

# Application of nonlinear methods to the study of ionospheric plasma

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**Abstract.** Most of the processes taking place in the auroral region of Earth's ionosphere are reflected in a variety of dynamic forms of the aurora borealis. In order to study these processes it is necessary to consider temporary and spatial variations of the characteristics of ionospheric plasma. Most traditional methods of classical physics are applicable mainly for stationary or quasi-stationary phenomena, but dynamic regimes, transients, fluctuations, self-similar scaling could be considered using the methods of nonlinear dynamics. Special interest is the development of the methods for describing the spatial structure and the temporal dynamics of auroral ionosphere based on the ideas of percolation theory and fractal geometry. The fractal characteristics (the Hausdorff fractal dimension and the index of connectivity) of Hall and Pedersen conductivities are used to the description of fractal patterns in the ionosphere. To obtain the self-consistent estimates of the parameters the Hausdorff fractal dimension and the index of connectivity in the auroral zone, an additional relation describing universal behavior of the fractal geometry of percolation at the critical threshold is applied. Also, it is shown that Tsallis statistics can be used to study auroral ionosphere

## 1. Introduction

Dynamic modes, transients, fluctuations, self-similarity in open dissipative systems which include the magnetosphere-ionosphere system may be considered by the methods of nonlinear dynamics. Therefore, the development and application of investigation methods of nonlinear processes characteristics in such systems is now an actual task, particularly for the study of auroral structures of the Earth's ionosphere. Special interest is the development of methods for describing the spatial structure and temporal dynamics of the auroral zone based on the concepts of percolation theory and fractal geometry [1]. Over the past two decades, the fractal description is convenient and productive tool for solving various physical problems including problems of space physics, for instance, the study of processes on the Sun, solar wind, near-Earth stretched tail, the auroral structures. The main advantage of the fractal method is the versatility of the results and the independence of the nature of fractal structures in the region under consideration. In physical problems, the estimations of fractal characteristics of real objects are understood as the characteristics of the scaling properties of these objects which are valid for a certain range of scales. Also, Tsallis non-extensive statistical theory (q-statistics) [2], introduced as a extension of Boltzmann - Gibbs statistics, was recently used for the description of complex process in the



space plasma dynamics. For these reasons, in the present short review to describe of properties of auroral ionosphere, we use a geometric approach based on fractal theory and percolation theory as well as q-statistics. It is demonstrated that nonlinear methods are perspective and useful approach for studying near-Earth plasma.

## 2. Fractal model of ionospheric conductivity

We consider the E-region of nightside Earth's ionosphere at altitudes of 80 – 150 km and at latitudes where the major part of energetic particles precipitation is observed and these particles result in auroras. In this region, particle precipitation is the main cause of ionization in the nightside and, consequently, of increased conductivity. In E-layer of ionosphere, the electrons are magnetized and the ions are nonmagnetized. In this case, the expressions for the Hall and Pedersen conductivities are simplified and take the following form:  $\sigma_P = q^2 n / m_i \nu_{in}$  and  $\sigma_H = qn/B$  respectively.

The spatial fluctuations of the conductivity are described under the assumption that, according to experimental data, they obey a power-law distribution. We consider the ionosphere to be a "spongy" fractal medium the structure of which in the absence of other ionization sources is determined by the structure of precipitation of auroral particles. The rate of ionization caused by particle precipitation varies smoothly along the magnetic field lines; therefore, the nontrivial fractal structure can form only in the spatial distribution transverse to the magnetic field [3, 4]. Locally, in a given field line, the conductivity is determined by the parameters of the precipitating electron flux. Here we consider a quasi-stationary case assuming that changes of precipitation structure occur more slowly than a recombination processes in the considered layer of the ionosphere.

Diffusion processes on percolating fractal structures are significantly non-Gaussian. Due to holes, constrictions on the fractal set, the motion of a particle on it can change. Moreover, since the holes reveal themselves on a range of scales, a change takes place on all these scales. In order to evaluate this effect, we estimate both the Pedersen conductivity and Hall conductivity by means of a characteristic transverse length scale  $a$ . The minimum possible value of  $a$  is determined by the mean free path of the particles in the auroral zone ionosphere. It is necessary to know the fractal dimension  $d^f$  and the connectivity index of  $\theta$  to describe the auroral ionosphere. We use the expression for the coefficient of turbulent transport that describes the kinetics of random processes on fractal geometry [1]:

$$D \sim \frac{\langle r^2 \rangle}{\tau} \left( \frac{\tau}{t} \right)^\mu \quad (1)$$

In the expression (1),  $\tau = a/v_T$  is the collision time,  $\langle r^2 \rangle$  is the mean-square distance of the particle,  $\mu$  is the fractal exponent of diffusion, which is expressed as  $\mu = d_s/d^f = 2/(2 + \theta)$  where  $d_s$  is a spectral fractal dimension [5]. Fractal dimension  $d_s$  of a fractal set is the ratio of the Hausdorff dimension  $d^f$  to the minimal Hausdorff dimension of paths  $d_s = d^f/d_\theta = 2d^f/(2 + \theta)$ . The index  $\mu$  is in fact the inverse Hausdorff dimension of geodesic lines  $d_\theta$  [6], namely  $\mu = 1/d_\theta$ . To estimate the conductivity, we need an expression for the carrier concentration (density):  $n \sim a^{d^f-d}$ , where  $d$  is the Euclidean dimension. Thus, the Pedersen conductivity dependence of the characteristic spatial scale  $a_p$  (the mean free path of particle) becomes:

$$\sigma_p \sim \frac{q^2 n}{m_i \nu_{in}} \propto n \tau \sim n \frac{a_p}{D} \propto a_p^{d_P^f-d} * a_p * a_p^{-\mu+1} \sim a_p^{2-\mu+d_P^f-d} \quad (2)$$

Here,  $d_P^f$  is the fractal dimension that specifies the Pedersen current in (2), that is, the motion of ions;  $a_p$  is the characteristic length scale of the Pedersen current (mean free path of ions). In obtaining estimate (2), it is assumed that time for percolation case is estimated as  $\tau \propto a_p/D$  in the expression (2) rather than  $\tau \sim a_p^2/D$  which is valid for classical regular case.

Now we obtain an estimate for the Hall conductivity through the spatial scale  $a_H$ :

$$\sigma_H \sim \frac{qn}{B} \propto n \propto a_H^{d_H^f - d} \quad (3)$$

where  $d_H^f$  is the fractal dimension that characterizes the Hall current which is determined by the motion of electrons.

An important special case is the region of intense field-aligned currents. Since the field-aligned current  $j_{||}$  generates transversal currents in the ionosphere and the corresponding transverse electric fields  $E_{\perp}$ , we have  $j_{||} \simeq \nabla_{\perp}(\Sigma \cdot E_{\perp})$ . If  $a$  is the transverse characteristic scale, the change of potential drop in the ionosphere at this scale is  $\Delta\varphi_{\perp} \sim j_{||}a^2/\Sigma$ , and the change of field-aligned parallel potential drop is  $\Delta\varphi_{||} \sim j_{||}$ . Also, we use semi-empirical estimations of the conductivity depending on the flux and the kinetic energy of precipitating electrons [3]. Thus, we have second estimation (see [3] for detail information). Equating the exponents of the two estimates for the Pedersen conductivity, we obtain:  $\frac{1}{3}d_P^f - \frac{1}{3}\theta = 2 - \mu + d_P^f - d$ . If it is considered that the Euclidean dimension is  $d = 2$ , the consistency condition is one of two equations needed to unambiguously determine the parameters  $d_P^f$  and  $\theta_P$ . To close the system, criticality condition of percolation threshold (the Alexander-Orbach conjecture) ensuing from the universal value theorem can be taken as the second equation [5, 6]. Criticality condition relates uniquely the fractal parameters of the system: the Hausdorff dimension  $d^f$  and the connectivity index  $\theta$  near the nonequilibrium quasi-stationary states. The parameter  $\Lambda$  characterizes the geometry of the percolation transition and determines the minimal fractional number of degrees of freedom that a particle must have to pass through a region under consideration in the process of random walks. The set of points visited by the particle then forms a percolating fractal network with the spectral dimension  $d_s = \Lambda \approx 1,327$ . Criticality condition leads to the independence of the fractal geometric characteristics from microscopic properties of the medium. In this case, this condition allows us to estimate the critical values of fractal parameters that are necessary for the closure of the magnetosphere-ionosphere current system. This follows from the inequalities:  $(2d^f)/(2 + \theta) \geq \Lambda$ . Taking into account that  $\mu = d_s/d^f = 2/(2 + \theta)$ , we deduce for the fractal dimension  $d_P^f$  and the connective index  $\theta_P$ :  $d_P^f \geq 1.38$ ,  $\theta_P \leq 0.09$ . The obtained values fully describe the fractal geometry of Pedersen current near the percolation threshold. For percolation of Pedersen current, enhanced conductivity regions should fill in only a fractal subset with the fractal dimension  $1.38 \leq d_P^f \leq 2$ .

In the same way we find the estimates of fractal characteristics of the Hall conductivity in the ionosphere:  $d_H^f \leq 1.85$ ,  $\theta_H \leq 0.77$ . The fractal topology of the Hall current of auroral ionosphere region on the percolation threshold is given by obtained fractal parameters. It is interesting to note that the limit threshold fractal dimension  $d_H^f$  is very close to the Hausdorff dimension 1.89 of the set, titled "Cantor cheese" (two-dimensional "Cantor cheese" is also sometimes called "Sierpinski carpet"). Sierpinski carpet has remarkable property because it specifies the maximum Hausdorff dimension of the fractal set, all of whose points can be visited without self-crossings [6]. It should be noted that since the directions and heights of the Pedersen and Hall currents are different, the same precipitation can create structures with different dimensions for the Pedersen and Hall conductivity [4].

The obtained threshold values of the fractal Hausdorff dimension and the connectivity index are characteristic of a path-connected set for Pedersen conductivity. A set is called path-connected if along with any pair of points it also contains a path connecting these points [6]. The Hausdorff dimension of a path-connected fractal set has  $d^f \geq d_{\theta} \geq 1$  and  $\theta > 0$  that take place for Pedersen conductivity. Note that for fractal sets are not path-connected if the condition  $0 \leq d^f < d_{\theta} < 1$  holds as well as  $\theta < 0$ . For the Hall conductivity using different empirical relationships, there is such a state where a negative connectivity index  $\theta < 0$  but

fractal dimension  $d^f \geq 1$ . Such interesting class of fractal objects are called as asymptotically path-connected set [4, 6].

It is well known that evaluation of indices that characterize the self-similarity in the experiment is usually performed by one-dimensional series, namely, time series of measurements (for example, along the trajectory of the satellite) or section of a two-dimensional distribution (for example, the auroral glow). In these cases, we are dealing with a statistically self-affine series. Fractal properties of the self-affine series are commonly described by Hurst index  $H$ , which is associated with the fractal dimension of the set  $d^f$  as  $H = 2 - d^f$ . The index  $H$  defined by the power dependence of the increment variance on the scale of  $\sigma^2(\varepsilon) = |\varepsilon|^{2H}$  or by the power-law form of the power spectrum  $f^{-\alpha}$ , where  $f$  is the (spatial) frequency and  $\alpha = 2H + 1$ . Thus, the fractal structure characteristics of the Pedersen conductivity  $d_p^f \geq 1.38$  have following value:  $H_p = 2 - d_p^f \leq 0.62$  hence  $\alpha = 2H_p + 1 \leq 2.24$ . Values of the Hurst index  $H$  and the index  $\alpha$  obtained from the various experimental data in the auroral region (satellite and ground-based data) agree well with our theoretical fractal parameters (detailed study of these comparisons are given in [4]). Correspondence of analytical estimates and different experimental results indicates availability and validity of the fractal and percolation methods. It can be concluded that fractional calculus is an effective approach modeling of phenomena in auroral zone plasma induced by nonlinear processes.

### 3. q-statistics

We demonstrate for the first time that q-statistics [2] can be used to study the Earth's auroral region. In order to show that q-statistics may apply for study of auroral ionosphere, we use pulsating aurora event on 2011-12-03 from 22:00UT which were observed by Apatity all-sky camera [7]. The obtained data (where background was subtracted and bright stars were deleted) were divided into ten time intervals and for each interval we calculated the value of parameter  $q$ . The parameter  $q$  is estimated using maximum likelihood. Specifically the maximum likelihood estimates (MLEs) have been determined from a solution of the stationary points of the log likelihood function. Parameter  $q$  indicates a departure from Gaussian distribution. In the limit  $q \rightarrow 1$  we have usual Gaussian statistics.

Our study shows that when bright auroras are observed, there is a strong deviation from unity of the parameter  $q$ . Otherwise, the value of  $q$  tends to unity, that is, to Gaussian statistics. This is due to the fact that during auroral glow ionospheric processes are substantially non-Gaussian and strong intermittency occurs. Also, it is established a good correlation between the change in values of  $q$  and flatness. While aurora glows are observed, flatness and parameter  $q$  grow and vice versa. In fact, it is seen that the Tsallis statistics can be used to study of non-Gaussian process, intermittency, along with flatness and/or probability density function (PDF).

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