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Algorithms for Calculating Alternating Infinite Series

Hector Luna Garcia[1]**and Luz Maria Garcia**
Área de Física Atómica Molecular Aplicada, CBI, UAM-A, Av. San Pablo No. 180, Col. Reynosa, Azcapotzalco, México, D. F. 02200, México

E-mail: 5553972854.hl@gmail.com

Abstract. This paper are presented novel algorithms for exact limits of a broad class of infinite alternating series. Many of these series are found in physics and other branches of science and their exact values found for us are in complete agreement with the values obtained by other authors. Finally, these simple methods are very powerful in calculating the limits of many series as shown by the examples.

1. Introduction

In a previous paper [1] some novel algorithms which made use of polygamma functions for exact limits of a large branch of infinite series were introduced; moreover, the Laplace transform is used to find the sum of some of the infinite series. However, because of their importance in physics, remained pending deal the exact limits of alternating series which converge. The infinite series, including the alternate, are an important part of a course of mathematical analysis. In particular, alternating series $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}$ is convergent and is the Leibnitz definition for the irrational π number. To consider the infinite series in the literature is natural to introduce some convergence criteria due to Cauchy, Kummer, D'Alembert, and Gauss authors [2], which ensure us the convergence of infinite series. In addition, special forms of the term of the alternating series there are some criteria or tests that help quickly conclude its convergence: absolute convergence, ratio test and the root criteria [3]. However, the application of these criteria does not allow us to know their limits. In this paper, we present some algorithms to compute the exact limit of a broad class of alternating series which converge. This work is divided in four parts: This introduction, Section 2 illustrates the method; in Section 3 the method is generalized. Finally, Section 5 presents conclusions and future projects.

2. Special Cases of Alternantes Series

2.1 The infinite Series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{a^n}$.

This series is denoted as:

[1] Hector Luna Garcia, lghm@correo.azc.uam.mx.

$$S(b)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{b^n}, \tag{1}$$

where $b>1$ is an integer. Using the Feynman identity,

$$1/A=\int_0^{\infty}e^{-Ax}dx\ , \tag{2}$$

where $A>0$, the infinite series in equation (1) becomes

$$S\left(b\right)=\sum_{n=1}^{\infty}\int_0^{\infty}\left(-1\right)e^{-b^n x}dx. \tag{3}$$

Introducing the sum into the integral in equation (4), and changing $y=e^{-b^n x}$ variables, one has

$$S\left(b\right)=\int_1^0\sum_{n=1}^{\infty}\left(-\frac{1}{b}\right)^n dy=\int_1^0\lim_{N\to\infty}\left[\left(-\frac{1}{b}\right)-\left(-\frac{1}{b}\right)^N\right]\left[1-\left(-\frac{1}{b}\right)\right]^{-1}dy\ . \tag{4}$$

Considering now $N\to\infty$, the result is

$$S\left(b\right)=\int_0^1\frac{1}{\left(1+b\right)}dy=\frac{1}{\left(1+b\right)}. \tag{5}$$

Equation (5) is the most general expression result of this section. As an application, we present the following examples:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^n}=\frac{1}{3},\ \sum_{n=0}^{\infty}\frac{(-1)^{n}}{2^n}=\frac{2}{3},\text{ and, }\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{5^n}=-\frac{5}{6}.$$

2.2 The Infinite Series $\sum_{n=1}^{\infty}\frac{(-1)^n}{a^{2n}}$

Following the same algorithm as in the previous subsection

$$S\left(2n,b\right)=\sum_{n=1}^{\infty}\frac{(-1)^n}{b^{2n}}=-\frac{1}{\left(1+b^2\right)}. \tag{6}$$

As an application, the authors present the following examples

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{3^{2n}}=-\frac{1}{10}\text{, and, }\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}}=\frac{4}{5}.$$

2.3 The Infinite Series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n+a}$.

For this series denoted, $S(n,a)$ one has

$$S(a)=\sum_{n=1}^{\infty}(-1)^n\int_0^{\infty}e^{-(n+a)x}dx\ , \tag{7}$$

where we have used the Feynman's formula (2).
Then, interchanging the order of addition and integral, the equation (7) becomes

$$S(a)=\int_0^{\infty}e^{-ax}\sum_{n=1}^{\infty}\left(-e^{-x}\right)^n dx=\int_0^{\infty}e^{-ax}\lim_{x\to\infty}\left[\left(-e^{-x}\right)-\left(-e^{-x}\right)^N\right]\left[1-\left(-e^{-x}\right)\right]^{-1}dx\ . \tag{8}$$

In the limit as $N\to\infty$, the result is

$$S(a)=\int_0^\infty \left[e^{-ax}\left(-e^{-x}\right)\right]\left(1+e^{-x}\right)^{-1}dx, \tag{9}$$

Changing $z=e^{-x}$ variables, the equation (9) finally is expressed as:

$$S(a)=\int_1^0 u^a\left(1+u\right)^{-1}du\,. \tag{10}$$

At this point, it is important to mention that integral given by equation (9), cannot be written in terms of Eulerian functions [4] (polygamma functions), due to the plus sign in the binomial denominator. In contrast to those discussed in the Ref. [1] previously mentioned. However, the equation (9) is the most general expression result of this section. As one application, we present the following examples

$\sum_{n=0}^\infty \frac{(-1)^n}{2n+1}=\frac{\pi}{4}, \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}=\ln 2\,, \sum_{n=1}^\infty \frac{(-1)^{n+1}}{3n-2}=\frac{1}{3}\left(\ln 2+\frac{\pi}{\sqrt{3}}\right)$, and,

$$\sum_{n=1}^\infty \frac{(-1)^{n+1}}{4n-3}=\frac{1}{4\sqrt{2}}\left(\pi+2\ln\left(\sqrt{2}+1\right)\right).$$

This series is found in Ref [5].

3. Series of type $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{(n+b)^2}$.

This series is denoted as

$$S\left(2,b\right)=\sum_{n=1}^\infty \frac{(-1)^{n+1}}{(n+b)^2}\,. \tag{11}$$

The Feynman identity applied to this case is

$$1\big/A^2=\int_0^\infty xe^{-Ax}dx\,. \tag{12}$$

Following the same procedure as was used in section 2, leads to the following result:

$$S(2,b)=\int_0^\infty \left[xe^{-(a+1)x}\right]\left(1+e^{-x}\right)^{-1}dx=\int_1^0 \left(\ln z\right)z^a\left(1+z\right)^{-1}dz\,, \tag{13}$$

where $z=e^{-x}$, the variable change is used.
Applying equation (13) the already reported result is obtained

$$\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}.$$

3.1 Series of type $S(2k,n)=\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^{2k}}$, with $k=1,2,...$.

The Feynman identity applied to this case is

$$1\big/A^m=\tfrac{1}{(m-1)!}\int_0^\infty x^{m-1}e^{-Ax}dx\,. \tag{14}$$

Following the same procedure as was used in section 2, leads to the following result:

$$S(2k,n)=\tfrac{1}{(2k-1)!}\int_0^\infty x^{2k-1}e^{-x}\left(1+e^{-x}\right)^{-1}dx=\tfrac{1}{(2k-1)!}\int_0^1 \left(-\ln z\right)^{2k-1}\left(1+z\right)^{-1}dz\,. \tag{15}$$

Applying equation (15), we have

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12},$$

whose value was found in section 2. As another examples, we find

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4}=\tfrac{1}{6}\int_1^0(\ln z)^3(1+z)^{-1}\,dz=\tfrac{1}{6}\left(\tfrac{7\pi^4}{120}\right)=\tfrac{7\pi^4}{720},$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^6}=\tfrac{1}{120}\int_1^0(\ln z)^5(1+z)^{-1}\,dz=\tfrac{1}{120}\left(\tfrac{31\pi^4}{252}\right)=\tfrac{31\pi^6}{30240}.$$

This series is found in Ref [5].

3.2 Series of type $S(2k+1,n)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^{2k+1}}$, with $k=1,2,\ldots$.

Following the same procedure, leads to the following result:

$$S(2k+1,n)=\tfrac{1}{2^{2k+1}(2k)!}\int_0^{\infty}x^{2k}e^{-\frac{x}{2}}\left(1+e^{-x}\right)^{-1}dx=\tfrac{1}{2^{2k+1}(2k)!}\int_0^{\infty}(\ln z)^{2k}\,z^{-\frac{1}{2}}(1+z)^{-1}\,dz\,. \quad (16)$$

Changing $z=w^2$ variables, the equation (16) finally is expressed as:

$$S(2k+1,n)=\tfrac{1}{2^{2k}(2k)!}\int_0^1\left(\ln w^2\right)^{2k}\left(1+w^2\right)^{-1}dw \quad (17)$$

Finally one has the following examples

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}=\tfrac{1}{8}\int_0^1\left(\ln w^2\right)^2\left(1+w^2\right)^{-1}dw=\tfrac{1}{8}\left(\tfrac{\pi^3}{4}\right)=\tfrac{\pi^3}{32},$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^5}=\tfrac{1}{384}\int_0^1\left(\ln w^2\right)^4\left(1+w^2\right)^{-1}dw=\tfrac{1}{384}\left(\tfrac{5\pi^5}{4}\right)=\tfrac{\pi^5}{1536},\text{ and,}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^7}=\tfrac{1}{46080}\int_0^1\left(\ln w^2\right)^6\left(1+w^2\right)^{-1}dw=\tfrac{1}{46080}\left(\tfrac{61\pi^7}{4}\right)=\tfrac{61\pi^5}{184320}.$$

This series is found in Ref [5].

5. Conclusions

The limits of a broad class of alternating series which converge were obtained using some novel algorithms. These algorithms are general and can be used to evaluate many de these series that are important in different branches of the physics, as well as in other relevant science applications. Moreover, we can apply these algorithms to other types of alternating series, not considered in this article.

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