Some More Solutions of Burgers' Equation

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Abstract. In this work, similarity solutions of viscous one–dimensional Burgers' equation are attained by using Lie group theory. The symmetry generators are used for constructing Lie symmetries with commuting infinitesimal operators which lead the governing partial differential equation (PDE) to ordinary differential equation (ODE). Most of the constructed solutions are found in terms of Bessel functions which are new as far as authors are aware. Effect of various parameters in the evolutional profile of the solutions are shown graphically and discussed them physically.

Introduction

Burgers' equation is widely used in the physical problems to model various phenomenon such as modelling of gas dynamics [1], traffic flow, investigating the shallow water waves [2, 3], examining the chemical reaction diffusion model [4], turbulent flow patterns of soil water [5], shock waves in hydrodynamics turbulence [6]. First time, it was also used in the context of a statistical theory of turbulent motion of fluids [7, 8].

One–dimensional viscous Burgers' equation has a non–linear convection term and a second order viscous diffusion term so it becomes a simplified form to one–dimensional analogue of the Navier–Stokes equations. Initially, Burgers' equation was presented by Bateman [9] and later describing a mathematical model of turbulence [1, 8, 10]. Due to extensive work [1, 8] of Burgers it is known as Burgers' equation. Besides its fundamental non–linearity, exact solutions of the Burgers' equation have been obtained for a wide range of initial and boundary conditions [6]. Numerical solutions of Burgers' equation were found impractical for small viscosity due to slow convergence of solutions [11].

Although the literature devoted to Burgers' equation is indeed enormous [12, 13]. For the equation, A. Ouhadan et al. [13] obtained some exact solutions namely exponential, rational and periodic solutions forming a new class of Lie point symmetries through its maximal sub algebras. Furthermore, exact solutions of the forced Burgers' equation are constructed [14] making use of the modified Hopf–Cole transformation for stationary and transient external forces. Motivation of these researches leads us to find the exact solutions of Burgers' equation by using similarity transformations method (STM).

Nevertheless, due to a lot of physical significance of Burgers' equation in real life problems, we have an unquestionable interest to get exact solutions by using STM. The STM of differential equations depends on the Lie groups theory of invariance [15–16]. Indeed, the invariance enables us to reduce the number of variables by one. The applications of the method for various problems of fluid dynamics can be studied [17–20].

The goal of this research is to get the exact solutions of one dimensional Burgers' equation by using STM. In this work, authors have been focused on the exact solutions in terms of Bessel functions. Due to arbitrary choice of parameters, exact solutions are shown graphically and discussed them physically as far as possible. Some results [12, 13, 21] are deduced as special cases of our results.

Similarity Solutions and Discussions
Authors considered the Burgers' equation in the form

$$u_t + u\, u_x = \nu\, u_{xx}, \qquad \nu > 0, \tag{1}$$

where $u(x,t)$ is velocity and ν is kinematic viscosity of fluid. Eq. (1) is a quasilinear parabolic PDE. To apply STM on (1) one can consider one–parameter (ϵ) Lie group transformations

$$x^* = x + \epsilon\, \xi(x,t,u) + O(\epsilon^2), \quad t^* = t + \epsilon\, \tau(x,t,u) + O(\epsilon^2), \quad u^* = u + \epsilon\, \eta(x,t,u) + O(\epsilon^2), \tag{2}$$

where ξ, τ and η are the infinitesimals corresponding to x, t and u respectively which are to be determined in such a way that the Eq. (1) is invariant under the Lie group of transformations (2). Existence of such group allows the number of independent variables to be reduced by one.

Applying Lie group theory [15–16] on (1) yields the following infinitesimals

$$\xi = a_1\, xt - a_2\, x + a_3\, t + a_4, \quad \tau = a_1\, t^2 - 2a_2\, t + a_5, \quad \eta = a_1\, (x - ut) + a_2\, u + a_3, \tag{3}$$

where a_1, a_2, a_3, a_4 and a_5 are five arbitrary parameters. Further, Eq. (1) admits five dimensional Lie algebra generated by

$$X_1 = (x - ut)\frac{\partial}{\partial u} + xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t}, \quad X_2 = u\frac{\partial}{\partial u} - x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial u} + t\frac{\partial}{\partial x},$$

$$X_4 = \frac{\partial}{\partial x}, \quad X_5 = \frac{\partial}{\partial t}. \tag{4}$$

The generators X_1, X_2, X_3, X_4 and X_5 are linearly independent. In order to obtain exact solutions, one has to solve the following Lagrange system

$$\frac{dx}{a_1\, xt - a_2\, x + a_3\, t + a_4} = \frac{dt}{a_1\, t^2 - 2a_2\, t + a_5} = \frac{du}{a_1\, (x - ut) + a_2\, u + a_3}. \tag{5}$$

To get similarity solutions of Eq. (1) following cases can be raised:

(a)   (b)
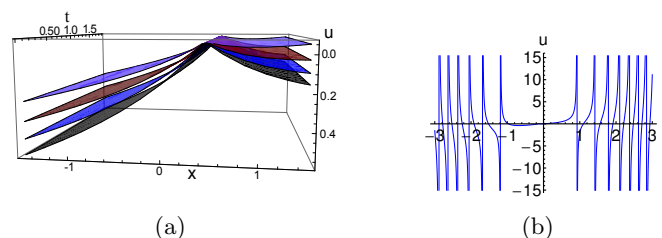
Figure 1. Variation in spatial structure of velocity profile for (10). (a) At $0 \leq t \leq 2$ for $\nu = 0.10\,(\text{magenta})$, $0.20\,(\text{red})$, $0.30\,(\text{blue})$, $0.40\,(\text{black})$; (b) At $t = 0.5$ for $\nu = 0.10$.

Case (I): If $a_1 \neq 0$ and treating $A_1 = -\frac{a_2}{a_1}$, $B_1 = \frac{a_3}{a_1}$, $C_1 = \frac{a_4}{a_1}$ and $D_1 = \frac{a_5}{a_1}$ leads to the following sub cases:

(Ia): If $D_1 \neq A_1^2$ then similarity form of the solution can be furnished as

$$u = \alpha + \frac{Xt}{\sqrt{t^2 + 2A_1 t + D_1}} + \frac{F(X)}{\sqrt{t^2 + 2A_1 t + D_1}}, \tag{6}$$

where $X = \frac{x + B_1 - \alpha(A_1 + t)}{\sqrt{t^2 + 2A_1 t + D_1}}$ is a similarity variable.

Substitution of (6) into (1) results the following equation after integration

$$2\nu F'(X) = F^2 - 2A_1 X F + D_1 X^2 + b_1, \tag{7}$$

where b_1 is an arbitrary constant of integration.

Eq. (7) can be transformed into an ODE of second order by taking $F = -\frac{2\nu}{v} \frac{dv}{dX}$

$$v''(X) + \frac{A_1 X}{\nu} v' + \left(\frac{D_1 X^2 + b_1}{4\nu^2}\right) v = 0. \tag{8}$$

Eq. (8) becomes Bessel's Equation [22] under the parametric restrictions $b = -\frac{A_1}{4\nu}$, $b_1 = 2A_1 \nu$ and $d^2 = \frac{D_1 - A_1^2}{16\nu^2}$.

If $D_1 < A_1^2$, then real solutions of Eq. (8) are not possible. On the other hand, if $D_1 > A_1^2$ then real solutions of (8) can be written as

$$v = X^{\frac{1}{2}} e^{bX^2} \left[k_1 J_{\frac{1}{4}}(X^2 d) + k_2 J_{-\frac{1}{4}}(X^2 d) \right], \tag{9}$$

where k_1 and k_2 are arbitrary constants of integration and $J_n(.)$ is the first kind Bessel function of order n.

Assuming $k = \frac{k_2}{k_1}$ ($k_1 \neq 0$), solution of (1) is furnished by

$$u(x,t) = \alpha + \frac{X}{\sqrt{t^2 + 2A_1 t + D_1}} \left[(t + A_1) - 4\nu d \frac{J_{-\frac{3}{4}}(X^2 d) - k J_{\frac{3}{4}}(X^2 d)}{J_{\frac{1}{4}}(X^2 d) + k J_{-\frac{1}{4}}(X^2 d)} \right] \tag{10}$$

Solution curves for expression (10) are shown in Fig. 1. In Fig. 1(a), the viscous damping effect is obvious because u decreases as ν increases from 0.10 to 0.40 at time t showing that (1) experiences a loss of momentum hence, loss of kinetic energy with respect to time. Also for all t solutions (10) show the degenerate nature near to $x = 0.06143$, $u = 0.4424$.

Apart from it, solution profiles in Fig. 1(a) at small viscosity $\nu = 0.10$ are smooth in the range $-1.50 \leq x \leq 1.50$ so these solutions lead to some physical applications as discussed in Miller [11]. This happens due to the presence of the term νu_{xx} in right hand side of Eq. (1). Furthermore, if such initial data for Eq. (1) is used which has finite discontinuity for small values of ν then u_{xx} becomes very large, and the viscous term becomes important whenever the discontinuity begins to form. This term keeps the solution smooth for all time, and determines the correct physical nature of the system.

Solution curves represented in Fig. 1(b) at $t = 0.5$ for $\nu = 0.1$ support to the fact that the solutions of Burgers' Eq. (1) are impractical for the small values of viscosity due to slow convergence of solutions [11]. It can also be observed that the number of shocks increases as ν decreases. So it is physically meaningful leading to a new research scope to capture the shocks and resolving them analytically or numerically.

(Ib): If $D_1 = A_1^2$, solution is given by

$$u = \frac{(C_1 - A_1 B_1)(2t + A_1)}{2(t + A_1)^3} + \frac{(x + B_1)}{(t + A_1)^2} t + \frac{F_1(X_1)}{(t + A_1)}, \tag{11}$$

with $X_1 = \frac{C_1 - A_1 B_1}{2(t+A_1)^2} + \frac{(x+B_1)}{t+A_1}$ as the similarity variable, while k_1 and k_2 are constants of integration, $b = -\frac{A_1}{4\nu}$ and $\alpha_1^2 = \frac{2}{9}\frac{(A_1 B_1 - C_1)}{\nu^2}$ are parameters.
So we obtain

$$F_1(X_1) = A_1 X_1 - 3\nu\alpha_1\sqrt{X_1}\left[\frac{J_{-\frac{2}{3}}(\alpha_1 X_1^{3/2}) - kJ_{\frac{2}{3}}(\alpha_1 X_1^{3/2})}{J_{\frac{1}{3}}(\alpha_1 X_1^{3/2}) + kJ_{-\frac{1}{3}}(\alpha_1 X_1^{3/2})}\right]. \tag{12}$$

We have taken $k = \frac{k_2}{k_1}$ (follows $k_1 \neq 0$). Thus by the help of Eqs. (11) and (12), the solution of (1) can be expressed as

$$u(x,t) = \frac{(x+B_1)}{(t+A_1)} + \frac{1}{(t+A_1)^2}\left[C_1 - A_1 B_1 - \frac{3\nu\alpha_1}{\sqrt{2}}\sqrt{2(x+B_1)(t+A_1) + (C_1 - A_1 B_1)}\right.$$
$$\left. \times \left(\frac{J_{-\frac{2}{3}}(\alpha_1 X_1^{3/2}) - kJ_{\frac{2}{3}}(\alpha_1 X_1^{3/2})}{J_{\frac{1}{3}}(\alpha_1 X_1^{3/2}) + kJ_{-\frac{1}{3}}(\alpha_1 X_1^{3/2})}\right)\right]. \tag{13}$$

Case (II): If $a_1 = 0$ and $a_2 \neq 0$ then one can obtain the similarity variable $X_2 = \frac{(x-A_2 t+l)}{\sqrt{C_2+2t}}$ (where $l = B_2 - A_2 C_2$, $A_2 = -\frac{a_3}{a_2}$, $B_2 = -\frac{a_4}{a_2}$ and $C_2 = -\frac{a_5}{a_2}$) with similarity form

$$u = A_2 + \frac{F_2(X_2)}{\sqrt{C_2+2t}}. \tag{14}$$

and corresponding ODE after integration can be written as

$$2\nu F_2'(X_2) = F_2^2(X_2) - 2X_2 F_2(X_2) + b_3, \tag{15}$$

Eq. (15) can be transformed into the following with help of $F_2(X_2) = -\frac{2\nu}{v}\frac{dv}{dX_2}$

$$v''(X_2) + \frac{X_2}{\nu}v'(X_2) + \frac{b_3}{4\nu^2}v(X_2) = 0. \tag{16}$$

Setting $b_3 = -4\nu$ and applying the transformation $v(X_2) = X_2 P(X_2)$ on (16) yields the solution of (1) as follows

$$u(x,t) = A_2 - \frac{2\nu}{X_2^2\sqrt{C_2+2t}}\left[X_2 + \frac{\exp\left(-\frac{X_2^2}{2\nu}\right)}{k + \int \frac{1}{X_2^2}\exp\left(-\frac{X_2^2}{2\nu}\right)dX_2}\right]. \tag{17}$$

On the other hand if **$b_3 = 0$** then particular solution is given by

$$u(x,t) = A_2 + \frac{1}{\sqrt{C_2+2t}}\left[\frac{\exp\left(-\frac{X_2^2}{2\nu}\right)}{c_2 - \frac{1}{2\nu}\int\exp\left(-\frac{X_2^2}{2\nu}\right)dX_2}\right] \tag{18}$$

It can also be noticed that solutions for $\nu = 0.10, 0.50, 0.70$ degenerate while coincide for $\nu = 0.10$ and 0.70. Solution (13) is an explicit expression involving Bessel functions. Multisolitons occur at $x = -0.25$ for (13) and represented by Fig. 2 taking $a_1 = 1.3517$, $a_2 = 0.8308$, $a_3 = 0.5853$, $a_4 = 0.5497$, $k = 0.9172$, $b = 1.5366$ and $b_2 = -0.1229$. Nature of solution curves have been represented for viscosity $\nu = 0.10$.
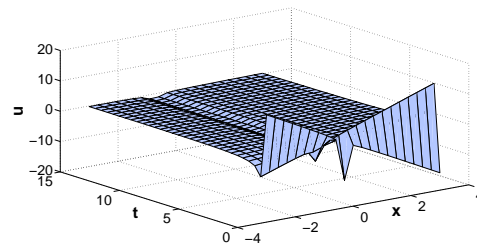
Figure 2. Multisoliton profile of (13) for $\nu = 0.10$

Conclusions

In summary, Lie groups theory reduces the one–dimensional Burgers' equation into an ODE which is solved analytically. Since we have adequate options to select different arbitrary constants applying the STM thus three exact solutions have been obtained explicitly. Solutions in the case (Ia) show viscous damping effect with time evolution which are of most physical significance and support to Miller [11] for a very small value of viscosity. Solutions in the case (Ib) are useful for shocks capturing of Burgers' Eq. (1). The quadrature form solutions are obtained in the case (II). Solutions in the cases (Ia), (Ib) and (II) are constructed in terms of Bessel functions. Three new solutions in the cases (Ia), (Ib) and (II) have been obtained. The exact solutions can serve as benchmark in analysis, accuracy testing of numerical results and their comparison.

1. References

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