Journal of Physics: Conference Series **585** (2015) 012008 doi:10.1088/1742-6596/585/1/012008

# Method of Integration for Equation of Two Balls in Dumped Collision

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Index terms: experimental mechanics, rolling hysteresis, photoelasticity

**Abstract**. The paper presents an integration method for the equation characteristic to a dumped collision. The equation which describes the dumped collision is nonlinear ODE of second order. The solution consists from two stages: in the first stage the equation order is reduced with a unity finding a prime integral. In this moment the maximum approach can be evaluated. Next, using a elementary numerical method the relation between approaching versus time is found. The method is applied for two equations which describe a dumped collision. The results obtained with the proposed method are in a perfect agreement with those from original paper. Finally, in order to emphasize the difference between the two models and the effect of dumping, the hysteresis loops for two values of the coefficient of restitution, corresponding to the two models are plotted to emphasize the difference are plotted.

## 1. Introduction

In multibody dynamics, collision or impact phenomena appear when velocities are changed suddenly. The collisions can be studied considering that the impact is instantaneously, using coefficients of restitution (COR) on one hand [1], or, considering that during impact process the kinematical and dynamical parameters present continuous variation, on the other hand.

The method considering the instantaneous character of impact has the main advantage of straightforward calculus, reduced in fact to algebra, but presents the essential inconvenience that cannot estimate the magnitude of forces occurring during collision. Additionally, a problem concerning the definition modality for the coefficient of restitution, (COR), arises. More precise, the technical literature mentions two modalities of defining the coefficient of restitution, [2], [3]. The first method is due to Newton and considers the kinematical definition of coefficient of restitution as the ratio, with opposite sign, of normal components of relative velocities of the initial contact points, after and before collision, respectively. Taking this modality of defining COR can lead to paradoxical situations. Such a circumstance is was exposed by Kane, [4], who presents the case of collision with friction of a mechanical system for which, the conclusion is that kinetical energy of the system is greater at final stage than initially. To eliminate this inconsistency, is required the acceptance of another hypothesis, specifically: the collision happens during a finite period of time, being characterised by two phases - the approaching and detaching phases. In this situation, the coefficient of restitution is defined according to Poisson as the ration between the percussion forces corresponding to detaching and approaching, respectively.

The challenge of describing the impact in a general form, considering all the parameters influencing the process, is practically impossible. From this cause, most of the works concerning the study of collisions treat systems considering more or less simplifying hypothesis. The simplest model met in elementary dynamics works is for the collision of two punctiform bodies. The next model regarding problem complexity considers centric collision of two spheres made of elastic, homogenous and isotropic materials.

Goldsmith, [5], presents a solution due to Timoshenko, [6], for a perfect elastic impact between two balls of different radii. The solution found by Timoshenko is based on the relation force-approach for the case of a system of two elastic balls. For this model Timoshenko found the analytical relation for maximum approaching and for the contact time. For the case of a more complicated model, that is the dumped collision, the problem becomes considerably complex. The simplest model describing the behaviour of this collision model is the Kelvin-Voigt model which was used for the dynamic system. Hunt and Crossley [7] showed that the model has the disadvantage of an open hysteresis loop and for the final stage of the impact process the two bodies attract each other instead of rejecting. They indicated that in order to obtain a closed hysteretic loop it is necessary to use a modified Kelvin-Voigt model with variable damping and elastic coefficients.

Lankarani and Nikravesh, [8], following the results of Hunt and Crossley, [7], obtained the equation for centric impact between two elastic balls.

## 2. Lankarani Nikravesh Model

The collision with dumping of two balls, as sketched in figure 1, is considered.

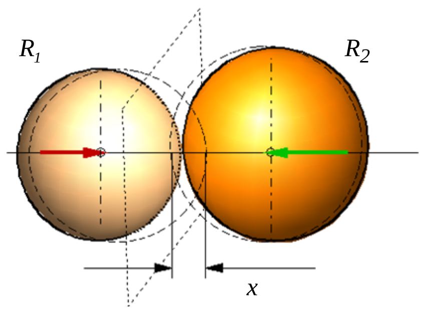


**Figure 1.** Two colliding balls.

The equation propose by Lankarani and Nikravesh, was obtained based, as in Timshenko's solution, on the force-displacement relation from the Hertz's problem to which was added the hypothesis that the dumping force during collision depends both on the approaching velocity of the bodies and on a power of bodies approaching. The tangible form of the equation obtained by the two authors is, [8]:

$$F = Kx^{\alpha}\left[1+\frac{3(1-e^2)}{4}\frac{\dot{x}}{v_0}\right] \quad (1)$$

where:

$e$ is coefficient of restitution defined by Newton as the ratio of the normal components of relative velocities of initial contact points for the detaching and approaching phase, respectively;

$x$ is the distance between the initial contacting points:

$v_0$ is initial impact velocity;

$\alpha$ is an exponent which for point contact is $\alpha = 3/2$;

*K* is a coefficient characterising the elastic proprieties and the geometry of the balls around the contact points. More exactly,

$$K = \frac{4}{3(\eta_1+\eta_2)}\left[\frac{R_1R_2}{R_1+R_2}\right]^{1/2} \tag{2}$$

and

$$\eta_{1,2} = \frac{1-\nu_{1,2}^2}{E_{1,2}} \tag{3}$$

Where $R_{1,2}$ are the balls radii, $E_{1,2}$ , $\nu_{1,2}$ the Young modulus and Poisson coefficients, respectively.

The equation 1 is a ordinary nonlinear differential equation. The equation has no analytic solutions. In the paper the authors don't give any information about the methodology of integration of this equation. They present only the results of integration under graphical form which show the influence of parameters $e$ and $v_0$ upon the kinematics and dynamical parameters and the forms of hysteresis loops. A key advantage of the model consists in presenting closed hysteresis loops closed in the origin.

## 3. Proposed method of integration

The authors suggest a method of integration for the equation (1), made in two steps:

1. The first step consist in finding an analytical relation between the normal approach and velocities:
2. The second step concerns numerical computing of time dependence of approach.

For the first step we use the method of the Horway and Veluswami, [9]. We write the equation (1) as:

$$\ddot{x}(t)+\gamma\, x(t)^n\, \dot{x}(t)+\chi\, x(t)^n = 0 \tag{4}$$

where upper dot represents time derivative. Using the notations:

$$M = \frac{m_1 m_2}{m_1+m_2} \tag{5}$$

where *M* the reduced mass of the system ( are the masses of the balls) and :

$$\chi = K/M\ ; \quad \gamma = K\frac{3(1-e^2)}{4v_0} \tag{6}$$

$$\frac{dv}{dx}\frac{dx}{dt} + \gamma\, x^{3/2} v + \chi\, x^{3/2} = 0 \tag{7}$$

$$\frac{dv}{dx}v + \gamma\, x^{3/2} v + \chi\, x^{3/2} = 0 \tag{8}$$

$$\chi\, x^{3/2} = -\frac{vdv}{1+\dfrac{\chi}{\gamma}v} \tag{9}$$

The general solution of the above equation is:

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$$\frac{2}{5}\chi x^{5/2} + C = \frac{\chi}{\gamma}v - \frac{\chi^2}{\gamma^2}ln(\chi + \gamma v) \tag{10}$$

In order to find a particular solution the initial conditions is imposed. At moment $t = 0$ we need $x = 0, v = v_0$ . So, for the approaching phase the equation (10) becomes:

$$\frac{2}{5}\chi x^{5/2} = -\frac{\chi}{\gamma}(v - v_0) + \frac{\chi^2}{\gamma^2}ln\frac{\chi + \gamma v}{\chi + \gamma v_0} \tag{11}$$

Denoting $t_m$ as the moment of reaching maximum approach $x_m$, from condition that at the moment $t = t_m$, $x = x_m$ and $v = 0$ we can find the maximum approach;

$$x_m = \left\{\frac{5}{2}\left[\frac{1}{\gamma}v_0 - \frac{\gamma}{\chi}ln\left(1 + \frac{\gamma}{\chi}v_0\right)\right]\right\}^{5/2} \tag{12}$$

For the detaching phase, the conditions imposed at $t = t_m$ are: $v = 0$, $x = x_m$ and conduct to the solution form for the coming off phase.

$$-\frac{2}{5}\chi\left[x'^{5/2}\right] - x_m^{5/2} = \frac{\chi}{\gamma}v - \frac{\chi^2}{\gamma^2}ln\left(1 + \frac{\gamma}{\chi}v\right) \tag{13}$$

In order to find the time dependence of $x$ on the approach interval we consider that the equation (11) give the dependence of $v$ as function of $x$.. This equation is a transcendental one and must be solved numerically. There are many software applications which can resolve it, most of these being based on the Runge-Kutta, [10] algorithm. In order to accurately describe the impact phenomenon, a large number of points should be considered. In order to reduce the computing time, a simple program was created for solving the equation by the bipartition method. Applying the Runge-Kutta method assumes that the functions are derivable and require much time compared to the bipartition method. Different forms of the relationship between velocity and displacement are obtained for the compression and expansion phases. The relations (11) and (13) are expressed under the form:

$$f(v,x) = \frac{2}{5}\chi x^{5/2} + \frac{\chi}{\gamma}(v - v_0) + \frac{\chi^2}{\gamma^2}ln\left(\frac{\chi + \gamma v_0}{\chi + \gamma v}\right) = 0 \tag{14}$$

where $x$ increases from zero to $x_m$

$$g(v', x') = \frac{2}{5}\chi\left[x'^{5/2} - x_m^{5/2}\right] + \frac{\chi}{\gamma}v' - \frac{\chi^2}{\gamma^2}ln\left(1 + \frac{\gamma}{\chi}v'\right) = 0 \tag{15}$$

and $x'$ decreases from $x_m$ to zero.

For the interval $[0, x_m]$ we choice a set of equidistant points $x_k$, $k = 0 \div n$, and for every $x_k$ we solve the equation (14) using bipartition method, with accuracy of $10^{-15}$. So we obtained $n$ pairs, $(x_k, v_k)$. Similarly, for equidistant points $x'_k$ for the detaching phase, pairs of points are obtained after numerical solving the equations (14) and (15). The velocity-displacement dependence is represented in figure 2, foe equidistant points from the range. As it can be noticed, in the vicinity of the point of maximum indentation, the points of the plot are more remote. To diminish this effect, it is necessary to choose the sequences $x_k$, $x'_k$ so that the points should be denser in the vicinity of point $x_m$. To this purpose, the points are dispersed inside the interval $[0, x_m]$ obeying the following law:

$$x_k = \frac{1-\alpha^{k/n}}{1-\alpha}, 0<\alpha<1; \quad x'_k = \frac{\beta^{k/n}-\beta}{1-\beta} \tag{16}$$

where $\beta = 1/\alpha$. For $COR = 0.999$, $n = 20$ and $\alpha = 1/200$, the pairs of points $(x_k, v_k)$ and $(x'_k, v'_k)$ are represented in figure 3.

One can observe that the points of the plots are more compact in the vicinity of the point $x_m$.

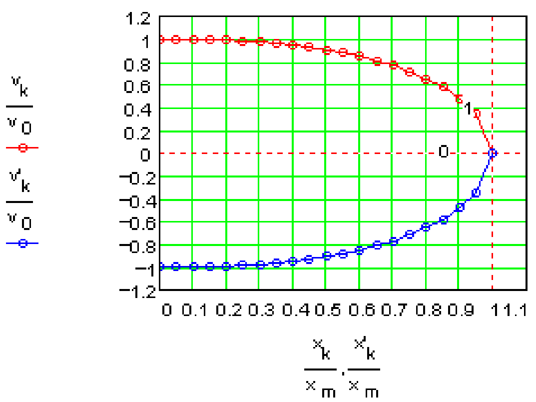

**Figure 2.** Velocity-displacement dependence for equidistant points

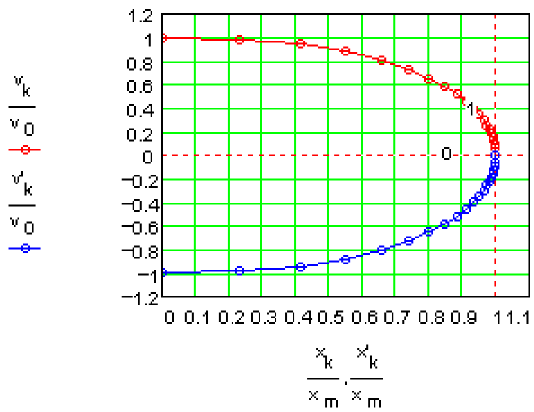

**Figure 3.** Velocity-displacement dependence for proposed method

The equation:

$$v = dx/dt \tag{17}$$

can now be integrated numerically, using Euler method. Therefore:

$$\begin{aligned} &for\ k=0,\ t_0=0, \\ &for\ k=1:n-1,\ t_k = t_{k-1} + \frac{x_k - x_{k-1}}{v_k} \end{aligned} \tag{18}$$

In equation (18) $k \neq n$ because $v_n \equiv 0$. The approaching time $t_m$ cannot be obtained using (18). Plotting the dependence $x_k = x_k(t_k), k = 0 \div n-1$, we obtained the curve plotted in figure 4, and it can be seen that the elapsed time between the moments correspondent to successive pairs $(x_k, v_k)$ increases as the moment of maximum indentation value $x_m$ is closer.

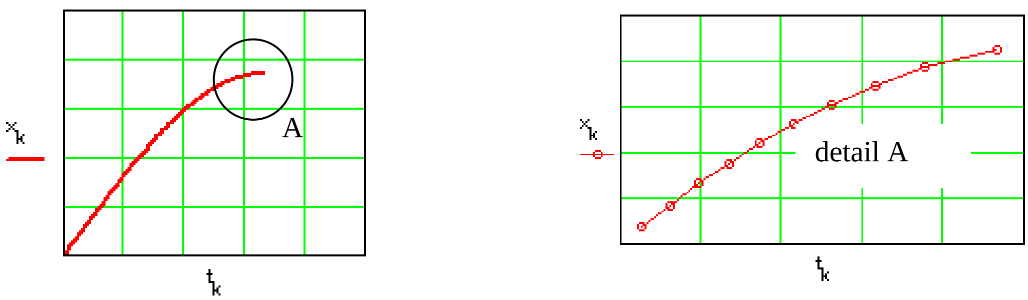


**Figure 4.** Indentation variation with time

The problem of finding the approaching time $t_m = t_n$ is important as the indentation time is one of the characteristic parameters of the impact process. In order to find the time $t_n$, the curve from Fig. 4 must be extrapolated. The shape of the curve from figure 4 suggested the next extrapolation possibilities.

1. The dependence $x = x(t)$ is approximated by a sinusoid having the form $x = A sin(\mu t)$, where the unknowns $A$ and $\mu$ are found from the condition of minimum for the function:

$$S(A,\mu) = \sum_{k=0}^{n-1} \left[ (A sin(\mu t_k) - x_k \right]^2 \tag{19}$$

By $t'_m$ is denoted the value found for the approaching time.

2. The dependence $x = x(t)$ is approximated by a sinusoid having the form $x = A sin(\mu t)$ but, from the beginning, the amplitude is imposed to $x_n$ and from the condition of minimum of the function:

$$T(\mu') = \sum_{k=0}^{n-1} \left[ \frac{x_m}{sin(\mu t''_m)} sin(\mu' t_k) - x_k \right]^2 \tag{20}$$

only the value of parameter $\mu'$ remains be found. In this case, the approaching time found is denoted by $t''_m$.

3. It is imposed the condition that the center of the circle passing through the last three points of the plot should be positioned on the vertical of the last point. Let the coordinates of the center of the circle be $(t_O, x_O)$. Practically, when the three points are very closed, the circumscribed circle is the osculator circle at the plot $x = x(t)$ in the point $(t_n, x_n)$.

$$\begin{cases} (t_{n-2} - t_O)^2 + (x_{n-2} - x_O)^2 = r^2 \\ (t_{n-1} - t_O)^2 + (x_{n-1} - x_O)^2 = r^2 \\ ((t_n - t_O)^2 + (x_n - x_O)^2 = r^2 \\ t_O = t_n \end{cases} \tag{21}$$

This last value for the approaching time is denoted $t'''$.

In order to decide which of the above three values of compressing time must be considered, it is accepted the assumption that that for values of coefficient of restitution closed to unity, the collision can be considered elastic. For the elastic collision there are analytical relations both for maximum approach and for compressing end expansion times, times having identical values for the elastic impact. Goldsmith provided the following relations:

$$t_{el} = \frac{x_{el}}{v_0} \int_0^1 \frac{1}{\sqrt{1 - x^{5/2}}} dx, \; x_{el} = \left[ \frac{5Mv_0^2}{4K} \right]^{2/5} \tag{22}$$

The tests for different values of COR close to unity and for a division of approaching distance in 20,000 points proved that the closest value to $t_{el}$ is the one obtained with the third hypothesis. For a value for coefficient of restitution $e = 0.999$, for the three alternatives of extrapolation, the following results were obtained: $t' / t_{el} = 1.041884$, $t'' / t_{el} = 1.024045$, $t''' / t_{el} = 0.999574$. By reporting to $t_{el}$ the times obtained, the dimensionless time obtained facilitates the comparison and thus, the third alternative $t_m \equiv t'''$ was chosen.

For the detaching phase, the equation can be integrated without difficulty using Euler method.

$$for\ k = 0,\ t'_0 = t_m,$$
$$for\ k = 1:n,\ t'_k = t'_{k-1} + \frac{x'_k - x'_{k-1}}{v'_k} \tag{23}$$

The Lankarani-Nikravesh model has the disadvantage, as the authors mention, of being appropriate only for quasielastic collision, for $e > 0.85$. The attempt of applying the model for the cases of smaller COR, $e < 0.85$, leads to contradictory aspects. The coefficient of restitution found based on relation (1), $e_{out} = v_f / v_0$, where $v_f$ is the relative velocity of contacting points for complete detaching, is greater than the initial coefficient and the difference between these two values increases with increasing internal friction.

Recently, Flores, [11] reached the conclusion that the Poincarè map of the model must be an ellipse. Following this conclusion, he proposes a new form for equation (1), to ensure for the coefficient of restitution $e_{out}$ identical values to the initial coefficient of restitution considered initially for the calculus of the coefficients of equation, namely:

$$F = Kx^{\alpha}\left[1 + \frac{8(1-e)}{5e}\frac{\dot{x}}{v_0}\right]. \tag{24}$$

In equation (24) the parameters have the same meaning as the ones from equation (1). The equation proposed by Flores differs from the equation proposed in [8] only by the values of coefficients and therefore, it can be integrated by the methodology described above.

## 4. Results

In order to validate the proposed method, there are presented the hysteresis curves and the impact force variation, for two values of impact coefficient, and the plot of dependence between the coefficient of restitution $e_{out}$ given by equations (1) and (24) as functions of the initial coefficient of restitution, $e_{in} \equiv e$.

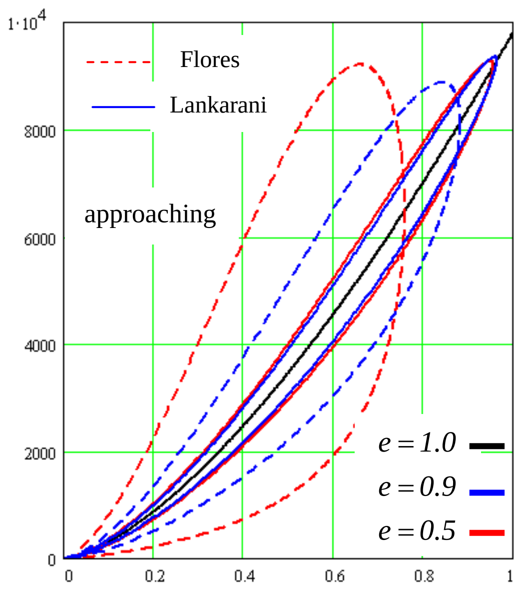


**Figure5**. Hysteresis loops.

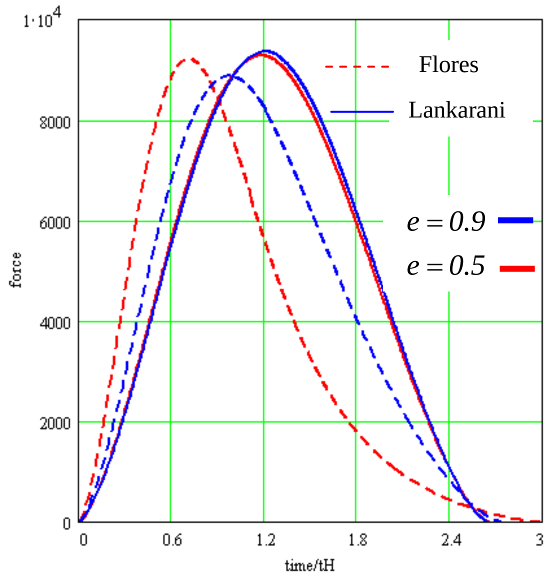


**Figure 6**. Impact forces versus time.

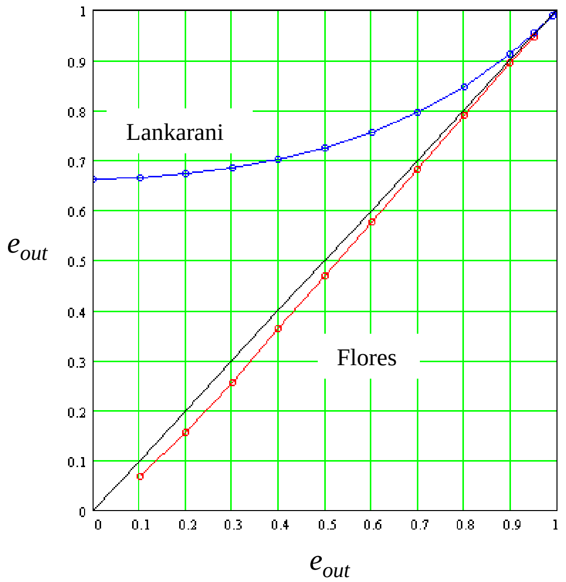


**Figure 7.** Dependences $e_{out} = e_{out}(e_{in})$.

The plots from figure 7 are identical to the ones presented by Flores and this confirms the exactness of the integration method for the equation.

## 5. Conclusions

The current paper presents a method for integration of the equation describing the dumped impact of two spheres.

The equation of the model is a second order nonlinear differential equation. The method for integration consists in two steps: first, finding by analytical calculus a first integral, allowing for the calculus of maximum indentation. The second stage concerns the numerical integration of first order nonlinear equation. The integration is made separately, as the impact process is divided in two phases: attaching or compression and detaching or expansion. The numerical procedure is based on Euler method. The precision of the results can be improved by an efficient domain partitioning for integration.

Finally, the variations with time for all collision characteristic parameters are obtained. The force variation in time is asymmetrical for approaching and detaching domains, and its maximum decreases with increasing coefficient of restitution.

The model presents a hysteresis loop, closed in the origin.

The agreement between the results obtained in the present work and the results from literature confirms the suitability of the integration method.

## Acknowledgments

This paper was supported by the project "Progress and development through post-doctoral research and innovation in engineering and applied sciences– PRiDE - Contract no. POSDRU/89/1.5/S/57083", project co-funded from European Social Fund through Sectorial Operational Program Human Resources 2007-2013.

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