Asymptotic solution of natural convection in a uniformly Joule-heating shallow cavity

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Abstract. The steady laminar two-dimensional Joule heating natural convection is investigated using asymptotical analysis, the fluid is in a rectangular cavity, the direct current contributes heat for heating the process medium by a pair of plate electrodes, the top wall is cooled with atmosphere and all the other walls are kept thermally insulated. The asymptotic solution is obtained in the core region in the limit as the aspect ratio, which is defined as the ratio of the vertical dimension of cavity to the horizontal dimension of cavity, goes to zero. The numerical experiments are also carried out to compare with the asymptotic solution of the steady two-dimensional Joule heating convection. The asymptotic results indicate that the expressions of velocity and temperature fields in the core region are valid in the limit of the small aspect ratio.

1. Introduction

Application of Joule heating in engineering can be found in various industrial processing,such as melting glasses in electric glass melters and for heating molten slag in electro-slag remelting process applied in the steel industry and nuclear waste treatment technology. Two types of body forces prevail in an electrically conducting liquid under Joule heating, they are the gravitational body force due to the non-uniform temperature field and the electromagnetic body force due to the interaction between self-induced magnetic field and moving charge carriers in the liquid.

With the development of the computer technology, many numerical simulations of Joule heating convection have been performed for rectangular geometries [1,2], However, to our knowledge, few asymptotic or analytical solutions of Joule heating convection have been reported, because the governing equations are nonlinear and coupled. The asymptotic or analytical solutions are very useful even for the newly rapidly developing computational fluid dynamics.

The purpose of this paper is to use the asymptotic analysis to examine convection and obtain an approximate analytical solution of the velocity and temperature field in uniformly Joule-heating shallow cavity.

2. Mathematical formulation of the problem

Consider the system described in Ref. [2] for simplicity, the fluid in the cube is heated by the electrode pair, which is assumed to be isopotential surfaces with an externally applied potential across them. All the boundaries of the pool are solid–fluid interfaces.

Mathematical model describing the state of an electrically conducting liquid pool in a cavity involves model equations for fluid flow, heat transfer and Maxwell's equations describing the electromagnetic field. Furthermore, if we examine the ratio $Ha^2/(Ra/Pr)$ we find that:

$$\frac{Ha^2}{Ra/\Pr} = \frac{\sigma\mu(\nu/L)}{\nu/\alpha}\cdot\frac{c_P\beta}{Lg} = \frac{\mathrm{Re}_m}{\Pr}\cdot\frac{c_P\beta}{Lg}$$

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For the type of applications envisioned here Re_m (the magnetic Reynolds number, which can be defined as the ratio of convective to diffusive transport of electromagnetic field) is order of 10^{-4}, one then needs to estimate the magnitude of ratio $(cp/\beta)/(gL)$, it gives a measure of the extent of buoyant force per unit temperature change for given thermal properties of the pool material, in order to appreciate the importance of the electromagnetic forces in the momentum balance, this parameter depends on the properties of the pool material and needs to be quite large in order for the effect of the induced magnetic field to be felt in the momentum balance. Otherwise it makes no sense to drop the effect of the induced magnetic field in Ohm's law while keeping it in the momentum balance.

Consequently if we define $U = \dfrac{\partial \psi}{\partial Y} \quad V = -\dfrac{\partial \psi}{\partial X}$,

Taking A as the aspect ratio $A=L/W$, Equations can be written as

$$\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} = -\omega$$

$$\frac{\partial \psi}{\partial Y}\frac{\partial \omega}{\partial X} - \frac{\partial \psi}{\partial X}\frac{\partial \omega}{\partial Y} = \frac{\partial^2 \omega}{\partial Y^2} - \frac{\partial^2 \omega}{\partial X^2} + \frac{Ra}{Pr}\frac{\partial \theta}{\partial X} \tag{1}$$

$$\frac{\partial \psi}{\partial Y}\frac{\partial \theta}{\partial X} - \frac{\partial \psi}{\partial X}\frac{\partial \theta}{\partial Y} = \frac{1}{Pr}\left(\frac{\partial^2 \theta}{\partial Y^2} + \frac{\partial^2 \theta}{\partial X^2}\right) + \frac{1}{Pr}\frac{1}{A}$$

3. Core flow and end region flow

With introduction of the characteristic horizontal scale $X=O(A^{-1})$, and

$$\left.\begin{array}{l} \theta = \theta_0 + A\theta_1 + A^2\theta_2 + \cdots, \\ \psi = \psi_0 + A\psi_1 + A^2\psi_2 + \cdots, \\ \omega = \omega_0 + A\omega_1 + A^2\omega_2 + \cdots, \end{array}\right\} \tag{2}$$

The systematic solution, valid for $A<<1$, i.e.

$$\psi = K_1\left(\frac{1}{24}Y^4 - \frac{1}{12}Y^3 + \frac{1}{24}Y^2\right) \tag{3}$$

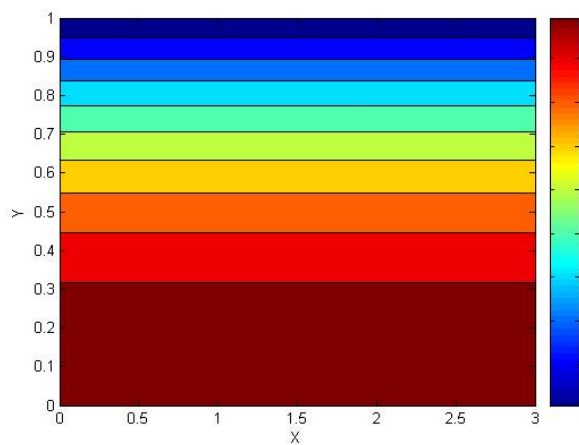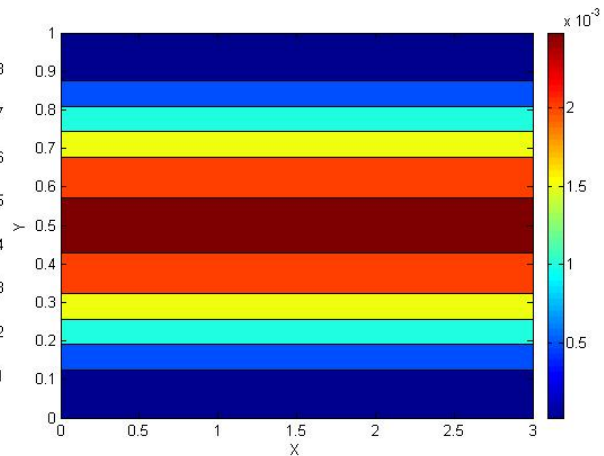$$\omega = -K_1\left(\frac{1}{2}Y^2 - \frac{1}{2}Y + \frac{1}{12}\right) \tag{4}$$

$$\hat{\theta} = -K_1\frac{Pr}{Ra}\hat{X} - \frac{1}{2}AX^2 + K_1^2 A\frac{Pr^2}{Ra}\left(\frac{1}{120}Y^5 - \frac{1}{48}Y^4 + \frac{1}{72}Y^3\right) + A^2 \tag{5}$$

Where $\hat{X} = AX, \hat{\theta} = A\theta$.

As in the core region, the solution can be obtained as a regular perturbation expansion in the small parameter A of the form, when $X \to \infty$, the solution is $\theta_0 = -Y^2 + A$, and

$$\lim_{X \to \infty} \psi_1 = -\frac{3}{2}\frac{Ra}{Pr}\left(\frac{1}{24}Y^4 - \frac{1}{12}Y^3 + \frac{1}{24}Y^2\right)$$

In order to clarify the physics meaning of the solutions above, we have plotted the isotherms and streamline contours in Fig. 1 and 2 from the above solutions. It should be noted that all of the results are plotted as square figures, hence, for cavities with small aspect ratios the horizontal length scale is substantially compressed. Fig. 1 reveals that the maximum liquid temperature occurs at the adiabatic bottom wall, this supports the present physics observation of internal heating condition, the upper part, the lower temperature. Centro-symmetry property ($Y=0.5$) is shown in Fig. 2 which is anticipated the convection roll in the cavity from physics point.

Fig. 1 At $O(1)$ isotherms $X = \infty$

Fig. 2 Streamlines $X = \infty$

4. Numerical simulation and comparison

Numerical experiments are carried out to compare simulation results with the asymptotic solution of the steady two-dimensional Joule-heating convection. The validation of the code for the simulation was performed in our previous works [2], and there is no need to repeat here.

Fig. 3 is a plot of the theoretically and numerically determined velocity profiles u/K_1 on the centre-line $X = 1/2A$ as a function of Y. The numerical data are expressed with the dotted line and solid line is for the theoretical result in the figure.
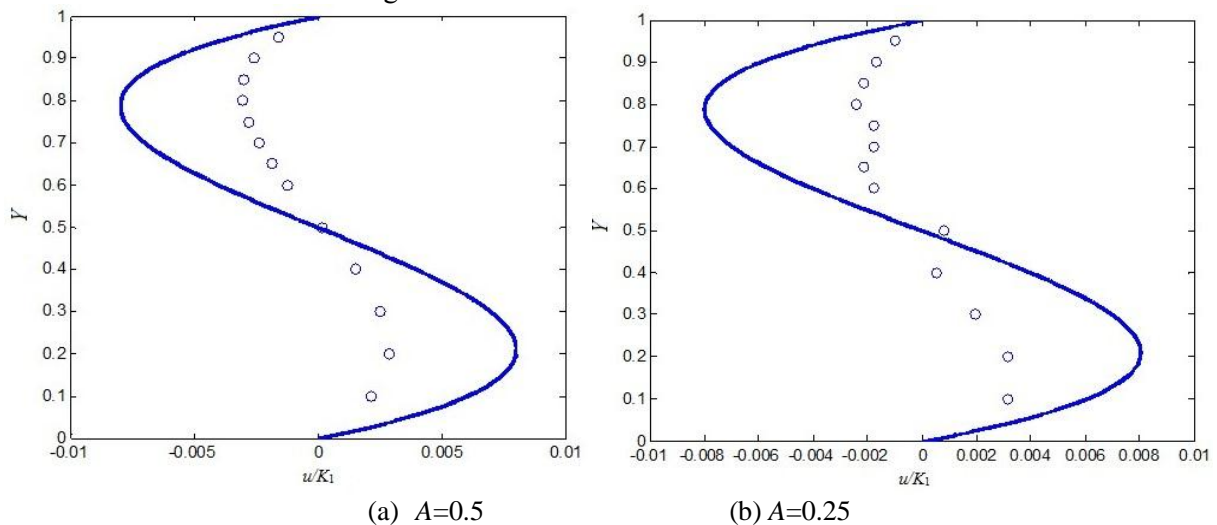

(a) A=0.5   (b) A=0.25

Fig. 3 Comparison of normalized horizontal velocity at X=1/2A with theoretical parallel-flow profile (solid line). Ra=15000 ,Pr=1

For both cases, the analytical solution and numerical results present the same symmetry characteristics, from the asymptotic profile, the flow obeys to asymmetry by the vertical mid-plane of the cavity, the horizontal velocity is zero on this plane. The flow field consists of anti-clockwise and clockwise convection rolls in the left and right cavity respectively. It is found that the asymptotical solutions are not suitable in the Y=0.2 and Y=0.8 plane, In the asymptotical solutions, the terms $O(A^2)$ are neglected, therefore, the error between the asymptotical solutions and the simulation results increases. Further consideration of core solution of equation (5) indicates that a plot $\theta + K_1 \dfrac{Pr}{Ra} X$ as a function of Y yield the fifth-order polynomial $f(Y)$.Fig. 4 is a plot of this function evaluated at the centre-line 1/2A.

Predicted isotherms show that the isotherms are more uniform close the liquid surface which means the conduction mechanism dominates. The numerical data in lower part of cavity is somewhat different from the theoretical profile, as the temperature in the lower part of cavity is not linear. In the asymptotical solutions, in order to get K_1 the term of fifth order of temperature is neglected, therefore, it is found that there is error between the asymptotical solutions and the simulation results.
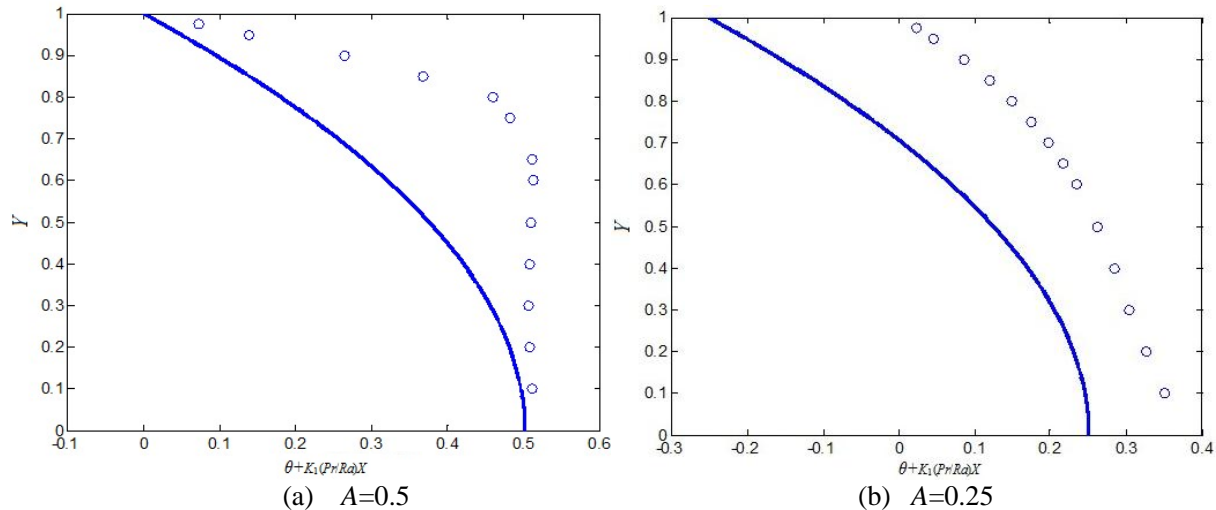

(a) A=0.5   (b) A=0.25

Fig. 4 Comparison of numerically determined temperature Θ at X= 1/2A with theoretical core temperature (solid line).

5. Conclusion

Approximate solutions of steady laminar two-dimensional Joule heating natural convection are obtained in the core region in the limit as the aspect ratio goes to zero.

Numerical experiments are carried out to compare simulation results with the asymptotic solution under the same conditions.

The asymptotical solutions show that the temperature and the velocity in the core region have the same trend as the numerical solutions respectively.

The asymptotical solutions show the flow patterns and isotherms are steady symmetric with respect to the mid-plane of cavity, which have agreement with the simulation results and physics meaning of convection.

The reason of discrepancies between the asymptotic results and the numerical simulations is that the higher order terms are neglected in the asymptotical solutions.

References:

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