

# Order and Disorder Evolution Determined by Non-Linear Processes

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**Abstract.** We show that a low dimensional system can evolve toward an infinite dimensional ergodic system. We also show the existence of the reverse process, that is, a large-dimensional ergodic system that evolves toward a low-dimensional ordered system.

## 1. Introduction

Order and disorder are related to coherence and de-coherence concepts, respectively (see for example Refs. [1, 2]). Consequently, exploring the de-coherence processes gives rise to the understanding of the entropy increase evolution toward maximum value in thermal equilibrium. In this work, by exploring recursive maps, we discover that in addition to the de-coherence evolution, there is an opposite process referred to as re-coherence. Re-coherence describes a system that originally possessed a disordered component, but evolves toward absolute coherence. A demonstration of a low dimensional system that experiences such a re-coherence evolution was recently introduced by us [3]. In this paper we show that re-coherence and de-coherence have the same origin, that is, nonlinear maps.

Two contradicting processes determine a system of nonlinear dynamics where if these two processes reach a finite number of equilibrium states the associated map is regarded as a regular map. Otherwise, the map is considered to be chaotic [4]. We show that chaotic maps are associated with ergodicity evolution while regular maps are responsible for the re-coherence evolution.

## 2. Primary Definitions

### 2.1. Low Dimensional Complementary Maps

We start with a low dimensional nonlinear map and we show how it expands into an infinite number of degrees of freedom with an ergodic nature.

Let  $f_R(x_n)$  be an iterative map with a control parameter  $R$  such that

$$x_{n+1} = f_R(x_n), \quad x \in [0, 1] \quad (1)$$

where all values of  $x$  are within the interval  $[0, 1]$ . We refer it as an **x-map**.

We introduce coherence by introducing a **complementary map**, referred to as the **y-map**, as follows[3]

$$y_{n+1} \equiv 1 - f_R(1 - y_n), \quad (2)$$



where it is easy to see that  $\forall n \ y \in [0, 1]$ .

For example, the  $y$ -map for the logistic map

$$x_{n+1} = Rx_n(1 - x_n) \quad (3)$$

becomes

$$y_{n+1} = 1 - Ry_n(1 - y_n). \quad (4)$$

In general the complementary maps can evolve independently provided that the initial variables  $x_0$  and  $y_0$  are determined separately. Otherwise, when the initial conditions are coordinated such that  $x_0 + y_0 = 1$  the maps are considered to be **initially coherent**.

In a pure mathematical sense it can be shown that coherence, namely the relation  $x_n + y_n = 1$ , is conserved during the iterations.

Consider initially correlated maps,

$$y_0 = 1 - x_0. \quad (5)$$

Correspondingly, the next iteration yields

$$\begin{aligned} x_1 &= f_R(x_0) \\ y_1 &= 1 - f_R(1 - y_0) = 1 - f_R(x_0) = 1 - x_1 \\ \Rightarrow y_1 &= 1 - x_1. \end{aligned} \quad (6)$$

Thus, the initial relation  $y_0 = 1 - x_0$  is replicated to the  $n = 1$  iteration such that  $y_1 = 1 - x_1$ . Obviously, in a pure mathematical manner, this correlation is maintained for all  $n$  iterations

$$\text{if } y_0 = 1 - x_0 \text{ then } \forall n \ y_n = 1 - x_n. \quad (7)$$

## 2.2. States Representation of the Nonlinear Maps

Suppose that we have some 2-D system with the **generating states**  $|1\rangle$  and  $|\emptyset\rangle$ [5]. These generating states span a primary low dimensional basis that under the influence of chaotic maps, expands toward an infinite dimensional space.

The complementary maps are embedded in the generating states through the states coefficients. Since the coefficients are defined under the complex space we need to define at least two sets of complementary maps: The ordinary maps of eqs. 1 and 2 that defines the coefficients real part and what we refer as **hidden** maps that defines the states phases.

The hidden maps are defined as follows:

$$\begin{aligned} p_{n+1} &= \phi_\rho(p_n) \\ \tilde{p}_{n+1} &= 1 - \phi_\rho(p_n) \\ \forall n \quad 0 &\leq p_n \leq 1. \end{aligned} \quad (8)$$

The phase  $\theta_n$  is defined as

$$e^{i\theta_n} = \sqrt{p_n} + i\sqrt{\tilde{p}_n}. \quad (9)$$

Altogether, we define the initial generating states

$$\begin{aligned} |0\rangle &= \sqrt{x_0}e^{i\theta_0} |1\rangle + \sqrt{y_0} |\emptyset\rangle \\ |\tilde{0}\rangle &= -\sqrt{y_0}e^{i\theta_0} |1\rangle + \sqrt{x_0} |\emptyset\rangle, \end{aligned} \quad (10)$$

where we considered  $\theta_0$  as a relative phase. We assume that the initial maps  $x_0, y_0$  and  $p_0, \tilde{p}_0$  are unitary correlated meaning that  $x_0 + y_0 = p_0 + \tilde{p}_0 = 1$ .

We assume that the states interact with an external environment that modifies the states coefficients in according to the recursive-complementary maps to generate the iterating states

$$\begin{aligned} |n\rangle &= \sqrt{x_n}e^{i\theta_n} |1\rangle + \sqrt{y_n} |\emptyset\rangle \\ |\tilde{n}\rangle &= -\sqrt{y_n}e^{i\theta_n} |1\rangle + \sqrt{x_n} |\emptyset\rangle \end{aligned} \quad (11)$$

### 3. Description of Maps Evolution

Part of the following was published in Ref. [5].

Recursive nonlinear maps have the tendency of reaching constant values, provided that they are not in the chaotic regimes. If the maps reach single values, denoted by  $x_\infty, y_\infty$  and  $\theta_\infty$ , we obtain that any initial basis of states terminates at the unique basis

$$\begin{aligned} |n \rightarrow \infty\rangle &= \sqrt{x_\infty} e^{i\theta_\infty} |1\rangle + \sqrt{y_\infty} |\emptyset\rangle \\ |\tilde{n} \rightarrow \infty\rangle &= -\sqrt{y_\infty} e^{i\theta_\infty} |1\rangle + \sqrt{x_\infty} |\emptyset\rangle. \end{aligned} \quad (12)$$

For example, when  $R < 3$  the logistic maps reach the single set  $x_\infty = \frac{R-1}{R}$  and  $y_\infty = \frac{1}{R}$ . Consequently, for real coefficients we obtain states that, except for the  $x_n = 0, 1$ -values, always converge into the unique basis of states

$$\begin{aligned} |\infty\rangle_R &= \sqrt{\frac{R-1}{R}} |1\rangle + \sqrt{\frac{1}{R}} |\emptyset\rangle \\ |\tilde{\infty}\rangle_R &= -\sqrt{\frac{1}{R}} |1\rangle + \sqrt{\frac{R-1}{R}} |\emptyset\rangle. \end{aligned} \quad (13)$$

This basis of states spans a 2-D Hilbert space.

#### 3.1. The Maps Dimension Extension Toward Chaos and Ergodicity

The  $\phi$  and the  $f$  maps are considered to be independent, meaning, that at the same time one set of maps can evolve in a regular manner, while the other can behave chaotically. For that reason we restrict our discussion only to a single set, that is, the  $f$ -maps keeping in mind that the proceeding arguments valid for both maps.

It is known that for a large variety of maps, increasing the strength parameter  $R$  such that the map evolves from regular toward chaotic behaviors is followed by universal steps.

The first step (relatively small  $R$ ) is characterized by maps that are converging into a single pair of stable points " $x^{(1)}$ " and " $y^{(1)}$ " (or  $\theta^{(1)}$ ) calculated as

$$\begin{aligned} x_{n+1} = x_n &\equiv x^{(1)} \Rightarrow x^{(1)} = f_R(x^{(1)}) \\ y_{n+1} = y_n &\equiv y^{(1)} \Rightarrow y^{(1)} = 1 - f_R(1 - y^{(1)}). \end{aligned} \quad (14)$$

For a higher level of  $R$ , the maps no longer converge into single values as they reach a state of two alternating values. Nevertheless, it is possible to redefine a map which retrieves a single stable value as follows

$$\begin{aligned} x_{n+2} = x_n &\equiv x^{(2)} \Rightarrow x^{(2)} = g(R, x^{(2)}) \\ g &\equiv f \circ f \\ y_{n+2} = y_n &\equiv y^{(2)} \Rightarrow y^{(2)} = \tilde{g}(R, 1 - y^{(2)}) \\ \tilde{g} &\equiv (1 - f) \circ (1 - f). \end{aligned} \quad (15)$$

For higher strength parameter the maps terminate with four alternating values characterized by the maps  $G = g \circ g$  and  $\tilde{G} = \tilde{g} \circ \tilde{g}$  and the proceeding eight splits are determined by the  $G \circ G$  and  $\tilde{G} \circ \tilde{G}$  maps. Increasing  $R$ , the same splitting continues until it reaches the chaotic regime for which it is impossible to redefine maps that are stabilized at a single value.

Suppose we start with the regular initial conditions for the 2-D Hilbert space. For  $l = 1$  the map reaches a final value. For  $l = 2$  there are two final values. Thus, in order to retrieve a single value we use the double map  $g = f \circ f$ . If  $n$  represents some time interval that passes between the  $f$ -iterations, the composed map  $g$  corresponds with a time interval which is twice the original  $f$ -time. We now show that this interval doubling corresponds with an extension of the 2-D Hilbert space dimension.

We start with the basic map  $f$  (or  $\phi$ ) applied in the  $l = 2$  - regime where in the steady state it alternates between two values.

Suppose that we start with an initial condition that is one of the two final states. For  $n = 0, 1, 2, 3, \dots$  the maps split into two types - even series and odd series iterations. Even series iterations are characterized by

$$\begin{aligned}
 &\text{Even series} \\
 &x_{2n+2} = f_R(x_{2n}) \quad (\text{The series } x_0, x_2, x_4, \dots) \\
 &y_{2n+2} = 1 - f(y_{2n}) \quad (\text{The series } y_0, y_2, y_4, \dots) \\
 &\text{Odd series} \\
 &x_{2n+3} = f_R(x_{2n+1}) \quad (\text{The series } x_1, x_3, x_5, \dots) \\
 &y_{2n+1} = 1 - f(y_{2n-1}) \quad (\text{The series } y_1, y_3, y_5, \dots).
 \end{aligned} \tag{16}$$

The  $l = 4$  space possesses the following degrees of freedom:

$$\begin{aligned}
 &\text{For } n = 0, 1, 2, 3, \dots \\
 &x_{4n+4} = f_R(x_{4n}) \quad (\text{The series } x_0, x_4, x_8, \dots) \\
 &x_{4n+5} = f_R(x_{4n+1}) \quad (\text{The series } x_1, x_5, x_9, \dots) \\
 &x_{4n+6} = f_R(x_{4n+2}) \quad (\text{The series } x_2, x_6, x_{10}, \dots) \\
 &x_{4n+7} = f_R(x_{4n+3}) \quad (\text{The series } x_3, x_7, x_{11}, \dots).
 \end{aligned} \tag{17}$$

Counting the  $l = 4$  map degrees of freedom we obtain the four branches of eq. 17. We assume that the three former  $l = 2$  branches also participate in the degree of freedom counting, thereby obtaining the  $l = 4$  space dimension  $d = 2^1 + 2^2 + 2^4 = 26$ .

In general we assume that the  $l$ -space dimension is  $d_l = \sum_{i=0}^{l/2} 2^{2^i}$ .

The corresponding states are  $|\nu_l\rangle_{i,j}$  and  $|\tilde{\nu}_l\rangle_{i,j}$  where

$$\begin{aligned}
 |\nu_l, \nu_\lambda\rangle_{i,j} &= \sqrt{x_{\nu_l}} \exp\{i\theta_{\nu_\lambda}\} |\emptyset\rangle + \sqrt{y_{\nu_l}} |1\rangle \\
 |\tilde{\nu}_l, \nu_\lambda\rangle_{i,j} &= -\sqrt{y_{\nu_l}} \exp\{i\theta_{\nu_\lambda}\} |\emptyset\rangle + \sqrt{x_{\nu_l}} |1\rangle \\
 \nu_l &= ln + i, \quad i = 0, 1, \dots, l-1 \\
 \nu_\lambda &= \lambda n + j, \quad j = 0, 1, \dots, \lambda-1
 \end{aligned} \tag{18}$$

where the symbols  $i$  and  $j$  identify a states in the extended space. For example  $l = 1$  yields the single sequence of iterating states  $|n\rangle$  and  $|\tilde{n}\rangle$ . In that case the two time sequences are the same. For  $l = 2$  or  $\lambda = 2$ , we obtain two sets of iterating states where the even states corresponds with the  $i = 0$  states while the  $i = 1$  is associated with the odd series.

### 3.2. Evolution Toward Ergodicity

We show that the transition to a longer iteration time step (such as the time doubling that happens in the transition from the  $f$  to the  $g$  maps) corresponds with enlarging the system dimension.

In the chaotic regime the time interval is defined to be infinitely long and the space dimension infinitely large. For that everlasting time, the iteration concept becomes meaningless. In other words, at some stage, the  $x$ ,  $y$  values lose the evolution history and thereby become random numbers with the appropriate evolved corresponding distribution. The infinite space dimension defines the associated ensemble. In other words, a single everlasting time step possesses an infinite independent initial possible values and the apparent time evolution calculation is just a new random selection.

We note that although the maps' tendency toward randomness is a primary characteristic, it does not appear at once; it takes some time for the random errors to take control over the

system. Nevertheless, randomness is the final stage in the variables' evolution that can be associated with the coherence loss during the evolution. This chaotic stage can be associated with thermodynamic equilibrium. We note that the final distribution is calculated by considering an ensemble which is composed of all  $x, y$  values regardless of their iteration order. In that sense, the original 2-D space that evolved into a random infinite random space describes the ergodicity process.

The transition from time evolving system toward random ensemble can be introduced as follows: Suppose that  $N_{max}$  is the final iteration being measured. Large values of the scaled time intervals  $\nu_l$  or  $\nu_{lambda}$  are crucial for the system to reach a final situation (see eqs. 180). Consequently, small values of  $l$  or  $\lambda$  requires large values of  $n$  with small iteration steps  $ln$  or  $\lambda n$ . In addition, the conditions  $i = 0, 1, \dots, l-1$  or  $j = 0, 1, \dots, \lambda-1$  yields a relatively low dimensional space. However, for large number of  $l$  or  $\lambda$ , the values of  $\nu_l$  or  $\nu_\lambda$  can reach high values even for a meaningless time interval ( $N_{max} = 1$ ). These high values are obtained due to the possible high values of  $i$  or  $j$  which stand for the space high dimensionality.

### 3.3. The Re-Coherence Process

Suppose that we have an ensemble of variables uniformly distributed in the interval  $[0, 1]$ . At  $t = 0$  ( $n = 0$ ) we randomly pick two variables and expose them to the double nonlinear **regular** maps as defined by eqs. 1,2. At this initial time the chosen random variables serve as initial conditions  $x_0$  and  $y_0$  for the  $x - y$  maps. Assuming that the chaotic maps that generated the ensemble are no longer active, these two picked initial variables will iterate towards one pair of the finite number of values as described by eqs. 14-17. We can now repeat this experiment many times but the result will always be the same. For simplicity, we will no focus on maps that converge into the single pair values,  $x_\infty, y_\infty = 1 - x_\infty$ . Regardless of the ensemble chosen member, all the ensemble values will converge into the same pair  $x_\infty, y_\infty = 1 - x_\infty$ . In that sense the infinite dimensional random ensemble is reduced into a 2-D space. In other words, if we wait long enough, every measurement we conduct will end up with the same reading. In the states language the random ensemble will reduce into an infinite number of identical pairs of basis states as described in eqs.12. The fact that every selected pair out of the random ensemble eventually generates a state justifies naming the process **re-coherence**.

## 4. Summary

System evolution toward Ergodicity is a known fact in statistical mechanics. However, it only describes a partial feature of the word, namely, inanimate systems. In this paper, we showed the existence of an opposite process, namely ergodic systems that evolve toward order. We can say that both the ergodicity and these *reorder* time evolutions are the outcome of the same process namely recursive nonlinear maps. If ergodicity is the outcome of maps defined in the chaotic regime, it is only reasonable to assume that regular maps are also active in nature and, in particular, there are processes that ergodic systems become ordered. Since ergodic systems describe the inanimate word, we cannot avoid the conclusion that the reordered processes describe animate systems creation as was suggested long ago by Schrödinger[6, 7]. In biology such an evolution is referred to a **Biopoiesis** process[8].

- [1] R. Alicki, A. Ozinski, P. Pakonski and K.I. Zyczowski, (2004) *J. Phys. A: Math. Gen.* **37** 51575172.
- [2] A. N. Soklakov and R. Schack (2002) *Phys. Rev. E*, **66**, 036212.
- [3] Y. Roth (2014), *J. Phys.: Conf. Ser.* **490** 012055.
- [4] H. Schanz, T. Dittrich, and R. Ketzmerick *Phys. Rev. E*, **71**, 026228 (2005).
- [5] Y. G. Roth, **Advances in Pure Mathematics**, **4**, (2014).
- [6] E. Schrödinger (1944), *What Is Life?*. (Cambridge University Press 1992).
- [7] M. P. Murphy, Luke A. J. O'Ne *What is Life? The Next Fifty Years: Speculations on the Future of Biology*, (Cambridge University Press, 1997).
- [8] A. K. Lal, *Astrophysics and Space Science*, **317** 267-278 (2008).