

Microscopic black hole stabilization via the uncertainty principle

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Abstract. Due to the Heisenberg uncertainty principle, gravitational confinement of two- or three-rotating particle systems can lead to microscopic Planckian or sub-Planckian black holes with a size of order their Compton wavelength. Some properties of such states are discussed in terms of the Schwarzschild geodesics of general relativity and compared with properties computed via the combination of special relativity, equivalence principle, Newton's gravitational law and Compton wavelength. It is shown that the generalized uncertainty principle (GUP) provides a satisfactory fit of the Schwarzschild radius and Compton wavelength of such microscopic, particle-like, black holes.

1. Introduction

The Black Hole Uncertainty Principle [1, 2, 3] seeks for a connection between the Heisenberg Uncertainty Principle (HUP) on microscopic scales, expressed via the Compton wavelength, and the event horizon of black holes, commonly referring to macroscopic scales [1, 2, 3]. These considerations have led to the concepts of Generalized Uncertainty Principle and Generalized Event Horizon [1, 2, 3, 4]. One implication is that there could exist sub-Planckian black holes with a size of order their Compton wavelength [1, 2, 3, 4]. It has been thus proposed that there may exist some link between black holes and elementary particles [2].

In this work we examine the gravitational confinement of two or three relativistic light particles rotating around their center of mass and we use the Schwarzschild geodesics of GR in conjunction with the HUP to investigate the properties of such states. It is found that such rotational states exist and their properties are similar with those computed by treating the two- or three- rotating particle motion via the combination of HUP, special relativity, Newton's gravitational law and the equivalence principle [5, 6, 7]. Interestingly when the rest mass of the two or three light relativistic particles is in the range of neutrino masses ($\sim 5 \cdot 10^{-2} \text{ eV}/c^2$), then the mass of the gravitationally confined microscopic Planckian or sub-Planckian rotating particle states are in the hadron ($\sim 1 \text{ GeV}$) mass range [5, 6, 7].

2. SR treatment

The SR treatment of the three rotating particle model (Fig.1) has been presented already elsewhere [5, 7]. In brief the special relativity equation of motion for a circular orbit [8, 9]



is used in conjunction with Newton's gravitational law to obtain

$$\gamma m_o \frac{v^2}{r} = \frac{Gm_g^2}{\sqrt{3}r^2}, \quad (1)$$

where m_g is the gravitational mass of each light particle, which according to the equivalence principle equals its inertial mass, m_i . The latter is given by

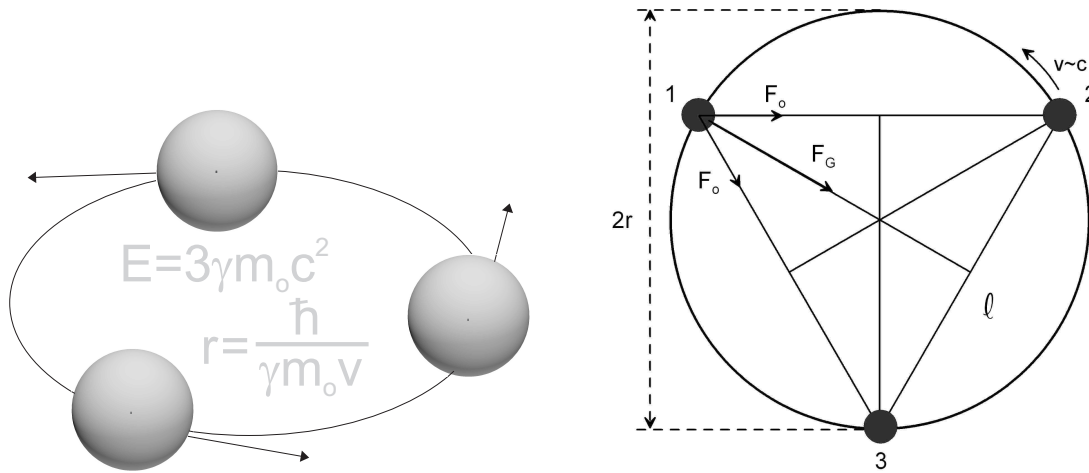


Figure 1. Schematic of the model of three rotating particles, together with the expressions for the rest energy and the radius of the bound state (left). The three particles are moving at a constant tangential velocity \mathbf{v} in a circle of radius r around their center of mass and are equally spaced. F_o is the attraction force between two particles at distance $\ell = \sqrt{3}r$ and F_G is the resultant radial force (right).

$$m_i = \gamma^3 m_o, \quad (2)$$

where $\gamma(= (1 - v^2/c^2)^{-1/2})$ is the Lorentz-factor, as shown already in 1905 for linear motion [10] and more recently for arbitrary particle motion [5], including circular orbits. It must be noted that the transverse mass γm_o , which is equal to the relativistic mass, is not an inertial mass, since the force computed from the product of mass and acceleration it is not invariant [5, 7, 8]. Combining (1) and (2) one obtains

$$r = \frac{Gm_o}{\sqrt{3}c^2} \gamma^5 \frac{\gamma^2}{\gamma^2 - 1}, \quad (3)$$

or, equivalently, for $\gamma \gg 1$:

$$r/r_s = (1/2\sqrt{3})\gamma^5, \quad (4)$$

where $r_s(= 2Gm_o/c^2)$ is the Schwarzschild radius.

Upon combining with the de Broglie wavelength equation

$$\lambda = \frac{\hbar}{\gamma m_o v}, \quad (5)$$

and assuming $r = \lambda$ one obtains from (4) and (5) that

$$\gamma^3 m_o = 3^{1/4} m_{Pl}, \quad (6)$$

where $m_{Pl} (= (\hbar c/G)^{1/2})$ is the Planck mass, therefore

$$\gamma^6 = 3^{1/2} \left(\frac{m_{Pl}}{m_o} \right)^2. \quad (7)$$

Very interestingly one observes from (6) that the mass of each rotating particle equals $3^{1/4} m_{Pl}$, regardless of the value of m_o . It also follows from (6) that the gravitational force acting on each of the three rotating particles is given by

$$F = \frac{G m_o^2 \gamma^6}{\sqrt{3} r^2} = \frac{G m_{Pl}^2}{r^2} = \frac{\hbar c}{r^2}, \quad (8)$$

which is a factor of $\alpha^{-1} (= \epsilon \hbar c / e^2 \approx 137)$ larger than the electrostatic force $e^2 / \epsilon r^2$ between two unit electric charges at a distance r . This is the expected value of the strong force at the same distance [11]. The ratio of this force to the nonrelativistic ($\gamma \approx 1$) Newtonian force, F_N , at the same distance is given by

$$\frac{F}{F_N} = \gamma^6 = 3^{1/2} \left(\frac{m_{Pl}}{m_o} \right)^2. \quad (9)$$

Since the mass of composite state formed by the three rotating particles equals $3\gamma m_o$, it follows from (7) that

$$m = 3\gamma m_o = 3^{13/12} m_o^{2/3} m_{Pl}^{1/3}. \quad (10)$$

Solving this equation for m_o one obtains

$$m_o = \frac{(m/3)^{3/2}}{3^{1/8} m_{Pl}^{1/2}}. \quad (11)$$

Considering as the composite particle mass m the mass of a neutron ($939.565 \text{ MeV}/c^2$) and as m_o the mass of one of the constituent quarks one finds

$$m_o = 0.0437 \text{ eV}/c^2, \quad (12)$$

which, surprisingly, is within the current uncertainty limits ($0.05 \pm 0.01 \text{ eV}/c^2$) of the rest mass of the heaviest neutrino [12]. The corresponding relativistic mass γm_o equals $313 \text{ MeV}/c^2$ which falls in the range of effective quark masses [13]. Consequently, as already shown [5, 6, 7], the mass of quarks can be modeled successfully by the mass of gravitationally confined neutrinos.

3. GR treatment

It is interesting to examine if the key results of the SR treatment, e.g. equations (6) and (9) can be obtained using the theory of general relativity (GR) where the masses are considered fixed and not velocity dependent.

In order to apply the Schwarzschild geodesics equation of GR [14] to the rotating neutrino problem one must adjust the physical model of Fig. 1 to the standard geometry of the Schwarzschild metric which involves a light test particle of mass m^* rotating around a central mass M . This can be done via the model shown in Figure 2. First we note that in the three-rotating particle model the force exerted to each particle is given by $F_N = G m_o^2 / (\sqrt{3} r^2)$, therefore the Newtonian potential energy due to the other two particles is given by

$$V_N = -G m_o^2 / (\sqrt{3} r). \quad (13)$$

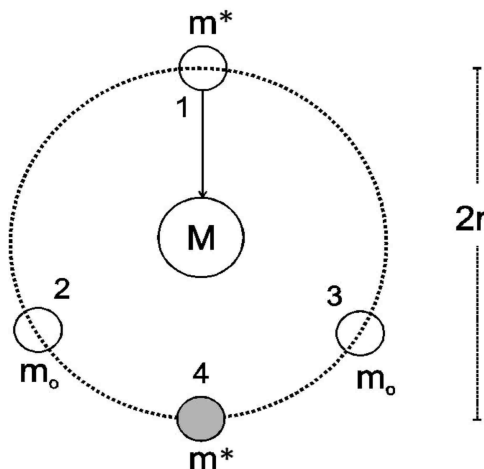


Figure 2. GR model for the three-neutrino state; $m^* = 2^{1/2}3^{-1/4}m_o$.

Since the same cyclic motion of particle 1 due to particles 2 and 3 can be obtained by substituting particle 1 with a new particle of mass m^* and by also substituting particles 2 and 3 by a particle 4 of mass m^* in the antidiometric position of 1, it follows that V_N must also equal the potential energy of 1 due to 4, i.e. $V_N = -Gm^{*2}/2r$. From this and equation (13) it follows $m^* = 2^{1/2}3^{-1/4}m_o$. We then consider the one-dimensional Schwarzschild effective potential, $V_s(r)$ with $M \gg m^*$

$$V_s(r) = -\frac{GMm^*}{r} + \frac{2^{-1/2}3^{1/4}L^2}{2m_or^2} - \frac{2^{-1/2}3^{1/4}GML^2}{c^2m_or^3}, \quad (14)$$

where L is the angular momentum. Setting $L = \hbar$ one obtains

$$\frac{V_s(r)}{(m^*c^2/2)} = -\frac{r_s}{r} + \frac{a^2}{r^2} - \frac{a^2r_s}{r^3}, \quad (15)$$

where $r_s (= 2GM/c^2)$ is the Schwarzschild radius of the central mass M and $a (= \hbar/m^*c)$ is the Compton wavelength of the rotating mass m^* .

The effective potential is not the actual potential experienced by the rotating particle. It is the potential of an identical particle in ordinary one-dimensional nonrelativistic mechanics which causes the same one-dimensional motion as the radial motion of the actual rotating particles [14]. The derivative $F = dV_s/dr$, termed effective force, is thus the force acting on this identical particle in ordinary one-dimensional nonrelativistic motion.

Two circular orbits are obtained when the effective force $F = dV_s(r)/dr$ is zero, which upon differentiation of (15) gives

$$r^{\pm} = \frac{a^2}{r_s} \left[1 \pm \sqrt{1 - \frac{3r_s^2}{2a^2}} \right]. \quad (16)$$

The larger one of this roots (a minimum in V_s) is very large ($10^{24}m$) and irrelevant in the present model. The smaller one (a maximum in V_s) is unstable.

When the rotational radius r is smaller than r^- , which is close to the Schwarzschild radius, r_s , then the effective force suggests that the formation of a black hole with an event horizon at r_s is energetically favored (Fig. 3).

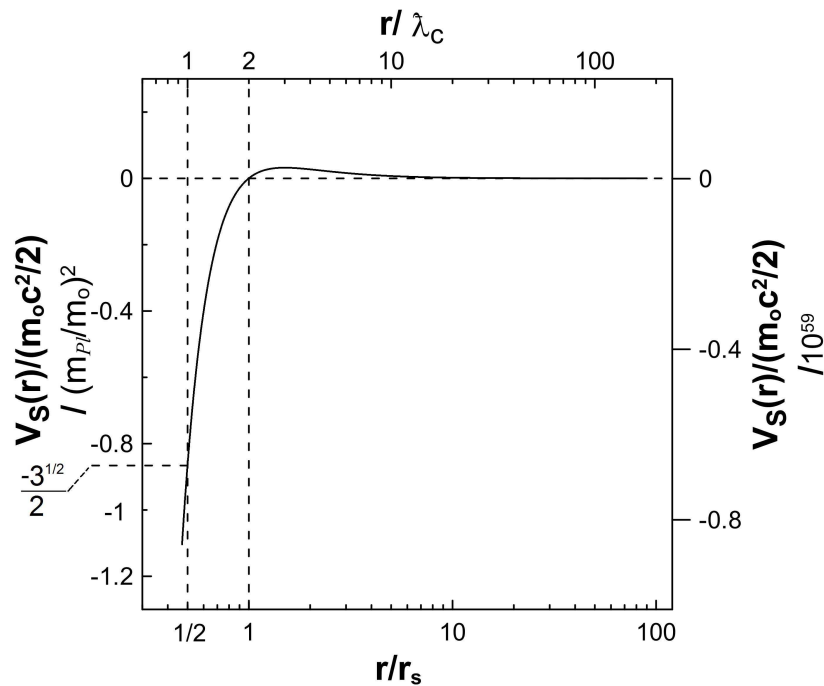


Figure 3. Plot of equation (15).

Differentiation of eq. (15) gives

$$F_s(r) = dV_s(r)/dr = (m^* c^2/2) \left[\frac{r_s}{r^2} - \frac{2a^2}{r^3} + \frac{3a^2 r_s}{r^4} \right]. \quad (17)$$

The first term in (17) corresponds to normal gravitational attraction according to Newton's gravitational law. Indeed, denoting this force by F_N and accounting for $r_s = 2GM/c^2$ one obtains

$$F_N(r) = \frac{m^* c^2}{2} \cdot \frac{2GM}{c^* r^2} = \frac{GMm^*}{r^2}. \quad (18)$$

The ratio $F(r)/F_N(r)$ of the effective force $F(r)$ to the Newtonian force F_N is thus given by:

$$F_s(r)/F_N(r) = 1 + \frac{a^2}{r_s^2} \left[3 \left(\frac{r_s}{r} \right)^2 - 2 \left(\frac{r_s}{r} \right) \right]. \quad (19)$$

Using the definition of a , r_s and $m_{Pl} (= \hbar c/G)^{1/2}$ and recalling $m^* = 2^{1/2} 3^{-1/4} m_o$ we obtain

$$F_s(r)/F_N(r) = 1 + \frac{m_{Pl}^4}{4M^2 m^{*2}} \left[3 \left(\frac{r_s}{r} \right)^2 - 2 \left(\frac{r_s}{r} \right) \right] = 1 + \frac{\sqrt{3}}{8} \cdot \frac{m_{Pl}^4}{M^2 m_o^2} \left[3 \left(\frac{r_s}{r} \right)^2 - 2 \left(\frac{r_s}{r} \right) \right]. \quad (20)$$

The minimum r value for which $F_N(r)$ can serve as a Newtonian centripetal force, and thus for which the ratio $F(r)/F_N(r)$ is defined, is the value $r_s/2$. This follows from

$$\frac{GMm^*}{r^2} = \frac{m^* v^2}{r} \quad \text{thus} \quad r = \frac{GM}{v^2}. \quad (21)$$

Accounting for $v < c$ it follows that indeed $r > r_s/2$. This is the minimum allowed r value which leads (Fig. 3) to the lowest possible value of $V_s(r)$ and which according to eq. (15) is given by

$$V_s(r)_{min} = (m^*c^2/2) \left[-2 - \frac{4a^2}{r_s^2} \right] \approx -(m^*c^2/2) \frac{4a^2}{r_s^2} = -m^*c^2 \frac{m_{Pl}^4}{2M^2m^{*2}} = -m^*c^2 \frac{\sqrt{3}}{4} \cdot \frac{m_{Pl}^4}{M^2m_o^2} \quad (22)$$

When the limit $r = r_s/2$ coincides with the Compton wavelength limit, $\lambda_c (= \hbar/Mc)$, imposed by the Heisenberg uncertainty principle (HUP), i.e. when

$$r_s/2 = \frac{GM}{c^2} = \frac{\hbar}{Mc} = \lambda_c, \quad (23)$$

it follows

$$M = \left(\frac{\hbar c}{G} \right)^{1/2} = m_{Pl} \quad (24)$$

This result is consistent with that extracted from the *SR* treatment (eq. 8) which shows that the mass keeping, via its gravitational attraction, each rotating particle to its orbit is the Planck mass m_{Pl} .

Substituting in (19) with $r = r_s/2$ and accounting for $a^2/r_s^2 \gg 1$ one obtains

$$F_s(r)/F_N = 3^{1/2} \left(\frac{m_{Pl}}{m_o} \right)^2 \quad (25)$$

which, surprisingly, is the same result as that obtained via the special relativistic treatment (eq. 9).

It thus appears that $r_s/2$ provides a satisfactory generalized event horizon (GEH), at least when used in conjunction with the Heisenberg uncertainty principle (HUP). At $r = r_s/2 = \lambda_c$ both the Schwarzschild geodesics and the Heisenberg uncertainty principle are satisfied for $M = m_{Pl}$.

4. Generalized uncertainty principle

These results are consistent with the generalized uncertainty principle (GUP) discussed by Adler [1], Carr [2, 3], Vagenas [4] and others, i.e.

$$\Delta x \geq \frac{\hbar}{\Delta p} + L_p^2 \frac{\Delta p}{\hbar} \quad (26)$$

where Δx is the position uncertainty, p is the momentum and $L_p (= (G\hbar/c^3)^{1/2})$ is the Planck length. Using in (26) $\Delta p = Mc$ one obtains

$$\Delta x \geq \frac{\hbar}{Mc} + \frac{G\hbar}{c^3} \cdot \frac{Mc}{\hbar} = \lambda_c \left(1 + \frac{GM^2}{\hbar c} \right) = \lambda_c \left(1 + \frac{M^2}{m_{Pl}^2} \right) \quad (27)$$

Consequently for $M = m_{Pl}$ one obtains $\Delta x = 2\lambda_c$, thus $\Delta x/2 = \lambda_c$ and, therefore, in view of eq. (23), i.e. $r_s/2 = \lambda_c$, it follows $\Delta x = r_s$. Consequently the uncertainty in position equals r_s , i.e. the event horizon of the black hole and therefore for $M = m_{Pl}$, the use of the GUP in conjunction with $\Delta x = r_s$ leads directly to $\lambda_c = r_s/2$ and to the key result of equation (25) which shows the good agreement between the SR and GR treatments of the rotating particle model. This result, i.e. $\Delta x = r_s = 2\lambda_c$ also shows how the GUP leads to black hole stabilization forming bound states with some interesting properties, similar to those of hadrons.

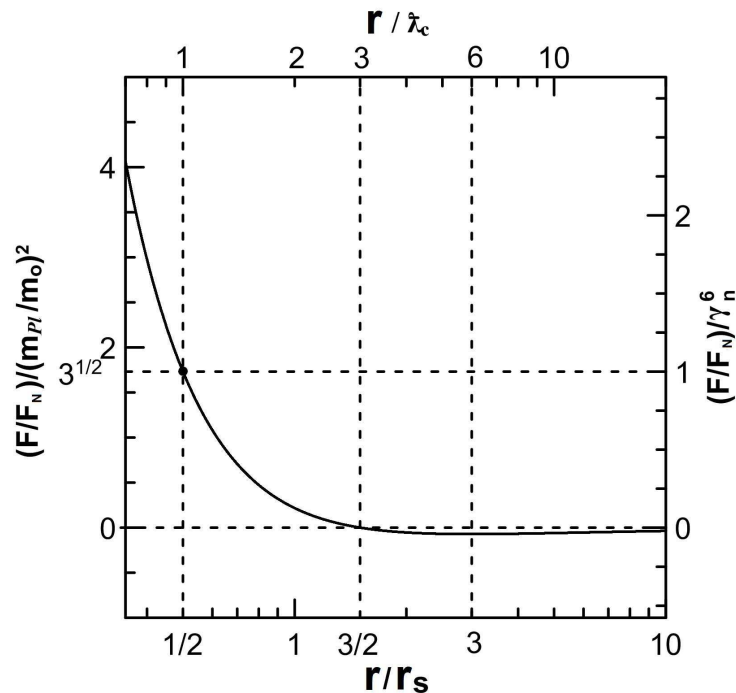


Figure 4. Plot of equation (20) for $M = m_{Pl}$.

For example, the effective force, F , at $r = r_s/2 = \lambda_c$ is equal (eqs 9 and 25) to the actual relativistic gravitational force, $\hbar c/r^2$, which keeps each particle in orbit in the SR treatment (eq. 8) and this force is a factor of $\alpha^{-1} (= \epsilon \hbar / e^2 \approx 137)$ stronger than the Coulombic force at the same distance, as expected for the strong force [11]. Also if one were to borrow from SR the inertial-gravitational mass expression $\gamma^3 m_o = 3^{1/4} m_{Pl} = 3^{1/4} M$ (eq. 6), then one can compute $\gamma (= 7.163 \cdot 10^9$ for $m_o = 0.04378 \text{ eV}/c^2$) and thus the relativistic mass $3\gamma m_o (\approx 939 \text{ MeV}/c^2)$ of the bound state, which is in the baryon mass range. Also upon using the above γ value in the Compton wavelength expression $\lambda_c = \hbar / \gamma m_o c$, one finds $\lambda_c = 0.63 \cdot 10^{-15} \text{ m}$, which is in the baryon radius range. It thus appears that indeed a link exists between microscopic black holes and elementary particles as already suggested [2].

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