

Alternate model of Chladni figures for the circular homogenous thin plate case with open boundaries

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Abstract. The wave equation is a direct but a complex approach to solve analytically for the Chladni figures, mainly because of the complications that non-smooth and open boundary conditions impose. In this paper, we present an alternate solution model based on the principle of Huygens-Fresnel and on the ideas of Bohr for the hydrogen atom. The proposed model has been implemented numerically and compared, with good agreement, to our own experimental results for the case of a thin homogenous circular plate with open boundaries.

1. Introduction

The Chladni experiment is commonly used as an undergraduate visualization of acoustical stationary waves in a plate [1], but the models that describes this figures also serves in other wave-like phenomena like quantum billiards [2]. The classical demonstration consists of harmonically exciting an elastic plate that has been sprinkled with sand or other granular media. The excitation will propagate in the plate and under certain conditions; it will form a stationary wave which is commonly represented mathematically as a separation of the wave equation solution into a purely temporal, and a purely spatial product. The effect on the plate is that there will be constant regions of zero amplitude and others zones vibrating at some intensity. The vibrant regions will propel the granular media in all directions until it falls in a region of zero amplitude maintaining that position. Because we are dealing with a stationary wave, the media will maintain a fixed pattern: the Chladni figure.

The standard way to analyze this type of figures is to solve the inhomogeneous Helmholtz equation using the appropriate boundary conditions. However, this approach is hard to implement, especially in the case of vibrating plates with non-smooth (irregular) open boundaries. The difficulty of this problem is reflected in the literature where most methods only solve for the plate irregularity. For instance, Amore presents a method for solving the Helmholtz equation using what he call “little sinc functions”; his method works for irregular and/or inhomogeneous membrane with fix boundary conditions only [3]. Muller [4] and Gander [5] utilizes finite element methods to find the eigenvalues of the homogenous Helmholtz equation; this approach excellent reproduce the Chladni figures for the rectangular plate case but the final product requires the use of a computer to make it useful.



The above examples are to show the difficulty of finding an analytical expression that describes Chladni figures. Therefore in this paper we present an alternate method based on the Huygens principle of wave propagation and on ideas from the Bohr postulate of quantized angular momentum for the hydrogen atom. The method was applied to the well-known case of the circular plate with excitation at the origin; then compared to familiar experimental results obtaining an acceptable level of agreement. We expect that this method can be extended to other geometries, irregularities and excitation point location.

The distribution of this paper is the following: In section 2, we give a brief view of the Huygens-Fresnel principle and the Bohr atom postulates. In section 3, we describe the postulates of the proposed model and apply them to describe the Chladni figures of the circular plate with excitation at the center. In section 4, we show the numerical implementation of the model and compare it to our own experimental results. Finally, in section 5, both conclusions and future work are presented.

2. Huygens-Fresnel Principle and Bohr postulate

In order to better expose the alternate model; in this section we concisely describe the principles and postulate that serve as a basis for our prototype.

2.1. Huygens-Fresnel Principle[6]

In an effort to explain reflection, refraction, interference and diffraction of light, Huygens proposed in 1678 that every point in a wave-front can be considered as a source of secondary waves that propagate forward at speed of light; the following wave-front is constructed from the envelop of all these secondary waves.

Afterwards in 1816, Fresnel introduced an obliquity factor to the Fresnel principle that gives preponderant amplitude to the secondary waves in the direction of propagation. In order to achieve better experimental verification, Fresnel also made some assumptions about the amplitude and phase of the secondary waves. Summing all this up, the mathematical expression of the Hugins-Fresnel model can be written:

$$\psi(\vec{r}) = \frac{ik}{2\pi} \int_S \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} F(\theta) dr', \quad (1)$$

where ψ represents the amplitude of the wave at \vec{r} , k is the wave number, $F(\theta)$ is the Fresnel factor and A is the amplitude at the origin \vec{r}_0 . It is interesting to note that the now called Huygens-Fresnel model predicted a bright spot of light in front of an opaque circular plate when illuminated from the back, this prediction caused the rejection of the model by the scientific community until was proved experimentally.

2.2. The Bohr atom[7]

At the beginning of the past century, it was known that matter is composed of negative and positive charged particles. Rutherford found experimentally that the positive charges where gather in dense clusters named nuclei. What followed was several models to explain how the negative charge is distributed respect the nuclei in order to conceive a minimum unit of matter called atom.

Bohr proposed a model where the electrons spin around the nuclei analogous to the planets around the sun. Even that this was a simple and logic model constructed from the Coulomb and Newton laws; it had an error based on Maxwell equations: an accelerated electron should emit radiation, making the electron lose energy and fall in to the nucleus.

In order to skip this problem, Bohr postulated that the electron could only spin around the nuclei in certain orbits in which it does not irradiate energy. These orbits are such that the angular momentum of the electron is an integer number of the Planck constant h . If we express

this postulate in terms of the linear moment p and substitute the Debroglie relation $\lambda = 2\pi\hbar/p$, then the permitted orbit of radius r follows

$$2\pi r = n\lambda, \quad (2)$$

where λ is the electron wavelength and n an integer number. From the expression above, we can see that the Bohr postulate implies that the permitted orbits are the ones that have a circumference equal to an integer of the electron wavelength. This is the idea that we are going to use in the proposed model.

3. The alternate model

As described before, Chlandi figures are stationary solutions to the wave equation, which is linear. This implies that the sum (superposition) of independent solutions can be used to construct a general solution. Therefore, we can propose a model as a linear combination of solutions that obey the following postulates:

1. The solution is composed in part of a divergent (principal) and convergent circular waves. The first wave travels from the excitation point up to the outer edge of the plate and the second wave, travels back with a phase shift proportional to the acoustical length path.
2. The principal wave excites the plate border creating secondary waves at every point around its perimeter. Every secondary wave reflects back in a direction perpendicular to the plate border following the Huygens-Fresnel principle.
3. The only frequencies that produce Chladni figures are those that form stationary waves around the perimeter of the plate in similar way to the stationary waves in the Bohr atom. This stationary wave also modulates the amplitude of the secondary waves of postulate 2. The sum of all these waves are what forms the Chladni figures.

The first postulate comes from the phenomenon of wave propagation and reflection by change of media. This is the case for Chladni plates, where a portion of the wave is transmitted to the air and the rest reflected back to the source.

The second postulate is a direct application of the Huygens-Fresnell principle with the consideration that the secondary waves are traveling back, perpendicular to the border perimeter and phase shifted in concordance to the distance from the excitation point.

The first two postulates are not sufficient because they would allow the formation of Chlandi patterns for any excitation frequency. Consequently, the third postulate limits the formation of Chladni patterns to excitation frequencies that yield stationary waves around the plate. In addition, the stationary wave modulates the amplitude of the secondary waves of postulate 2 at the border.

3.1 Circular plate

In this paper, we are going to apply the above postulates to a thin circular metal plate of radius R and excitation at the center. Let ψ represents the amplitude at any point inside the plate, then the mathematical expression in polar coordinates (r, θ) of the proposed model for this case reads

$$\psi(r, \theta) = \frac{A}{r} e^{ikr} + \frac{A}{r} e^{-ik(r+\varphi)} + A \int_0^{2\pi} \frac{e^{ik|\vec{r}-R\hat{\theta}|}}{|\vec{r}-R\hat{\theta}|} \cos(kR\theta) d\theta, \quad (3)$$

where A is the amplitude of the wave excitation point, r_0 , k is the wave number permitted by postulate 3, φ is the phase change mentioned in postulate 2. The first and second term represents the outgoing and reflected wave respectively. Observe that the reflected wave have a

phase shift φ , and the third term represents the infinite sum of waves around the plate modulated by an stationary wave represented by the cosine function.

4. Numerical implementation and experimental validation

The model presented in equation (3) was numerically implemented in MATLAB software and the code is available at the minimal request. The algorithm starts by first constructing a generalized circular wave function that represents a complex circular wave having the wave number, phase and origin as inputs. Second, we deploy and add two of these functions into a matrix, both with the same wave number and origin at the center, but with a phase difference φ between them.

Third, we use a cycle to sum multiple circular wave functions each having as origin a point in a circle subscribed on to the matrix. The amplitude of these outer waves is modulated by a cosine function with an argument that complies with postulate 3 with an angle measured respect the center of the matrix. The Fresnel factor was not implemented at this moment.

4.1. Experimental setup

The experimental setup consisted of a 15 inches radius (R) 20ga laminated plate placed over a mechanical vibrator (PASCO SF-2394), which was driven by a function generator (PASCO PI-9587). The procedure consisted in sprinkling common salt in the plate and sweeping the frequency of the function generator until a constant pattern was obtained.

4.2. Results

The literature describes that the Chladni figures for the circular plate consist on a combination of radial and concentric nodes. The following figure shows the case of one of these possible combinations; other cases are similar. The simulation are for $\lambda = 0.45R$, $n = 2$ and the correspondent experiment is shown to the right.

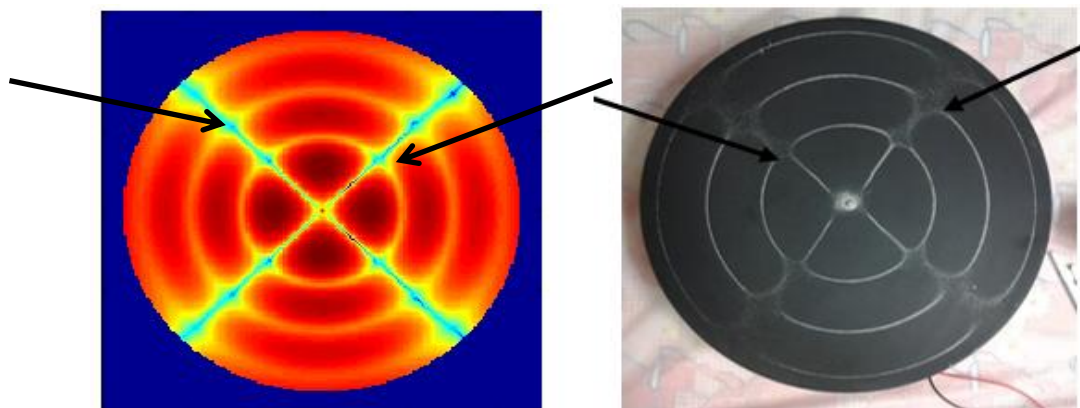


Figure 1. Simulation (left) and experiment (right) comparison, observe the similarity of the features where the radial and concentric nodes meet.

We can observe that the model reproduces with good level of accuracy the radial and concentric nodes. Furthermore, notice how the model also replicates the small features (arrows) that appear on the intersection of the nodes.

On figure 2, we observe a ripple effect on the experiment that we can reproduce on the simulation, in a comparable way, if we consider a non-integer value for n . If this is the case, we can consider this anomaly as a quasi-stationary case.

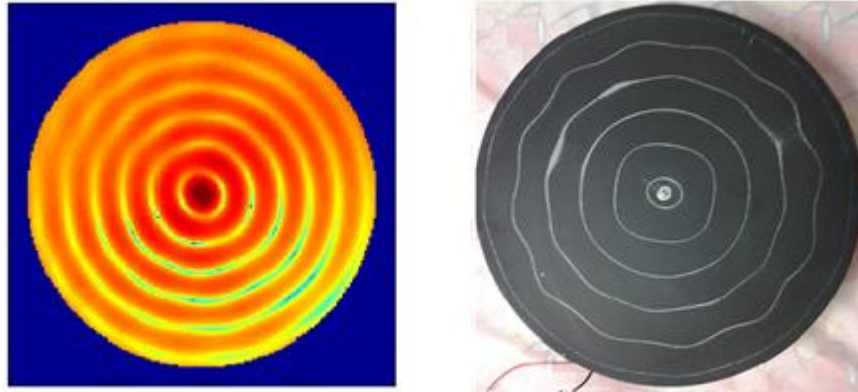


Figure 2. A possible explanation of the experimental ripple pattern (right) is a quasi-stationary state for $n = 1/5$ and $\lambda = 0.3R$, as shown in the simulation (left).

5. Conclusion and future work

We present a new model that reproduces the Chladni figures for the thin homogenous circular plate. The model uses a superposition of Huygens principles and ideas of the Bohr atom to obtain a mathematical expression. This expression was numerical implemented and confirmed experimentally with an adequate level of accuracy as it exhibited the radial and concentric nodes characteristic of the Chladni circular plate, and even reproduced the features at the intersection of the two types of nodes. The model also seemed to replicate a ripple patten observed experimentally.

In a future work, we are going to test the model for other geometrical figures such as conic sections and also for non-homogenous plates. In addition, the implementation of the Fresnel factor on the numerical simulation will be addressed.

References

- [1] Rossing T 1981 *Am. J. Phys.* **50** 271
- [2] Libisch, Rotter and Burgdörfer *Eur. Phys. J. Special Topics* **145** 245
- [3] Amore P 2008 *J. Phys. A: Math. Theor.* **41** 265206
- [4] Müller T 2013 *Eur. J. Phys.* **34** 1067
- [5] Gander and Kwok, *SIAM Rev.* **54** 573
- [6] Born M and Wolf E 1989 *Principle of Optics* (Pergamon Press) p.370
- [7] Beiser A 1995 *Concepts of Modern Physics* (McGraw-Hill, Inc) p.129
- [8] Ventsel and Krauthammer *Thin plates and shells Theory Analysis and Application* (Marcel Dekker, Inc.) p.106