

# On the Representation Theory of the Ultrahyperbolic BMS group $\mathcal{UHB}(2, 2)$ . I. General Results

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**Abstract.** The Bondi–Metzner–Sachs (BMS) group  $B$  is the common asymptotic group of all asymptotically flat (lorentzian) space–times, and is the best candidate for the universal symmetry group of General Relativity (G.R.).  $B$  admits generalizations to real space–times of any signature, to complex space–times, and supersymmetric generalizations for any space–time dimension. With this motivation McCarthy constructed the strongly continuous unitary irreducible representations (IRs) of  $B$  some time ago, and he identified  $B(2, 2)$  as the generalization of  $B$  appropriate to the ‘ultrahyperbolic signature’  $(+, +, -, -)$  and asymptotic flatness in null directions. We continue this programme by introducing a new group  $\mathcal{UHB}(2, 2)$  in the group theoretical study of ultrahyperbolic G.R. which happens to be a proper subgroup of  $B(2, 2)$ . In this short paper we report on the first general results on the representation theory of  $\mathcal{UHB}(2, 2)$ . In particular the main general results are that the *all* little groups of  $\mathcal{UHB}(2, 2)$  are compact and that the Wigner–Mackey’s inducing construction is exhaustive despite the fact that  $\mathcal{UHB}(2, 2)$  is not locally compact in the employed Hilbert topology. At the end of the paper we comment on the significance of these results.

## 1. Introduction

The best candidate for the universal symmetry group of General Relativity (G.R), in any signature, is the so called Bondi–Metzner–Sachs (BMS) group  $B$ . These groups have been described [1] for all possible signatures and all possible complex versions of GR as well. The original BMS group  $B$  was discovered by Bondi, Metzner and Van der Burg [2] for asymptotically flat space–times which were axisymmetric, and by Sachs [3] for general asymptotically flat space–times, in the usual Lorentzian signature.

In earlier papers [1, 4, 5, 6, 7] it has been argued that the IRs of the BMS group and of its generalizations in complex space–times as well as in space–times with Euclidean or Ultrahyperbolic signature are what really lie behind the full description of (unconstrained) moduli spaces of gravitational instantons. Kronheimer [8, 9] has given a description of these instanton moduli spaces for *Euclidean* instantons. However, his description only partially describes the moduli spaces, since it still involves *constraints*. Kronheimer does not solve the constraint equations, but it has been argued [1, 7] that IRs of BMS group (in the relevant signature) give an *unconstrained* description of these same moduli spaces.

A terminological remark is in order. In an arbitrary dimension, an ultrahyperbolic metric is a metric with signature  $(s, t)$  or  $(t, s)$  with  $s \geq t$  and for which  $t \neq 0, 1$ . The cases of  $t = 0, 1$  are called Riemannian and Lorentzian (or hyperbolic) metrics respectively. In dimension four, there remains only the case  $(2, 2)$ . McCarthy in [1] introduced  $B(2, 2)$  as the BMS group appropriate



to the ‘ultrahyperbolic signature’ and asymptotic flatness in null directions. The representation theory of  $B(2, 2)$  was initiated in [4] and [10]. In the present paper we introduce a new group  $\mathcal{UHB}(2, 2)$ , unnoticed in [1], in the group theoretical study of Ultrahyperbolic G.R., and report some general results on its representation theory derived in [11]. We also mention in passing some more results on the representation theory of  $\mathcal{UHB}(2, 2)$  derived in [12] and [13].

The fact that  $\mathcal{UHB}(2, 2)$  is a proper subgroup of  $B(2, 2)$  does not render the study of the representations of  $\mathcal{UHB}(2, 2)$  superfluous. In general, the following holds: Let  $G$  be a group,  $H$  be a subgroup and  $T(g)$  be an irreducible representation of the group  $G$ . Let  $T_H(g)$  be the restriction of  $T(g)$  on the subgroup  $H$ . In general, the representation  $T_H(g)$  is *not* irreducible. Moreover, there are irreducibles of  $H$  which *cannot* be extended to the whole group  $G$ . An example of this phenomenon is actually provided by the original BMS group. The Poincare group  $P$  is a subgroup of the BMS group  $B$ . Not all irreducibles of  $P$  are obtained by restricting the irreducibles of  $B$  to  $P$  (here, it is understood that both  $B$  and  $P$  are endowed with the Hilbert topology). There are irreducibles of  $P$ , namely the continuous spin irreducible representations, which *cannot* be extended to the whole group  $B$ , which in the Hilbert topology has no continuous spin irreducibles, all its irreducibles carry discrete spin.

In Section 2 we define the group  $\mathcal{UHB}(2, 2)$  and sketch the inducing construction which we use in order to construct the IRs of  $\mathcal{UHB}(2, 2)$ . In Section 3 we state the main general results on the representation theory of  $\mathcal{UHB}(2, 2)$  and comment on their significance.

## 2. The group $\mathcal{UHB}(2, 2)$ and the inducing construction

Recall that the ultrahyperbolic version of Minkowski space is the vector space  $R^4$  of row vectors with 4 real components, with scalar product defined as follows. Let  $x, y \in R^4$  have components  $x^\mu$  and  $y^\mu$  respectively, where  $\mu = 0, 1, 2, 3$ . Define the scalar product  $x.y$  between  $x$  and  $y$  by

$$x.y = x^0 y^0 + x^2 y^2 - x^1 y^1 - x^3 y^3. \quad (1)$$

Then the ultrahyperbolic version of Minkowski space, sometimes written  $R^{2,2}$ , is just  $R^4$  with this scalar product.

In [11] it is shown that

**Theorem 1** *The group  $\mathcal{UHB}(2, 2)$  can be realised as*

$$\mathcal{UHB}(2, 2) = L^2(\mathcal{P}, \lambda, R) \circledast_T G^2 \quad (2)$$

with semi-direct product specified by

$$(T(g, h)\alpha)(x, y) = k_g(x) s_g(x) k_h(w) s_h(w) \alpha(xg, yh), \quad (3)$$

where  $\alpha \in L^2(\mathcal{P}, \lambda, R)$  and  $(x, y) \in \mathcal{P}$ . For ease of notation, we write  $\mathcal{P}$  for the torus  $T \simeq P_1(R) \times P_1(R)$ ,  $P_1(R)$  is the one-dimensional real projective space, and  $\mathcal{G}$  for  $G \times G$ ,  $G = SL(2, R)$ . In analogy to  $B$ , it is natural to choose a measure  $\lambda$  on  $\mathcal{P}$  which is invariant under the maximal compact subgroup  $SO(2) \times SO(2)$  of  $\mathcal{G}$ .  $L^2(\mathcal{P}, \lambda, R)$  is the separable Hilbert space of real-valued functions defined on  $\mathcal{P}$ .

Moreover, if  $g \in G$  is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad (4)$$

then the components  $x_1, x_2$  of  $\mathbf{x} \in R^2$  transform linearly, so that the ratio  $x = x_1/x_2$  transforms fraction linearly. Writing  $xg$  for the transformed ratio,

$$xg = \frac{(\mathbf{x}g)_1}{(\mathbf{x}g)_2} = \frac{x_1 a + x_2 c}{x_1 b + x_2 d} = \frac{xa + c}{xb + d}. \quad (5)$$

The factors  $k_g(x)$  and  $s_g(x)$  on the right hand side of (3) are defined by

$$k_g(x) = \left\{ \frac{(xb+d)^2 + (xa+c)^2}{1+x^2} \right\}^{\frac{1}{2}}, \quad (6)$$

$$s_g(x) = \frac{xb+d}{|xb+d|}, \quad (7)$$

with similar formulae for  $yh$ ,  $k_h(y)$  and  $s_h(y)$ .

It is well known that the topological dual of a Hilbert space can be identified with the Hilbert space itself, so that we have  $L^{2'}(\mathcal{P}, \lambda, R) \simeq L^2(\mathcal{P}, \lambda, R)$ . In fact, given a continuous linear functional  $\phi \in L^{2'}(\mathcal{P}, \lambda, R)$ , we can write, for  $\alpha \in L^2(\mathcal{P}, \lambda, R)$

$$(\phi, \alpha) = \langle \phi, \alpha \rangle \quad (8)$$

where the function  $\phi \in L^2(\mathcal{P}, \lambda, R)$  on the right is uniquely determined by (and denoted by the same symbol as) the linear functional  $\phi \in L^{2'}(\mathcal{P}, \lambda, R)$  on the left. The representation theory of  $\mathcal{UHB}(2, 2)$  is governed by the dual action  $T'$  of  $\mathcal{G}$  on the topological dual  $L^{2'}(\mathcal{P}, \lambda, R)$  of  $L^2(\mathcal{P}, \lambda, R)$ . The dual action  $T'$  is defined by:

$$\langle T'(g, h)\phi, \alpha \rangle = \langle \phi, T(g^{-1}, h^{-1})\alpha \rangle. \quad (9)$$

A short calculation gives

$$(T'(g, h)\phi)(x, y) = k_g^{-3}(x)s_g(x)k_h^{-3}(y)s_h(y)\phi(xg, yh). \quad (10)$$

Now, this action  $T'$  of  $\mathcal{G}$  on  $L^{2'}(\mathcal{P}, \lambda, R)$ , given explicitly above, is like the action  $T$  of  $\mathcal{G}$  on  $L^2(\mathcal{P}, \lambda, R)$ , continuous. The ‘little group’  $L_\phi$  of any  $\phi \in L^{2'}(\mathcal{P}, \lambda, R)$  is the stabilizer

$$L_\phi = \{(g, h) \in \mathcal{G} \mid T'(g, h)\phi = \phi\}. \quad (11)$$

By continuity,  $L_\phi \subset \mathcal{G}$  is a closed subgroup.

In the inducing construction, described in detail in [11], [12] and [13], attention is confined to measures on  $L^{2'}(\mathcal{P}, \lambda, R)$  which are concentrated on single orbits of the  $\mathcal{G}$ -action  $T'$ . These measures give rise to IRs of  $\mathcal{UHB}(2, 2)$  which are induced in a sense generalising [14] Mackey’s [15, 16, 17, 18, 19, 20]. Remarkably [11] this inducing construction gives rise to *all* the IRs of  $\mathcal{UHB}(2, 2)$  when  $\mathcal{UHB}(2, 2)$  is equipped with the Hilbert topology. In a nutshell the inducing construction is realized as follows: Let  $\mathcal{O} \subset L^{2'}(\mathcal{P}, \lambda, R)$  be any orbit of the dual action  $T'$  of  $\mathcal{G}$  on  $L^{2'}(\mathcal{P}, \lambda, R)$ . There is a natural homomorphism  $\mathcal{O} = \mathcal{G}\phi_o \simeq \mathcal{G}/L_{\phi_o}$  where  $L_{\phi_o}$  is the ‘little group’ of the point  $\phi_o \in \mathcal{O}$ . Let  $U$  be a continuous irreducible unitary representation of  $L_{\phi_o}$  on a Hilbert space  $D$ . Every coset space  $\mathcal{G}/L_{\phi_o}$  can be equipped with a unique class of measures which are quasi-invariant under the action  $T$  of  $G^2$ . Let  $\mu$  be any one of these. Let  $D_\mu = L^2(\mathcal{G}/L_{\phi_o}, \mu, D)$  be the Hilbert space of functions  $f : \mathcal{G}/L_{\phi_o} \rightarrow D$  which are square integrable with respect to  $\mu$ . From a given  $\phi_o$  and any continuous irreducible unitary representation  $U$  of  $L_{\phi_o}$  on a Hilbert space  $D$  a continuous irreducible unitary representation of  $\mathcal{UHB}(2, 2)$  on  $D_\mu$  can be constructed. The representation is said to be induced from  $U$  and  $\phi_o$ . Different points of an orbit  $\mathcal{G}\phi$  have conjugate little groups and give rise to equivalent representations of  $\mathcal{UHB}(2, 2)$ .

### 3. Results and their significance

The main general results on the representation theory of  $\mathcal{UHB}(2,2)$  proved in [11] are the following

- (i) All the little groups  $L_{\phi_o}$  of  $\mathcal{UHB}(2,2)$  are compact
- (ii) The Wigner–Mackey’s inducing construction is exhaustive

Regarding the first result we note that the all the little groups  $L_{\phi_o}$  of  $\mathcal{UHB}(2,2)$ , being compact, are up to conjugation subgroups of the maximal compact subgroup  $SO(2) \times SO(2)$  of  $\mathcal{G}$ . They include groups which are finite as well as groups which are infinite, both connected and not–connected [11]. Therefore the construction of the IRs of  $\mathcal{UHB}(2,2)$  involves at the first instance the classification of all the compact subgroups of  $SO(2) \times SO(2)$ . This task, far from being trivial, was undertaken in [12] and [13].

Regarding the second result we note that the Wigner–Mackey’s inducing construction is exhaustive despite the fact that  $\mathcal{UHB}(2,2)$  is not locally compact in the employed Hilbert topology [11]. This result is rather important because other group theoretical approaches to quantum gravity which invoke Wigner–Mackey’s inducing construction (see e.g. [21]) are typically plagued by the non–exhaustiveness of the inducing construction which results precisely from the fact that the group in question is not locally compact in the prescribed topology. Exhaustiveness is not just a mathematical nicety: If the inducing construction is not exhaustive one cannot simply know if the most interesting information or part of it is coded in the irreducibles which cannot be found by the Wigner–Mackey’s inducing procedure. These results, compactness of the little groups and exhaustiveness of the inducing construction, not only are they significant for the group theoretical approach to quantum gravity advocated here, but also they have repercussions for the other approaches to quantum gravity [11].

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