On the generalized Hartman effect for symmetric double-barrier point potentials

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Abstract. We consider the scattering of a non-relativistic particle by a symmetrical arrangement of two identical barriers in one-dimension, with the barriers given by the well-known four-parameter family of point interactions. We calculate the phase time and the stationary Salecker-Wigner-Peres clock time for the particular cases of a double δ and a double δ' barrier and investigate the off-resonance behavior of these time scales in the limit of opaque barriers, addressing the question of emergence of the generalized Hartman effect.

Quantum tunneling times have been extensively investigated in the last decades, leading to several alternative definitions of transmission time (see [1] and references therein). A phenomenon that attracted particular interest is the so-called Hartman effect [2], which states that in the opaque limit (low probability of transmission) the tunneling time becomes independent of the barrier width. In a generalized version of the Hartman effect the tunneling time for a particle incident on a double-barrier potential also becomes independent of the inter-barrier distance in the opaque limit [3, 4]. Due to their counterintuitive nature and consequent conceptual difficulties, these effects have been investigated for a variety of tunneling times (phase time, dwell time, Larmor time, etc.). However, most of these investigations are restricted to rectangular barriers or (in the generalized case) double-δ barriers – thus, it is important to verify if such effects hold for other interactions, such as generalized point interactions.

Point interactions in one-dimensional quantum mechanics have been the source of significant theoretical and mathematical interest for providing the opportunity to investigate advanced tools such as regularization and renormalization, generally employed in quantum field theory, in a simpler scenario (see, e.g., [5]). From the point of view of applications, point-like interactions have long provided models of significant experimental relevance for lower dimensional systems in, e.g., atomic physics. The most general point interaction in one-dimensional quantum mechanics is given by a four-parameter family of interactions, as can be shown by either considering self-adjoint extensions (SAE) of the kinetic energy operator (see, e.g., [6, 7, 8]) or by using a distributional approach to the corresponding Schrödinger equation [9]. Recently this general four-parameter family of interactions in one dimension has been used to investigate tunneling times for a non-relativistic particle incident on a single scatterer [10].

In this work we calculate the phase and Salecker-Wigner-Peres (SWP) clock times for a non-relativistic particle tunneling through a *symmetrical* arrangement of two identical potential barriers given by the four-parameter generalized point interaction in one dimension (in this case the SWP clock time is equivalent to the dwell time – see below). Our main interest is in investigating the emergence of the generalized Hartman effect (GHE); thus we restrict ourselves to the δ and δ' interactions and analyze the behavior of those times in the opaque limit. For the double δ-potential the stationary tunneling times have already been addressed in the literature (see [11] and references therein), however, as we will see below it is worth revisiting the discussion of the GHE for this potential.

Let us consider the scattering of a non-relativistic particle of mass μ and energy E by two identical point scatterers localized at $z = 0$ and $z = L$. The point interaction at position z can be characterized by the boundary conditions (b.c.)

$$\Phi(z_+) = \Lambda\, \Phi(z_-), \qquad \text{with} \qquad \Phi(z) \equiv \begin{pmatrix} \psi(z) \\ \psi'(z) \end{pmatrix}, \tag{1}$$

where $\psi(z)$ indicates the wave function, $\psi'(z)$ denotes its space derivative and $\Phi(z_\pm) = \lim_{\epsilon\to 0^+} \Phi(z \pm \epsilon)$; Λ is the matrix

$$\Lambda = e^{i\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \quad \theta \in [0, \pi), \tag{2}$$

with $a, b, c, d \in \mathbb{R}$.

The four-parameter family of point interactions given by eq. (2) represents the most general point interactions consistent with conservation of the probability current in one dimension, as can be demonstrated by using a rigorous distributional approach to treat the corresponding Schrödinger equation [9], or as obtained by the SAE approach [6, 7, 8]. The interaction (1)-(2) includes the so-called δ and δ' interactions as well as other point interactions which do not have a simple characterization.

We also assume that the particle is coupled to a SWP clock [12, 13] which only runs when the particle is in the region $(0, L)$. As is well known, the effect of the particle-clock coupling is to introduce a small potential barrier of high $\mathcal{V}_m$ in the region $(0, L)$ [13, 14]. Then, for incidence from the left, the general solution of the time-independent Schrödinger equation can be written as (omitting the dependence on $\mathcal{V}_m$, for simplicity)

$$\psi_{\mathrm{I}}(z) = e^{ikz} + r(\mathbf{k})e^{-ikz}; \qquad \psi_{\mathrm{III}}(z) = t(\mathbf{k})e^{ikz}, \tag{3}$$

where ψ_{I} and ψ_{III} indicate the wave function in the regions $z < 0$ and $z > L$, respectively. In (3) we defined $\mathbf{k} \equiv (k, q)$, with $k \equiv \frac{1}{\hbar}\sqrt{2\mu E}$ and $q \equiv \frac{1}{\hbar}\sqrt{2\mu(E - \mathcal{V}_m)}$ being the respective wave numbers in regions I/III and II ($0 < z < L$).

It is convenient to express the transmission and reflection coefficients as

$$t(\mathbf{k}) = |t(\mathbf{k})|e^{i[\varphi_t(\mathbf{k}) - kL]} \qquad \text{and} \qquad r(\mathbf{k}) = |r(\mathbf{k})|e^{i\varphi_r(\mathbf{k})}. \tag{4}$$

The phase transmission time for this system can be immediately calculated as $t_p^T(k) = \hbar\frac{\partial}{\partial E}\varphi_t(k)$, where the limit $\mathcal{V}_m \to 0$ is implicit. Similarly, the phase reflection time is given by $t_p^R(k) = \hbar\frac{\partial}{\partial E}\varphi_r(k)$. We are also interested in the dwell time $\tau_D(k)$ (see, e.g., [1] and references therein), which can be obtained from the identity [14]

$$\tau_D(k) = |t(k)|^2 t_c^T(k) + |r(k)|^2 t_c^R(k), \tag{5}$$

where $t_c^{T(R)}(k)$ stands for the transmission (reflection) SWP clock time, which is given by $t_c^T(k) = -\hbar\left[\frac{\partial}{\partial \mathcal{V}_m}\varphi_t(\mathbf{k})\right]_{\mathcal{V}_m=0}$ (with a similar expression for the reflection SWP time).

In this work we are interested in the case in which both the barriers are symmetric (which implies that the whole potential is also symmetric). This corresponds to the restriction $\theta = 0$ and $a = d$ in (2) [9, 15]. In this case, after imposing the b.c. (1)-(2) at $z = 0$ and $z = L$ and taking into account the potential $\mathcal{V}_m$ in the region $(0, L)$ due to the interaction of the particle with the SWP clock, we obtain (from now on we use units such that $\hbar = 2\mu = 1$)

$$\varphi_t(\mathbf{k}) = \varphi_r(\mathbf{k}) - \frac{\pi}{2} = -\tan^{-1}\left\{\frac{2a(c - bk^2)q + \left[c^2 + b^2k^2q^2 - a^2(k^2 + q^2)\right]\tan(qL)}{2k\left[(a^2 + bc)q + a(c - bq^2)\tan(qL)\right]}\right\} . \quad (6)$$

Thus, it follows from (5), and using conservation of probability, that for these kinds of symmetric potential we have $\tau_D(k) = t_c^T(k) = t_c^R(k)$. It also follows that $t_p^T(k) = t_p^R(k)$. Such result, which holds for any symmetric regular potential, also holds for any symmetric arrangement of singular point interactions (not necessarily identical). Now we restrict ourselves to the double δ and double δ' interactions and focus on the opaque limit, with the purpose of analyzing the GHE.

The double δ-interaction. The (symmetric) double delta potential can be obtained by taking $a = d = 1$, $b = 0$, $c = \gamma$ and $\theta = 0$, with γ characterizing the strength of the interaction [8, 9, 15]. With these substitutions the phase time becomes

$$t_p^T(k)\big|_\delta = \frac{4k^4L + \gamma^3 + 2k^2\gamma(2 + L\gamma) - \gamma^3\cos(2kL) + 2k\gamma^2\sin(2kL)}{k\left[8k^4 + 4k^2\gamma^2 + \gamma^4 + \gamma^2(4k^2 - \gamma^2)\cos(2kL) + 4k\gamma^3\sin(2kL)\right]} , \quad (7)$$

and the dwell/clock time can be shown to be [11]

$$t_c^T(k)\big|_\delta = \frac{2k\left[2k^2L + \gamma + L\gamma^2 - \gamma\cos(2kL)\right] - \gamma^2\sin(2kL)}{8k^4 + 4k^2\gamma^2 + \gamma^4 + \gamma^2(4k^2 - \gamma^2)\cos(2kL) + 4k\gamma^3\sin(2kL)} . \quad (8)$$

In the limit $L \to 0$ the two barriers coincide at the origin and the above times reduce to the ones obtained in reference [10] for a single point scatterer with strength 2γ.

The opaque limit is characterized by very strong interactions, i.e., $\gamma \gg 1$. It is straightforward to show that in the *extreme* opaque limit, $\gamma \to \infty$, both the off-resonance phase and dwell/clock times vanish, regardless of L (assumed to be finite), and this is generally interpreted as indicating the presence of the GHE – see [11] and references therein. However, in the extreme limit $\gamma \to \infty$ there is absolutely no transmitted wave beyond the first barrier and the transmission phase delay φ_t becomes meaningless (in fact, in this situation the delta barrier becomes impenetrable [6, 9]). Accordingly, in this extreme limit the transmission phase and dwell/clock times cannot be understood as traversal times (see [1, 14] for a similar discussion in the context of rectangular barriers). Therefore, we follow the approach of reference [14] and investigate the opaque limit by considering the asymptotic behavior of the tunneling times when γ is very large, but still finite. The leading contributions to the off-resonance phase and dwell/clock times are respectively given by $t_p^T(k)\big|_\delta = \frac{1}{\gamma k}$ and $t_c^T(k)\big|_\delta = \frac{1}{\gamma^2}\left[kL\csc^2(kL) - \cot(kL)\right]$, and we observe that for the phase time it does not depend on the barrier separation L, while for the dwell/clock time there is a dependence on L. Therefore, by taking into account *only* the leading contributions to the transmission times in the opaque regime, when the probability of transmission is small but non-vanishing, it appears that the transmission phase time presents the GHE, whereas the dwell/clock time does not present such effect.

The double δ'-interaction. We consider a (symmetric) double δ' interaction as obtained from (1) and (2) by taking the parameters to be $\theta = 0$, $a = d = 1$, $b = \gamma$, $c = 0$, with γ giving the strength of the interaction [8, 9, 15] (alternative, if less common, definitions for a δ'-interaction exist – see, e.g., [16]).

The phase and the dwell/clock times are respectively given by

$$t_p^T(k)\big|_{\delta'} = \frac{2L(2+k^2\gamma^2)+\gamma(4+k^2\gamma^2)-k\gamma^2\left[k\gamma\cos(2kL)+2\sin(2kL)\right]}{k\left\{8+4k^2\gamma^2+k^4\gamma^4+k^2\gamma^2\left[(4-k^2\gamma^2)\cos(2kL)-4k\gamma\sin(2kL)\right]\right\}}\,, \tag{9}$$

$$t_c^T(k)\big|_{\delta'} = \frac{2(2L+\gamma+k^2L\gamma^2)-2\gamma\cos(2kL)+k\gamma^2\sin(2kL)}{k\left\{8+4k^2\gamma^2+k^4\gamma^4+k^2\gamma^2\left[(4-k^2\gamma^2)\cos(2kL)-4k\gamma\sin(2kL)\right]\right\}}\,. \tag{10}$$

Similarly to the case of the double δ barrier, in the *extreme* opaque limit $\gamma \to \infty$ both of the above times converge to zero. But, again, in this extreme limit the interaction corresponds to two impenetrable point barriers, and any quantity calculated in terms of transmitted waves does not have physical meaning. Thus, here we also consider the asymptotic behavior of the off-resonance times in the opaque regime, when γ is very large but finite – still allowing a non-vanishing probability of transmission. The leading contributions to the phase and dwell/clock times are respectively given by $t_p^T(k)\big|_{\delta'} = \frac{1}{\gamma k^3}$ and $t_c^T(k)\big|_{\delta'} = \frac{1}{\gamma^2 k^4}\left[\cot(kL)+kL\csc^2(kL)\right]$. Again, in what concerns *only* the leading contributions to the off-resonance times in the opaque limit, it seems that the GHE occurs for the phase time but not for the dwell/clock time.

It should be noticed, however, that for any finite interaction strength the next-to-leading contribution to the off-resonance phase time (of order $1/\gamma^2$) *does* depend on the barrier separation L for both the δ and δ' interactions[1]. That is, at the same order as the leading contribution for the clock/dwell times, the phase times also present a slow growth with L for any *finite* γ (here, "slow growth" means that it still allows superluminal velocities). This indicates that the GHE is just a mathematical consequence of taking the extreme opaque limit $\gamma \to \infty$, in which case any "transmission" time loses its physical meaning, as mentioned above.

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[1] A detailed numerical study of the dependence of the tunneling times with L, as well as an investigation of double-barrier point potentials more general than those considered in (6), will appear elsewhere.