease in the function $j_{\max}(\tau)$ is $\sim 1/\sqrt{\eta_0}\omega_0$, which usually equals to several field periods. Another situation is realized at high intensities which corresponds to high final dergee of ionization. In this case, $j_{\max}(\tau)$ decreases obeying the exponential law [1, 3]. This is connected with the fact that at high intensities, the ionization time τ_i increases with an increase in τ approximately as $\tau_i(\tau) \propto \sqrt{\tau}$, rather than obeying the linear law.

Let us now pass over to analysis of the dependence of the value $j_{\max}$ on the central wavelength λ of the laser pulse. For the convenience of the analysis, we will fix the number $N_{cyc} = \tau/T_0$ of field periods in the pulse. In this case, $j_{\max}$ increases linearly with growing λ,

$$j_{\max}(\lambda) \propto \lambda. \tag{24}$$

This increase in $j_{\max}$ is connected with the increase in electron concentration, which is proportional to λ at a fixed ratio τ/T_0. The efficiency of excitation of the low-frequency current (estimated in [1] in the framework of the quasistatic approach) is proportional to the square of normalized RCD. Thus, in the regime of low intensities, this efficiency is proportional to λ^2. Hence, one can conclude that the use of few-cycle laser pulses in the near and middle IR wavelength range can result in a significant increase of terahertz wave generation efficiency.

Finally, consider the dependence of $j_{\max}$ on the peak intensity $I = cE_0^2/8\pi$ (where c is speed of light in vacuum) of the laser pulse. For the case, when $w(E)$ is specified by formula 19,

$$j_{\max}(I) \propto \exp(-\beta\sqrt{I_a/I} - \nu\sqrt{I/I_a}), \tag{25}$$

where $I_a = 3.51 \times 10^{16}\text{W/cm}^2$ is the atomic unit of intensity. The sharp increase in $j_{\max}$ at low intensities is connected with a sharp (exponential) increase in the free-electron density when the peak intensity grows.

5. Conclusions

To conclude, we have derived simple closed-form analytical expression for the residual current density (RCD) excited in the process of gas ionization by few-cycle laser pulse

in the case of low peak intensity. The range of applicability of this formula is limited by the low values of the Keldysh parameter γ at the maximum of laser pulse envelope. This formula confirm the fact that at low dergee of ionization the dependence of RCD on the carrier-envelope phase (CEP) is a sinusoidal function and the optimal CEP is equal to $\pi/2$ and corresponds to the antisymmetric sine-like laser pulse.

It is shown that in the tunneling regime of ionization (at $\gamma \ll 1$), the value of maximal RCD $j_{\max}$ (normalized to the oscillatory current density), corresponding to the optimal phase, increases sharply (exponentially) with the growth of the peak pulse intensity. The character of the RCD decrease during the increase in the pulse duration τ depends on the field intensity. At low intensities, RCD decreases as τ increases obeying the Gaussian law. Also it is shown that the value of $j_{\max}$ increases linearly with an increasing wavelength (at a fixed number of field periods in the pulse). The efficiency of RCD excitation (determined by the squared value of $j_{\max}$) is proportional to λ^2. A fast increase in the efficiency of excitation of the low-frequency current with the increase in the pulse wavelength can be used to create sources of high-power THz radiation with the use of few-cycle laser pulses in the near and mid-infrared wavelength ranges.

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References

[1] Gildenburg V B and Vvedenskii N V 2007 *Phys. Rev. Lett.* **98** 245002
[2] Wu H C, Meyer-ter Vehn J and Sheng Z M 2008 *New J. Phys.* **10** 043001
[3] Silaev A A and Vvedenskii N V 2009 *Phys. Rev. Lett.* **102** 115005
[4] Kim K Y 2009 *Phys. Plasmas* **16** 056706
[5] Silaev A A and Vvedenskii N V 2009 *Phys. Scr.* **T135** 014024
[6] Kress M, Loeffler T, Thomson M D, Doerner R, Gimpel H, Zrost K, Ergler T, Moshammer R, Morgner U, Ullrich J and Roskos H G 2006 *Nat. Phys.* **2** 327
[7] Li C, Wang D, Song L, Liu J, Liu P, Xu C, Leng Y, Li R and Xu Z 2011 *Opt. Express* **19** 6783
[8] Bai Y, Song L, Xu R, Li C, Liu P, Zeng Z, Zhang Z, Lu H, Li R and Xu Z 2012 *Phys. Rev. Lett.* **108** 255004
[9] Liu P, Li R and Xu Z 2015 Phase evolution and thz emission from a femtosecond laser filament in air *Progress in Ultrafast Intense Laser Science XI* Springer Series in Chemical Physics ed Yamanouchi K, Nam C H and Martin P (Springer International Publishing) pp 175–193
[10] Kostin V A and Vvedenskii N V 2010 *Opt. Lett.* **35** 247–249
[11] Chen M, Pukhov A, Peng X Y and Willi O 2008 *Phys. Rev. E* **78** 046406
[12] Silaev A A, Ryabikin M Y and Vvedenskii N V 2010 *Phys. Rev. A* **82** 033416
[13] Alexandrov L N, Emelin M Y and Ryabikin M Y 2013 *Phys. Rev. A* **87** 013414
[14] Keldysh L V 1965 *Sov. Phys. JETP* **20** 1307–1314
[15] Vvedenskii N V, Korytin A I, Kostin V A, Murzanev A A, Silaev A A and Stepanov A N 2014 *Phys. Rev. Lett.* **112** 055004
[16] Tong X M and Lin C D 2005 *J. Phys. B: At. Mol. Phys.* **38** 2593
[17] Perelomov A M, Popov V S and Terentev M V 1966 *Sov. Phys. JETP* **23** 924–934
[18] Ammosov M V, Delone N B and Krainov V P 1986 *Sov. Phys. JETP* **64** 1191–1194