Wormholes from Chiral Fields

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Abstract. In this paper, Lorentzian wormholes with a phantom field and chiral matter fields have been obtained. In addition, it is shown that for different values of the gravitational coupling of the chiral fields, the wormhole geometry changes. Finally, the stability of the corresponding wormholes is studied and it is shown that are unstable (eg. Ellis's wormhole instability)

1. Introduction

A wormhole[1] is a topological object of space-time which correspond to a *shortcut* through space-time. Wormholes that could be crossed in both directions, are known as *traversable wormholes*. Transversable wormholes introduced in [1], were obtained when a phantom field was coupled to gravity. Their energy-momentum tensor would violate all (null, weak and strong) energy conditions; while the phantom field is a scalar field with a reversed sign in front of its kinetic term.

The action of the Einstein gravity coupled to a phantom field [2] and ordinary matter fields is of the form

$$S = \int \left[\frac{1}{16\pi G} \mathcal{R} + \mathcal{L}_{\mathrm{ph}} + \mathcal{L}_{\mathrm{ch}} + \mathcal{L}_{\mathrm{sk}} \right] \sqrt{-g} \, d^4x \tag{1}$$

where $\mathcal{R}$ is the curvature scalar, G is the Newton's constant and g is the determinant of the metric. Also, the Lagrangian of the phantom field ϕ is

$$\mathcal{L}_{\mathrm{ph}} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \,, \tag{2}$$

and the non-linear sigma model Lagrangian is

$$\mathcal{L}_{\mathrm{ch}} = \frac{\kappa^2}{4} \mathrm{Tr} \left\{ L_\mu L^\mu \right\} \tag{3}$$

where $L_\mu = \partial_\mu U U^\dagger$ and κ is a coupling constant. The U is the chiral field which is a function on the space-time manifold taking values in the Lie group SU(2). The last term in the action is the Skyrme term defined as

$$\mathcal{L}_{\mathrm{sk}} = \frac{1}{32e^2} \mathrm{Tr} \left\{ F_{\mu\nu} F^{\mu\nu} \right\} \tag{4}$$

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[1] Based on a talk given by T. Ioannidou

where $F_{\mu\nu} = [L_\mu, L_\nu]$ and e a coupling constant.

The variation of the action with respect to the metric leads to the Einstein equations

$$G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{5}$$

where the stress-energy tensor is

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_M - 2\frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} \ . \tag{6}$$

Note that $\mathcal{L}_M = \mathcal{L}_{\text{ph}} + \mathcal{L}_{\text{ch}} + \mathcal{L}_{\text{sk}}$ is the matter Lagrangian.

We consider static spherically symmetric wormhole solutions by assuming that

$$ds^2 = -A^2 dt^2 + d\eta^2 + R^2 d\Omega^2 \ , \tag{7}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ denotes the metric of the unit sphere, while A and R are functions of η. Note that the coordinate η takes values in $-\infty < \eta < \infty$. The limits $\eta \to \pm\infty$ correspond to the *disjoint asymptotically flat regions*.

In addition, the chiral field is taken to be of the form

$$U = \cos F + i \sin F \ \vec{e} \cdot \vec{\tau} \ , \tag{8}$$

where $\vec{e}$ is the unit vector field

$$\vec{e} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \ , \tag{9}$$

$\vec{\tau}$ is the Pauli matrices vector and $F(\eta)$ is the profile function.

For simplicity, we introduce the dimensionless quantities

$$\eta = \frac{\tilde{\eta}}{\kappa} \ , \quad R = \frac{\tilde{R}}{\kappa} \ , \quad \phi = \kappa\tilde{\phi} \ , \quad 4\pi G = \frac{\alpha}{\kappa^2} \tag{10}$$

which is equivalent to $\kappa = 1$ and $4\pi G = \alpha$ and we rename $\tilde{\eta} = \eta$, $\tilde{R} = R$, $\tilde{\phi} = \phi$.

2. Wormhole Topology

In order, to obtain wormhole solutions, we assume that the function R does not possess any zero. In particular, we assume that R behaves like $|\eta|$ in the asymptotic regions and possesses (at least) one minimum. Suppose $R'(\eta_0) = 0$ for some η_0 and $R(\eta_0) = r_0$. Then from the equations of motion we get that

$$R''(\eta_0) = \frac{1}{r_0}\left(1 - \alpha\sin^2 F_0 \left[2 + \frac{\sin^2 F_0}{e^2 r_0^2}\right]\right) = \frac{1}{r_0}\left(1 - \alpha/\alpha_{\text{cr}}\right) \ , \tag{11}$$

where $F_0 = F(\eta_0)$ and

$$\alpha_{\text{cr}} = \frac{1}{\sin^2 F_0 \left[2 + \frac{\sin^2 F_0}{e^2 r_0^2}\right]} \ . \tag{12}$$

Therefore, R has a minimum at η_0 if $\alpha < \alpha_{\text{cr}}$ and a maximum when $\alpha > \alpha_{\text{cr}}$. The first case corresponds to a wormhole since it describes a surface of minimal area which separates two asymptotically flat regions. In the second case the maximum is a local maximum since in the asymptotic regions $R = |\eta|$. This implies that there are two minima of R, one for $\eta < \eta_0$ and another for $\eta > \eta_0$. For a sequence of alternating minima and maxima the corresponding spatial

hyper-surfaces would also have two asymptotically flat regions divided by a throat. However, the geometry of the throat would be more complicated.

For simplicity let us choose $\eta_0 = 0$ which defines the position of the throat. In addition, we impose the following conditions at the throat

$$R(0) = r_0 \ , \quad R'(0) = 0 \ , \quad F(0) = F_0 \ . \tag{13}$$

The first choice correspond to the areal radius of the throat, and the second is the extremum condition. The third choice fixes the value of the chiral profile function at the throat, which is a free parameter. Let us consider the case where $F_0 = n\pi/2$, i.e. the wormholes are symmetric under the interchange of the asymptotic regions.

Finally, the following boundary conditions are considered

$$A(\eta \to \infty) \to 1 \ , \quad F(\eta \to \infty) \to 0 \ , \quad F(\eta \to -\infty) \to n\pi \ . \tag{14}$$

which sets the time scale, and results from requiring finite energy and conserved topological charge respectively.

Let us conclude, by identifying the topological charge of the solutions. Consider the chiral field U as a map of spatial slices of the wormhole space-time to the group manifold $SU(2) \sim S^3$. Since U takes constant values in both asymptotic regions the spatial slices become topologically equivalent to a three dimensional sphere, where the north and south pole correspond to the asymptotic regions $\eta \to +\infty$ and $\eta \to -\infty$, respectively. Thus the chiral field is a map between two three-spheres, and the topological charge is defined as the degree of the map. Due to the the chiral field ansäntz given by (8) and the above asymptotic boundary conditions for the profile function F with $n = 1$ the corresponding topological charge of our solution is one.

The stability of wormholes is crucial for their physical relevance. It is known that the Ellis wormhole possesses an unstable mode [3, 4]. Therefore, we investigate whether our solutions possess the Ellis instability. However, in our work [2], we make a careful choice of the gauge condition in order not to miss the unstable mode present in the Ellis wormhole. Unfortunately, all our wormholes are Ellis unstable, as discussed in the next section.

3. Conclusion
Morris-Thorne wormholes threaded by chiral fields that carry a conserved topological charge have been obtained. We consider static spherically symmetric solutions, which are symmetric under an interchange of the two asymptotically flat universes with respect to the throat.

We observe that when the chiral fields are described by the NLS model, the gravitational coupling constant α is the single free parameter. For small α the chiral fields have no influence upon the wormhole geometry, and the throat is characterized by a single minimal surface. As the gravitational coupling increases, the presence of the chiral fields influence the geometry of the wormhole. At a critical coupling α_{cr}, the throat becomes degenerate. Beyond α_{cr} the throat exhibits a set of three extrema, a (local) maximal surface and two minimal surfaces located symmetrically, one on each side. Therefore, the wormhole geometry develops a belly in the interior, which increases in size as α increases.

When the Skyrme term is added, another free parameter appears. Therefore, the gravitational coupling and the throat size can vary independently. As before increasing α influence the matter on the geometry of the throat, and the formation of an inner belly beyond a critical value α_{cr}. However, the splitting of the space-time into three parts is not as smooth as the NLS case, since at the maximal coupling the asymptotically flat universes touch the inner universe.

Let us conclude by stating that the chiral fields cannot stabilize the wormhole space-time. The instability of the Ellis solution is inherited by the wormholes threaded by chiral fields, although they *do* carry a topological charge. According to our observations this is not surprising

though, since, the topological charge may finally simply reside in a single of several disconnected space-time parts.

References

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